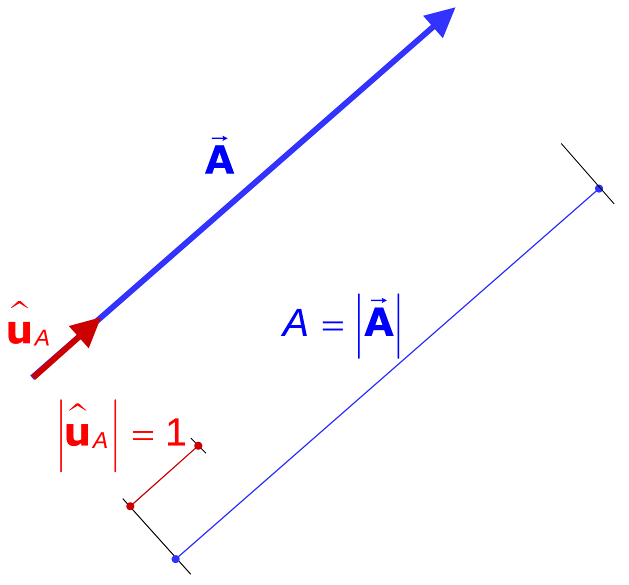
Vectores

Vector

Segmento orientado en el espacio (módulo, dirección y sentido).

$$\vec{\mathbf{A}} = \hat{\mathbf{u}}_A A$$

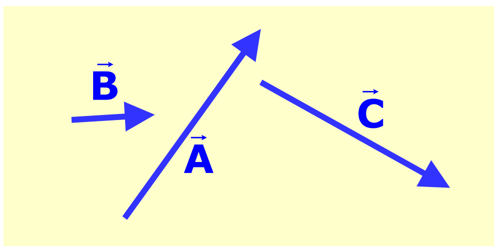


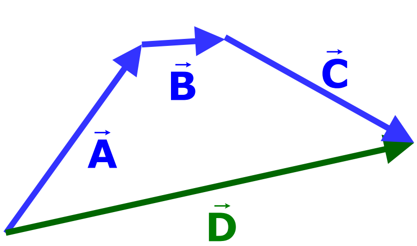
Ejemplo

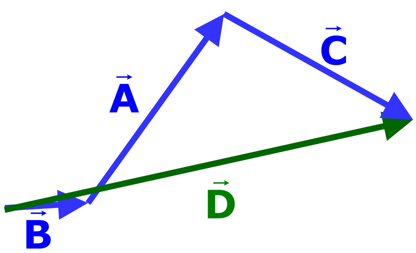
Vector velocidad
módulo: 500 km/h
sentido: el de avance

Vector fuerza gravitatoria
módulo: 600 kN
sentido: vertical descendente

Suma de vectores $\vec{D} = \vec{A} + \vec{B} + \vec{C}$

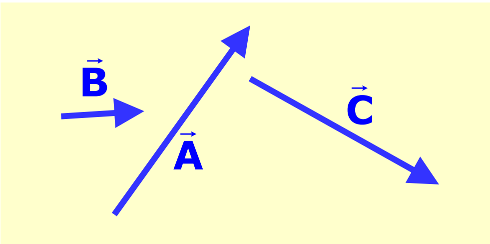


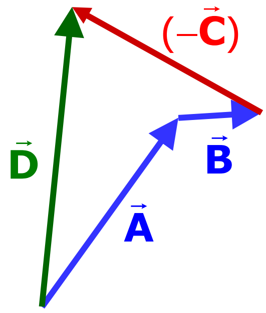




Suma y resta de vectores

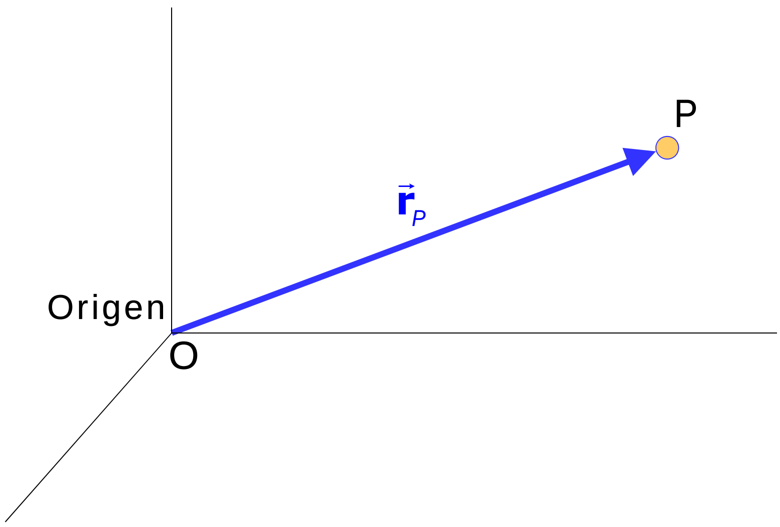
$$\vec{D} = \vec{A} + \vec{B} - \vec{C} = \vec{A} + \vec{B} + (-\vec{C})$$



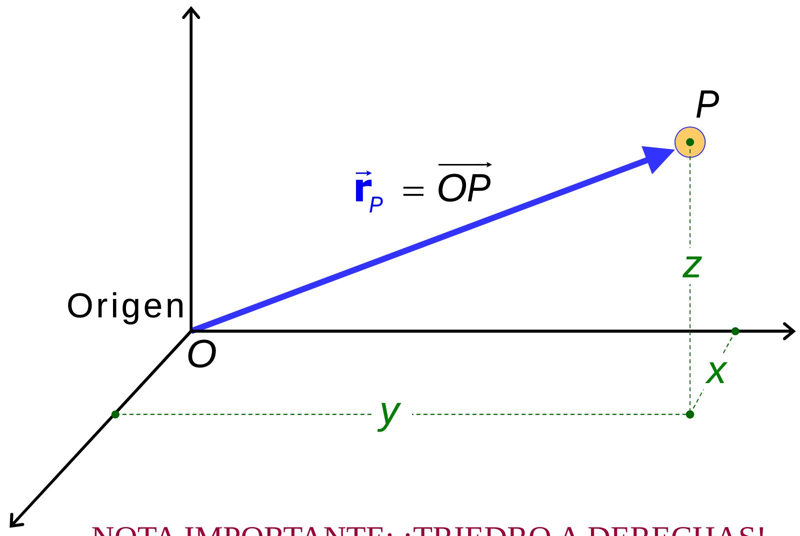


Vector de posición de un punto P

Vector cuyo origen es el origen de coordenadas y extremo el punto P

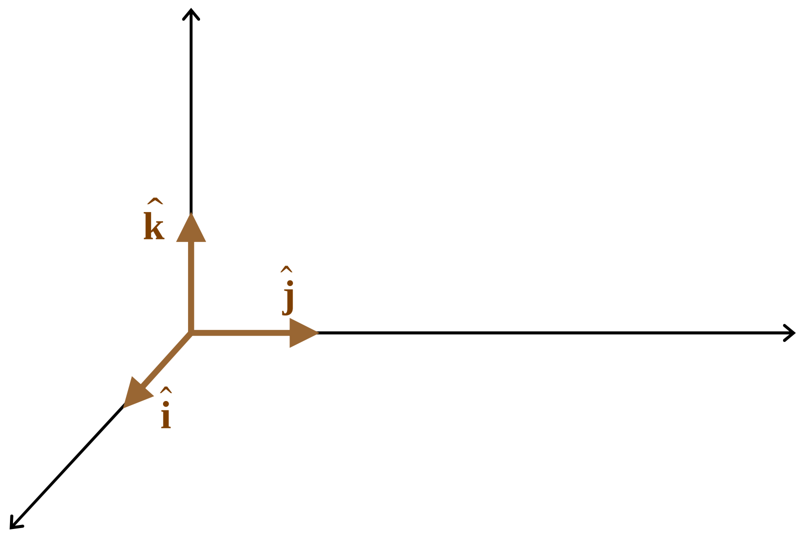


Coordenadas cartesianas

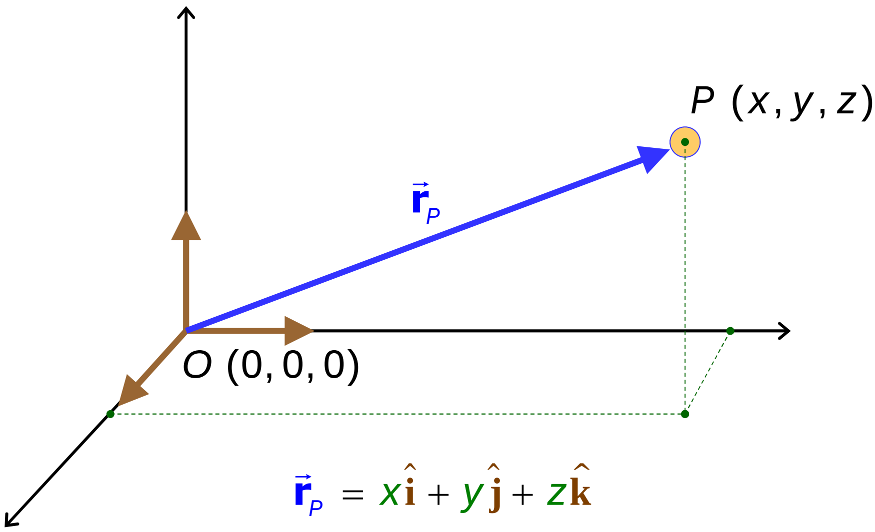


NOTA IMPORTANTE: ¡TRIEDRO A DERECHAS!

Coordenadas cartesianas

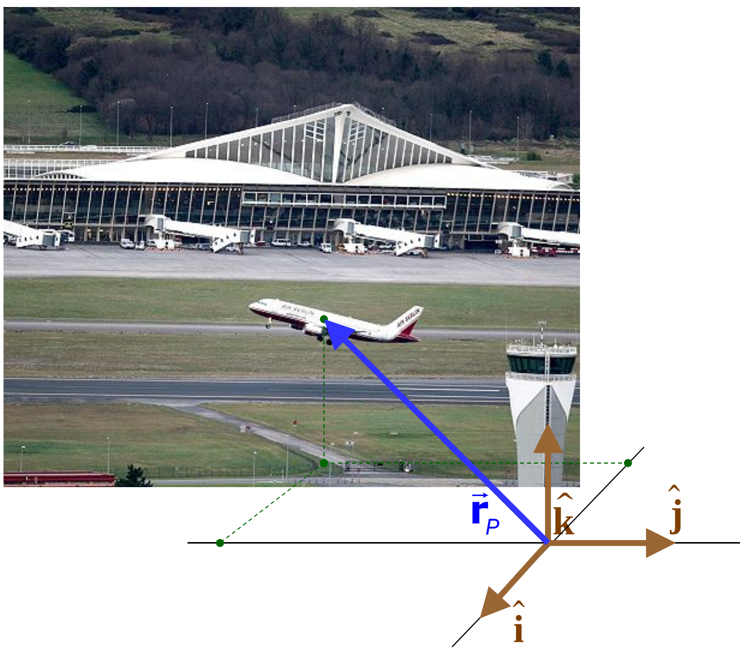


Coordenadas cartesianas

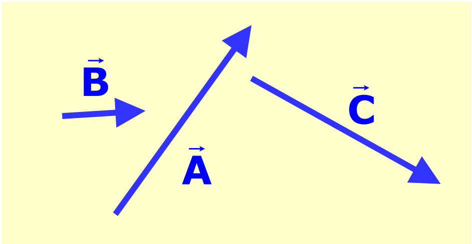


$$\vec{\mathbf{r}}_P = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

Ejemplo



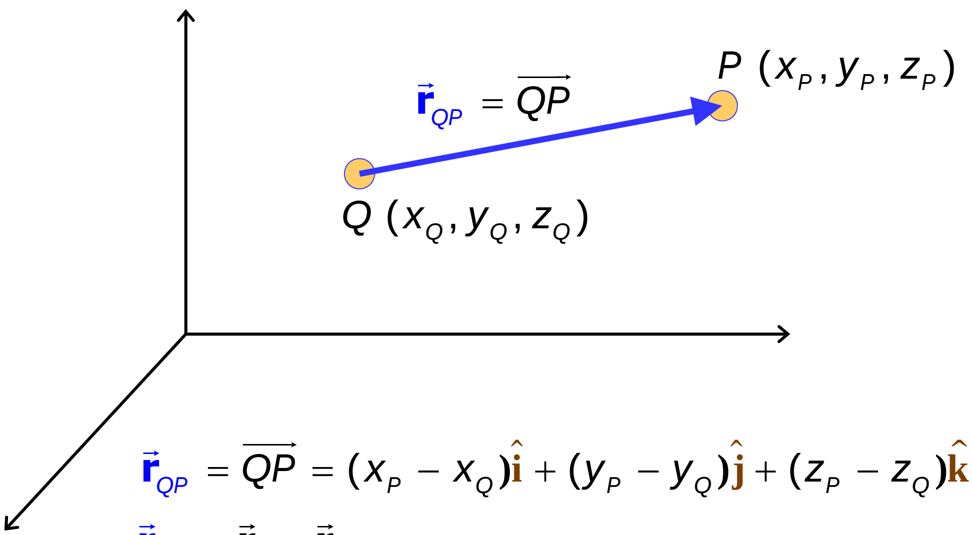
Coordenadas cartesianas



$$\vec{A} = x_A\hat{\mathbf{i}} + y_A\hat{\mathbf{j}} + z_A\hat{\mathbf{k}}$$
$$\vec{B} = x_B\hat{\mathbf{i}} + y_B\hat{\mathbf{j}} + z_B\hat{\mathbf{k}}$$
$$\vec{C} = x_C\hat{\mathbf{i}} + y_C\hat{\mathbf{j}} + z_C\hat{\mathbf{k}}$$

$$\vec{D} = \vec{A} + \vec{B} - \vec{C}$$
$$\vec{D} = (x_A + x_B - x_C)\hat{\mathbf{i}} + (y_A + y_B - y_C)\hat{\mathbf{j}} + (z_A + z_B - z_C)\hat{\mathbf{k}}$$

Coordenadas cartesianas

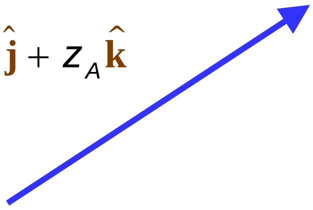


$$\vec{r}_{QP} = \overrightarrow{QP} = (x_P - x_Q)\hat{\mathbf{i}} + (y_P - y_Q)\hat{\mathbf{j}} + (z_P - z_Q)\hat{\mathbf{k}}$$
$$\vec{r}_{QP} = \vec{r}_P - \vec{r}_Q$$

Coordenadas cartesianas

Módulo de un vector

$$\vec{\mathbf{A}} = x_A\hat{\mathbf{i}} + y_A\hat{\mathbf{j}} + z_A\hat{\mathbf{k}}$$



$$A = \left|\vec{\mathbf{A}}\right| = \sqrt{x_A{}^2 + y_A{}^2 + z_A{}^2}$$

Multiplicación de vectores

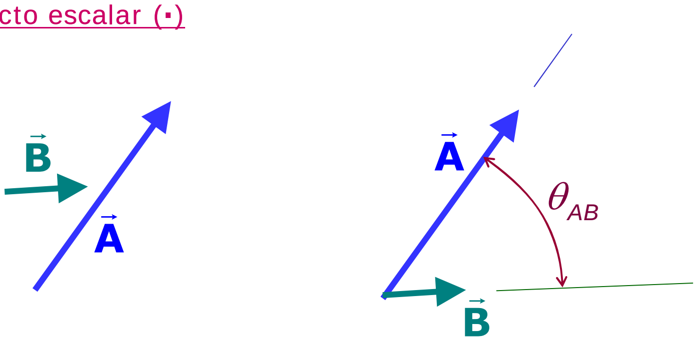
Multiplicación por un escalar

$$k\vec{\mathbf{A}} = \hat{\mathbf{u}}_A kA$$

$$k\vec{\mathbf{A}} = kx_A\hat{\mathbf{i}} + ky_A\hat{\mathbf{j}} + kz_A\hat{\mathbf{k}}$$

Multiplicación de vectores

Producto escalar (▪)

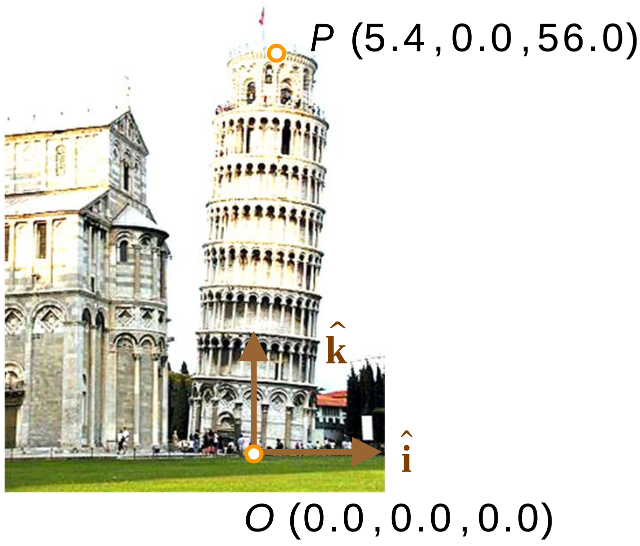


$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta_{AB}$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = x_A x_B + y_A y_B + z_A z_B$$

Nota: ¿cómo calcular el ángulo entre dos vectores? $\cos\theta_{AB} = \dfrac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{AB}$

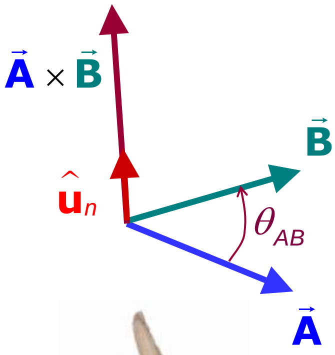
Ejemplo: ¿Ángulo de inclinación de la torre de Pisa?



Multiplicación de vectores

Producto vectorial (**x**)

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \hat{\mathbf{u}}_n AB \operatorname{sen} \theta_{AB}$$



$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix}$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (y_A z_B - z_A y_B)\hat{\mathbf{i}} + (z_A x_B - x_A z_B)\hat{\mathbf{j}} + (x_A y_B - y_A x_B)\hat{\mathbf{k}}$$

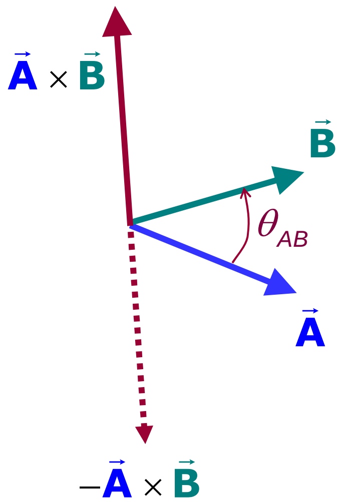
Ejemplo: ¿Módulo del vector V?

$$\mathbf{V} = (10\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + \mathbf{j}) =$$

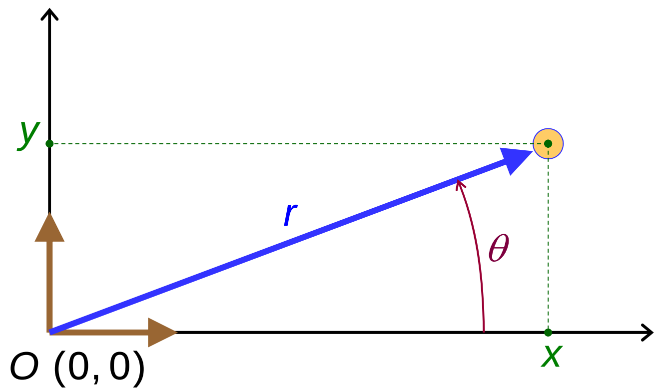
Multiplicación de vectores

Producto vectorial (**x**)

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$



Coordenadas polares (2D)

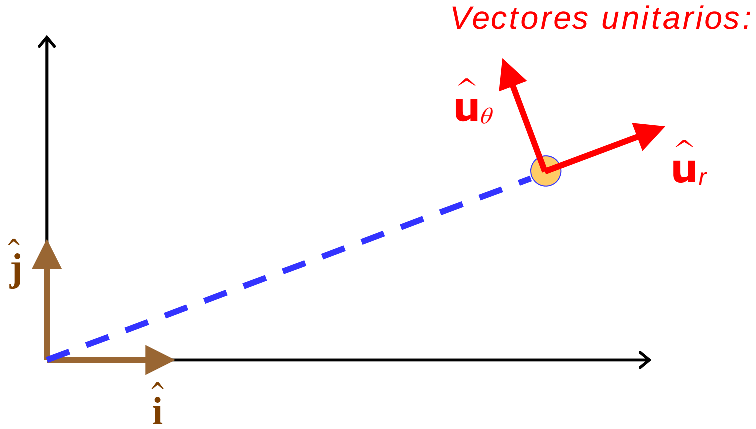


Coordenadas cartesianas: x, y

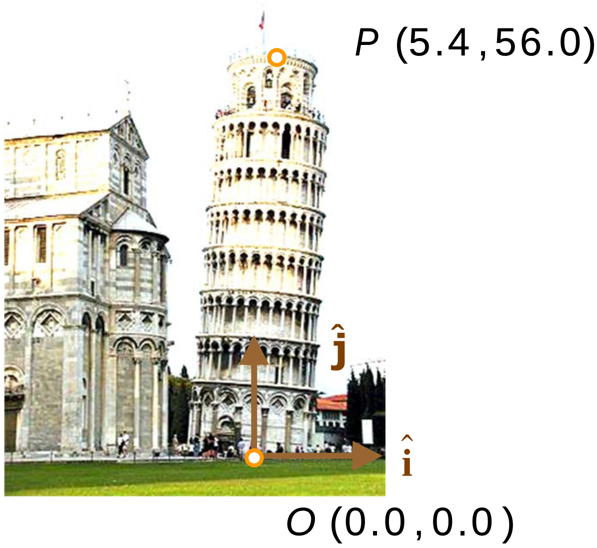
Coordenadas polares: r, θ

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

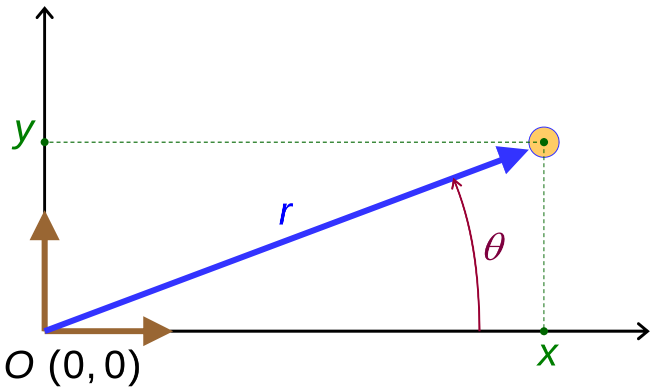
Coordenadas polares (2D)



Ejemplo: ¿Coordenadas polares del punto *P*?



Coordenadas cilíndricas (3D)

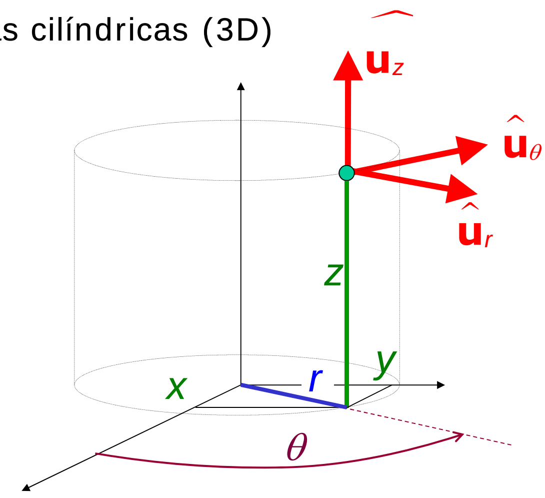


Coordenadas cartesianas: x, y, z

Coordenadas cilíndricas: r, θ, z

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

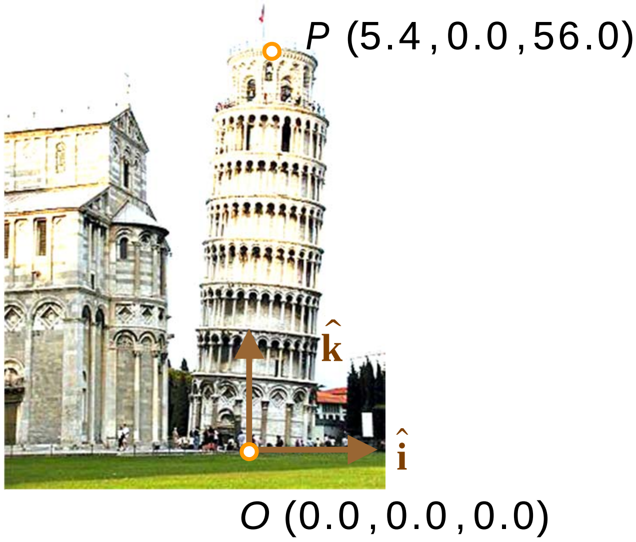
Coordenadas cilíndricas (3D)



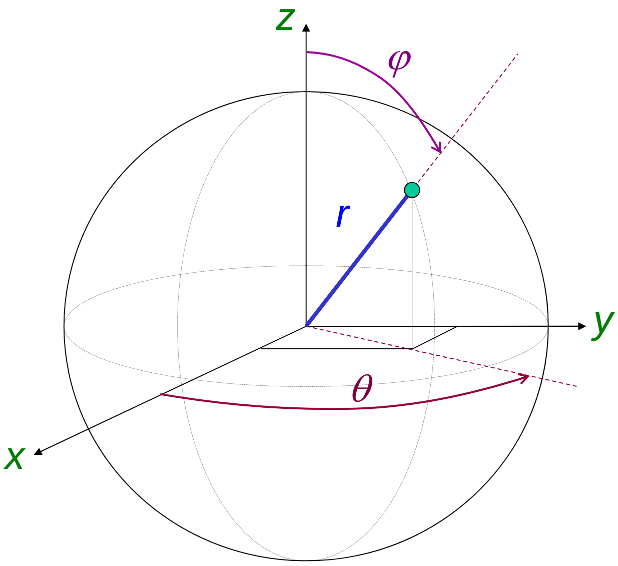
Coordenadas cartesianas: x, y, z

Coordenadas cilíndricas: r, θ, z

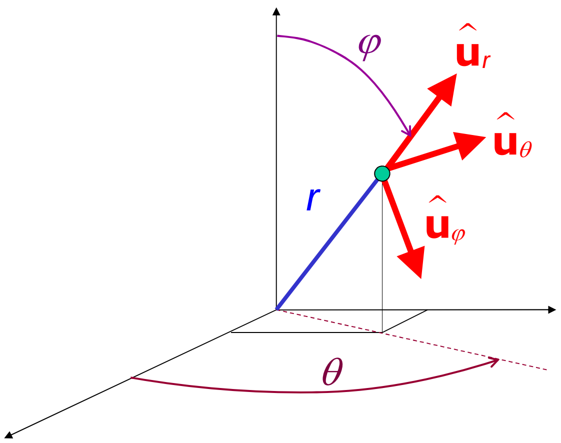
Ejemplo: ¿Coordenadas cilíndricas del punto *P*?



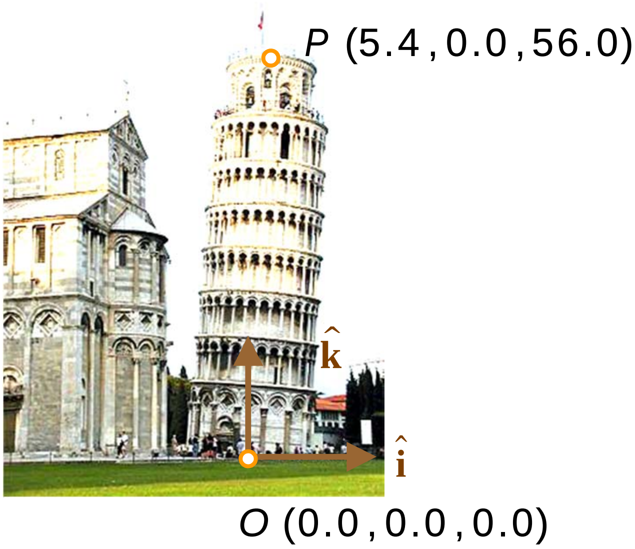
Coordenadas esféricas (3D)



Coordenadas esféricas (3D)



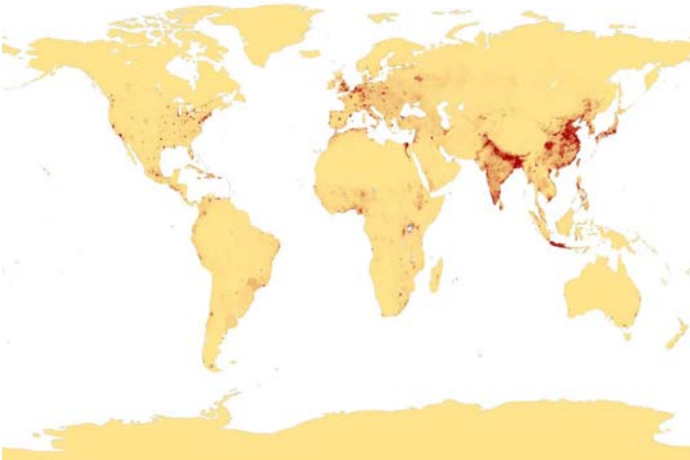
Ejemplo: ¿Coordenadas esféricas del punto *P*?



Campos escalares

Funciones escalares (campos escalares)

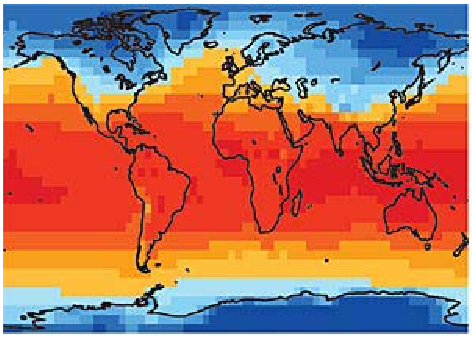
Ejemplo



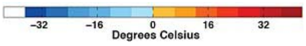
Densidad de población

Funciones escalares (campos escalares)

Ejemplo

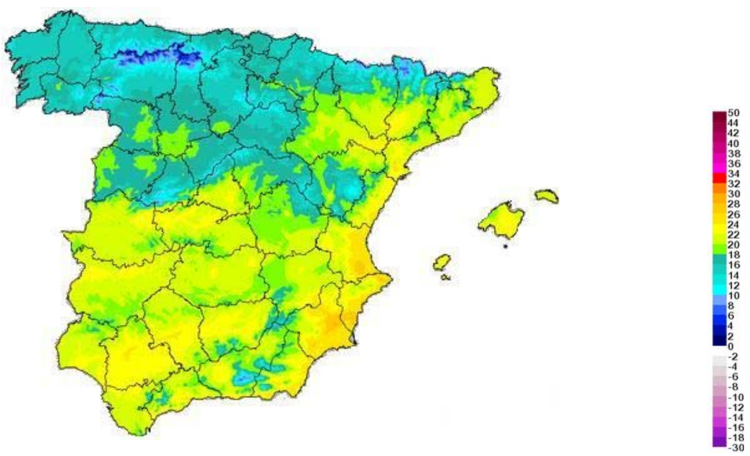


Temperaturas medias en febrero



Funciones escalares (campos escalares)

Ejemplo

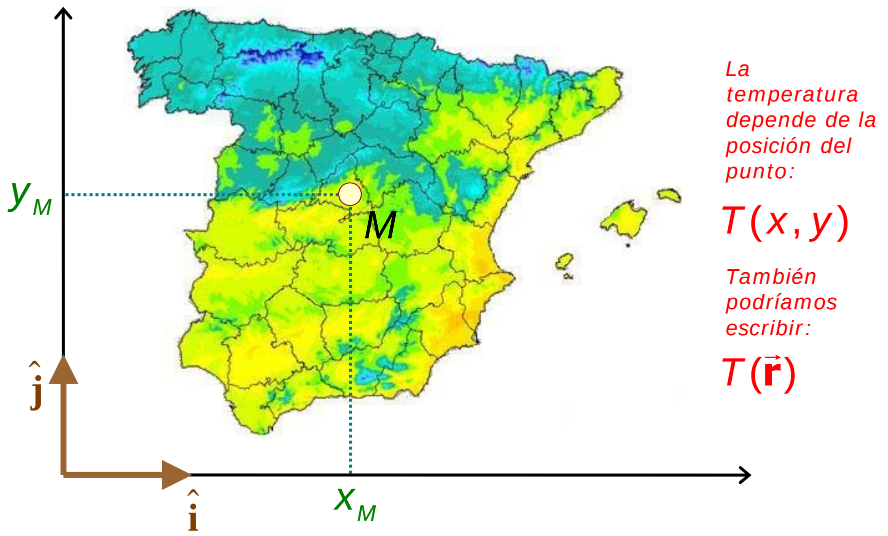


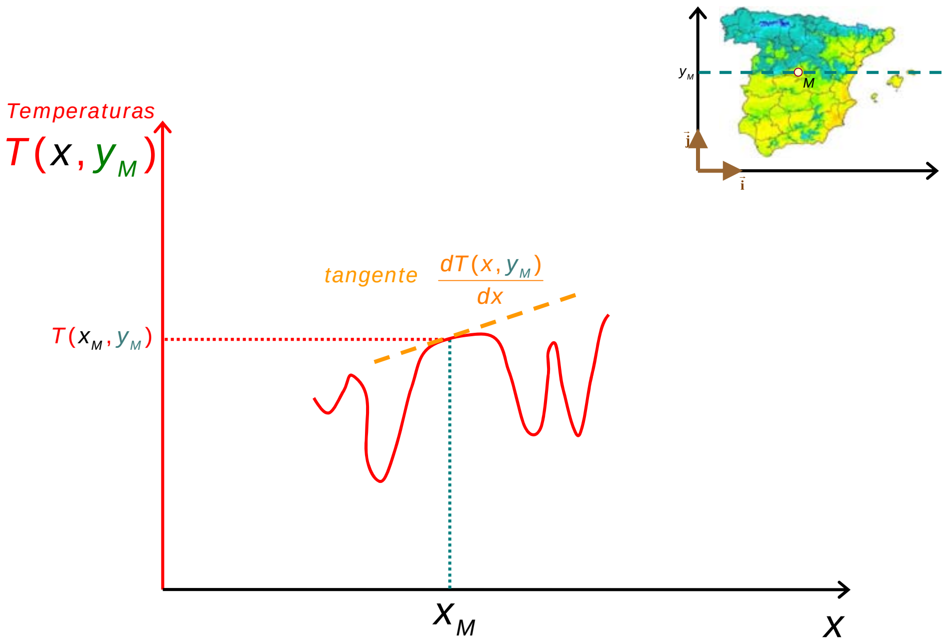
AEMet

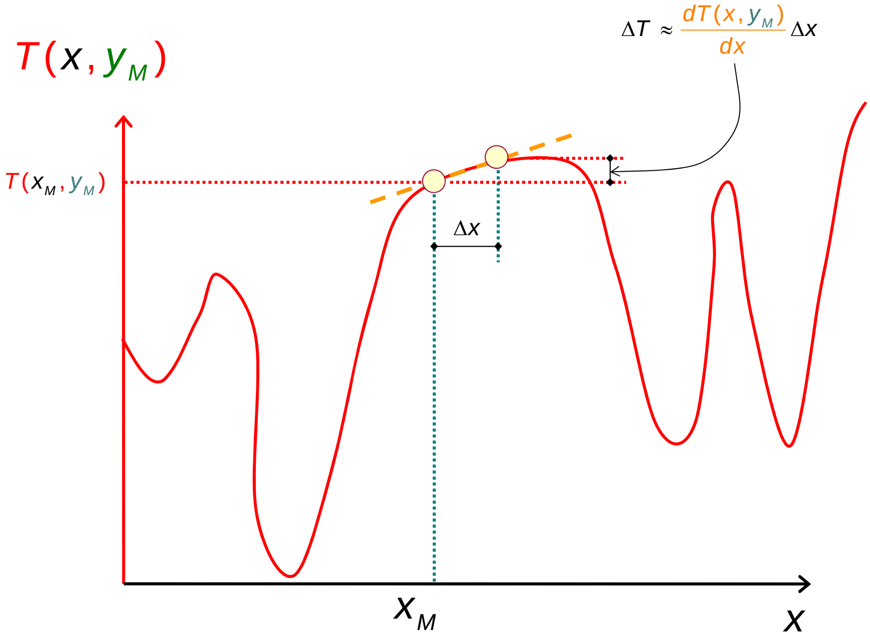
Temperaturas ~ 15:00h. 14/05/2009

Funciones escalares (campos escalares)

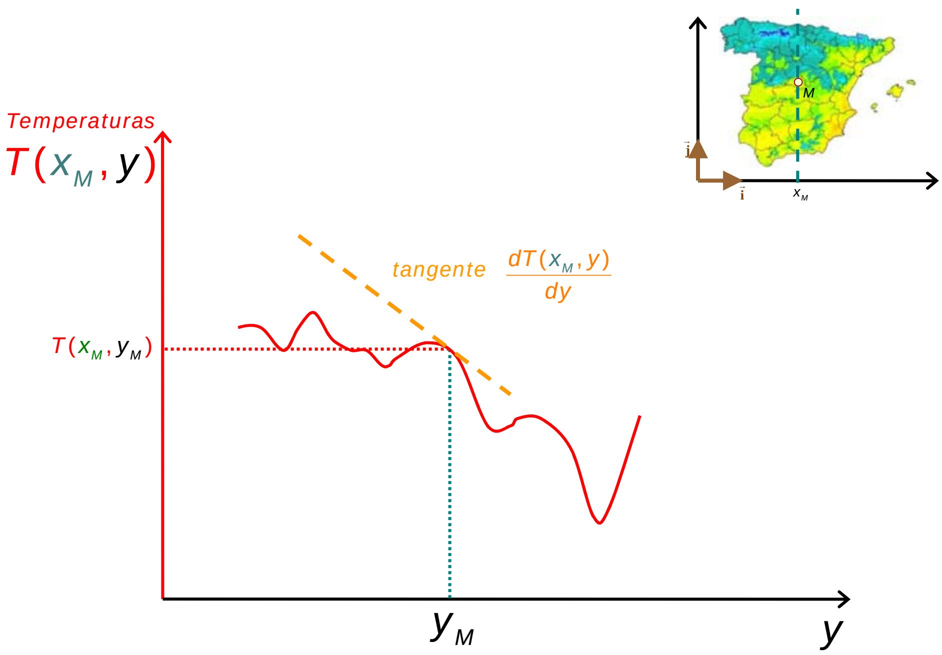
Ejemplo



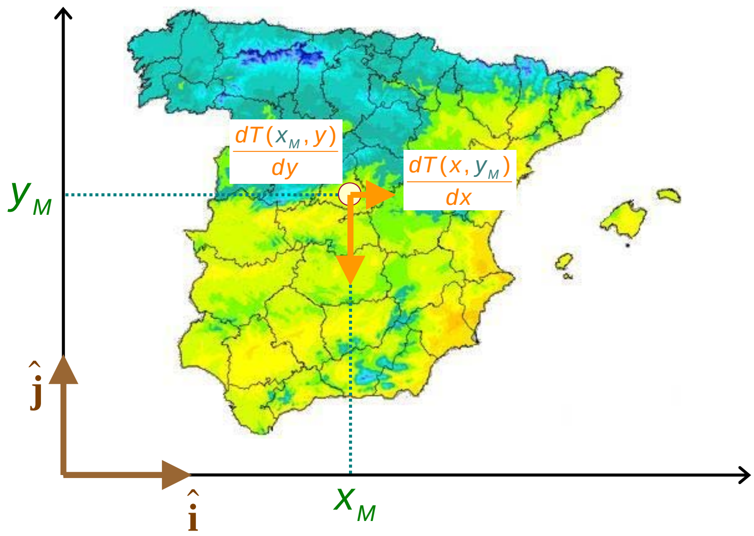




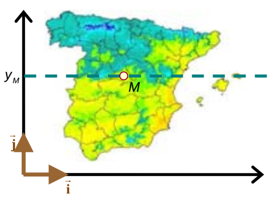
$\Delta T \approx \frac{dT(x, y_M)}{dx} \Delta x$
$T(x, y_M)$
$T(x_M, y_M)$
Δx
x_M
x



Temperaturas
$T(x_M, y)$
tangente
$\frac{dT(x_M, y)}{dy}$
$T(x_M, y_M)$
y_M
y
M
x_M

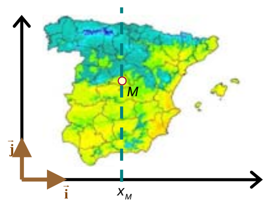


¿Qué hemos hecho?



$$\frac{dT(x, y_M)}{dx}$$

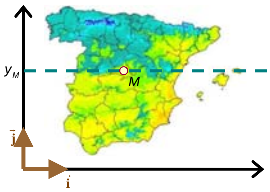
"Derivada parcial" de la temperatura T con respecto a la variable x, cuando la variable y es constante



$$\frac{dT(x_M, y)}{dy}$$

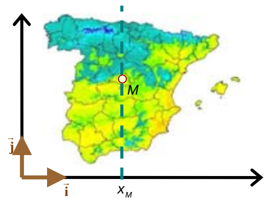
"Derivada parcial" de la temperatura T con respecto a la variable y, cuando la variable x es constante

"Derivadas parciales"



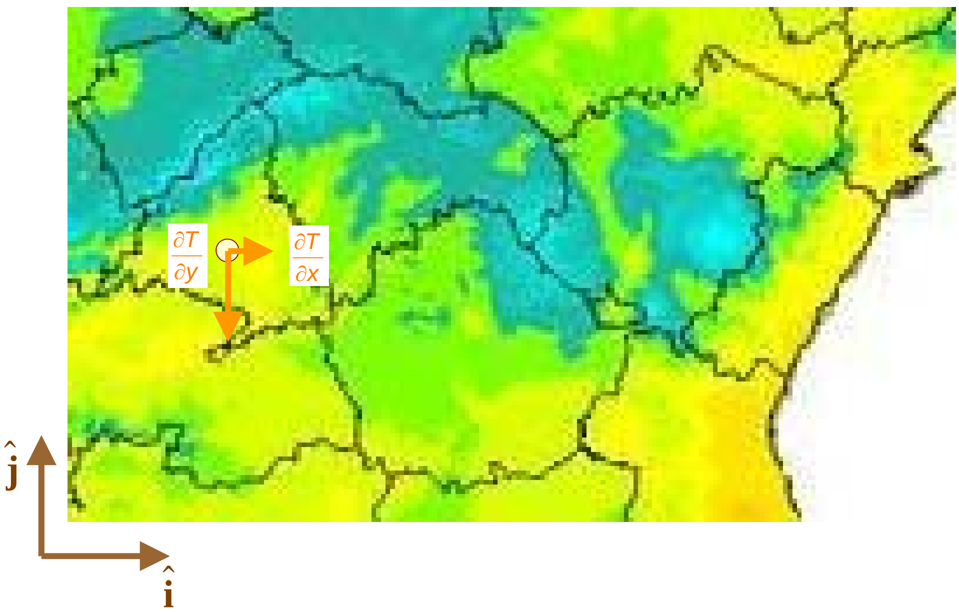
$$\frac{\partial T}{\partial x}$$

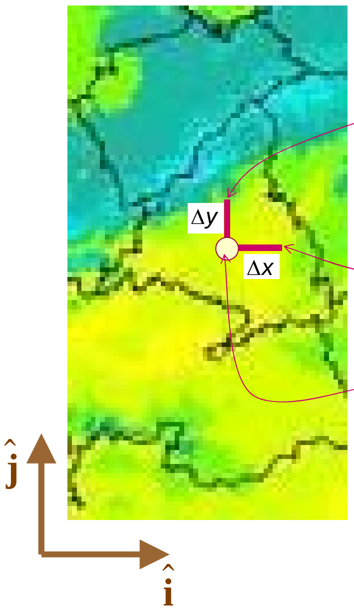
Derivada parcial de la temperatura T con respecto a la variable x, en el punto (x_M,y_M)



$$\frac{\partial T}{\partial y}$$

Derivada parcial de la temperatura T con respecto a la variable y, en el punto (x_M,y_M)

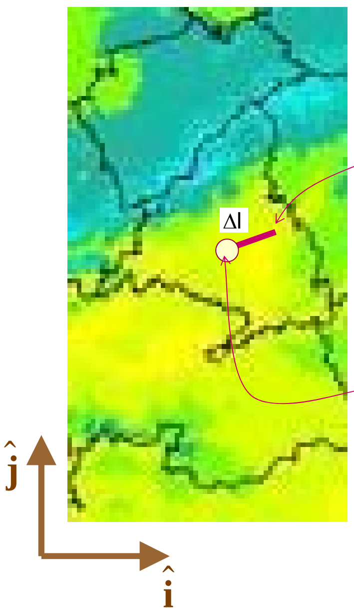




$$T(x_M, y_M) + \frac{\partial T}{\partial y}\Delta y$$

$$T(x_M, y_M) + \frac{\partial T}{\partial x}\Delta x$$

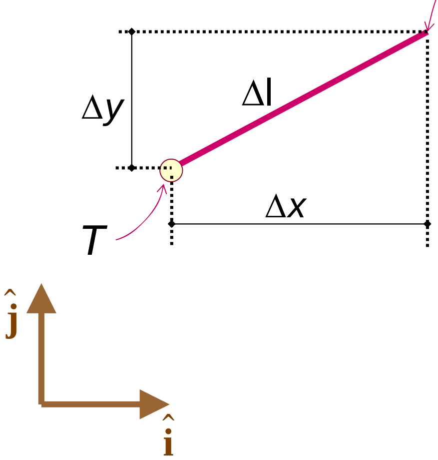
$$T(x_M, y_M)$$



$$T(x_M, y_M) + ¿\frac{\partial T}{\partial l}\Delta l\ ?$$

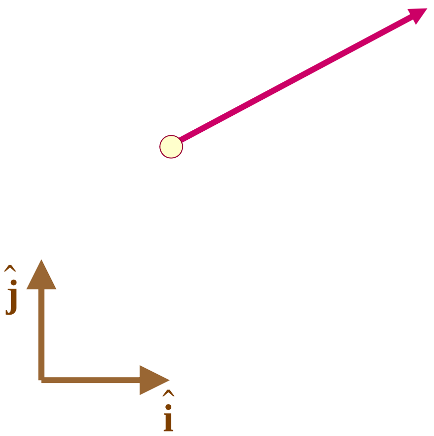
$$T(x_M, y_M)$$

$$\underbrace{T + \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y}_{\approx\, \Delta T}$$



$$\vec{\Delta \mathrm{l}} = \Delta \mathrm{l}\,\hat{\mathbf{u}}_{\mathrm{l}}$$

$$\vec{\Delta \mathrm{l}} = \Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}$$



$$\Delta T \approx \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y =$$

$$= \left(\frac{\partial T}{\partial x}\hat{\mathbf{i}} + \frac{\partial T}{\partial y}\hat{\mathbf{j}}\right)\cdot\left(\Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}\right)$$

$$\Delta T \approx \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y =$$

$$= \underbrace{\left(\frac{\partial T}{\partial x}\hat{\mathbf{i}} + \frac{\partial T}{\partial y}\hat{\mathbf{j}}\right)}_{\substack{\text{“Gradiente de T”} \\ \vec{\mathbf{grad}}\, T}}\cdot\underbrace{\left(\Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}\right)}_{\vec{\Delta \mathrm{l}}}$$

Gradiente de una función escalar

$$df = \frac{\partial f}{\partial \mathrm{l}} d\mathrm{l} = \overrightarrow{\mathbf{grad}}\, f \cdot \overrightarrow{d\mathrm{l}}$$

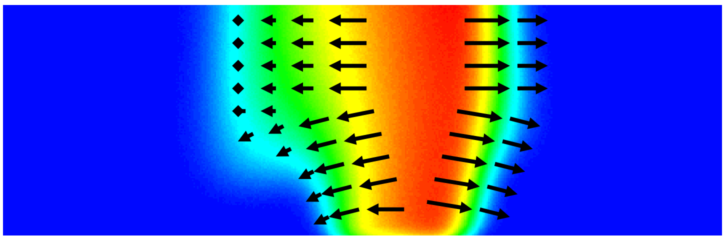
En el caso más general, tres dimensiones:

$$f(x, y, z)$$

$$\overrightarrow{\mathbf{grad}}\, f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

Significado físico del gradiente

En el espacio, si hay un campo escalar *f* no uniforme (no constante en el espacio) el lugar de los puntos con un mismo valor es una superficie (p.ej. isotermas, isobaras, superficies equipotenciales...). El gradiente ***gradf*** es perpendicular a estas superficies y está orientado en el sentido de los valores crecientes de *f*. Un valor grande de *f* significa fuertes variaciones en cortas distancias.



Temperaturas junto a una llama

Ejemplo: $f(x,y,z) = 2x + 3yz^2$

$$\frac{\partial f}{\partial x} = \frac{\partial(2x + 3yz^2)}{\partial x} = 2 + 0 = 2$$

$$\frac{\partial f}{\partial y} = \frac{\partial(2x + 3yz^2)}{\partial y} = 0 + 3z^2 = 3z^2$$

$$\frac{\partial f}{\partial z} = \frac{\partial(2x + 3yz^2)}{\partial z} = 0 + 3y2z = 6yz$$

$$\overrightarrow{\mathbf{grad}}\, f = 2\hat{\mathbf{i}} + 3z^2\hat{\mathbf{j}} + 6yz\hat{\mathbf{k}}$$

Ejemplo: $V(x,y) = \frac{\pi}{4}x^2y$

$$\overrightarrow{\mathbf{grad}}\, V = ?$$

$$\overrightarrow{\mathbf{grad}}\, V(2,3) = ?$$

(constante)

Ejemplo: $V(x,y,z) = a\,e^x \operatorname{sen}\left(\frac{\pi y}{4}\right)$

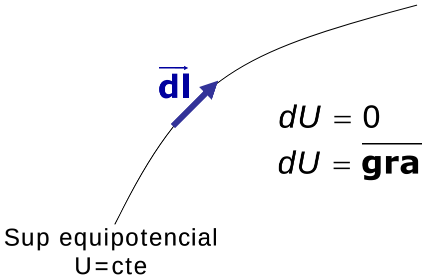
$$\overrightarrow{\mathbf{grad}}\, V = ?$$

$$\overrightarrow{\mathbf{grad}}\, V(1,1,0) = ?$$

Superficies equipotenciales:

En las que la función escalar U(x,y,z) es constante.

En ellas, se cumple que:

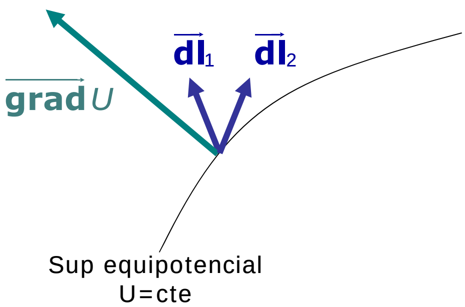


$$dU = 0 \qquad \Rightarrow \overrightarrow{\textbf{grad}}\, U \perp \vec{d\text{l}}$$
$$dU = \overrightarrow{\textbf{grad}}\, U \cdot \vec{d\text{l}}$$

El gradiente es perpendicular siempre a las superficies equipotenciales

El módulo de **grad***U*, |**grad***U*|, mide la variación de *U* en dirección perpendicular a las superficies equipotenciales. Además, esta dirección es la de máxima variación.

Demostración: *dU* será máxima cuando **grad***U* sea paralelo a **d**l.



Otros operadores diferenciales que estudiaremos:

Sea el campo vectorial:_

$$\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$$

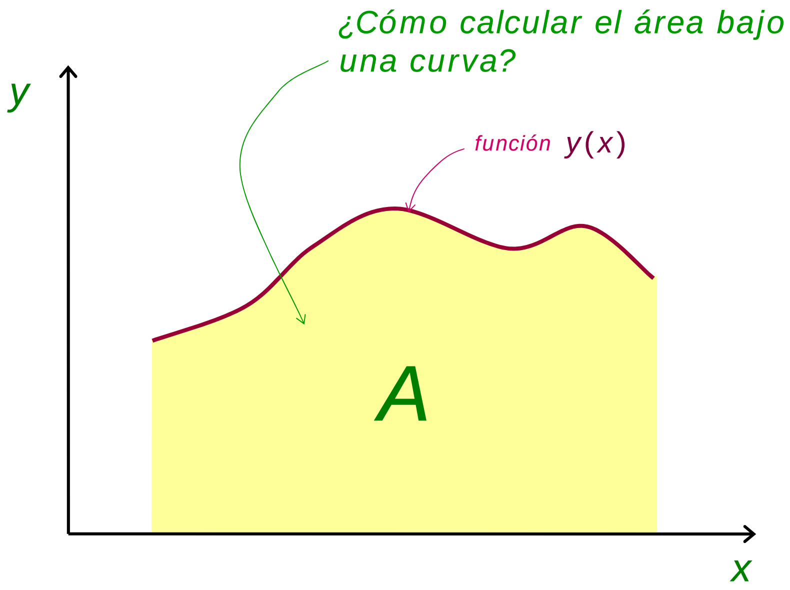
La divergencia del campo vectorial, en coordenadas cartesianas, es igual a:_

$$\mathrm{div}\vec{\mathbf{F}} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

El rotacional del campo vectorial , en coordenadas cartesianas, es igual a:

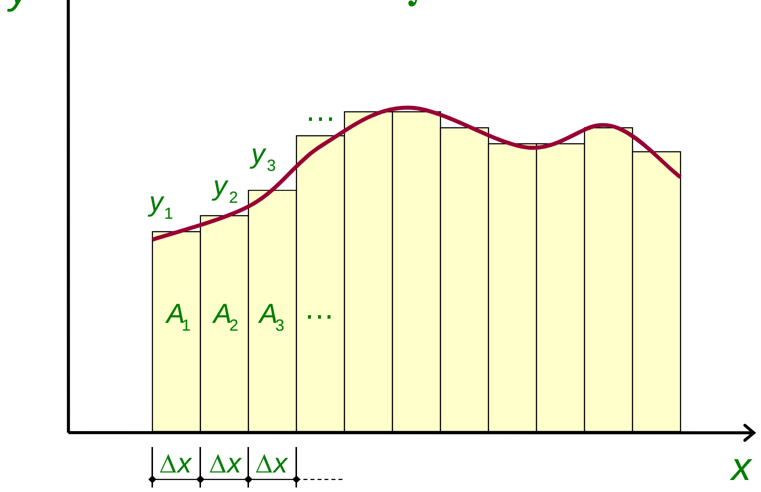
$$\overrightarrow{\mathbf{rot}\vec{\mathbf{F}}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{\mathbf{k}}$$

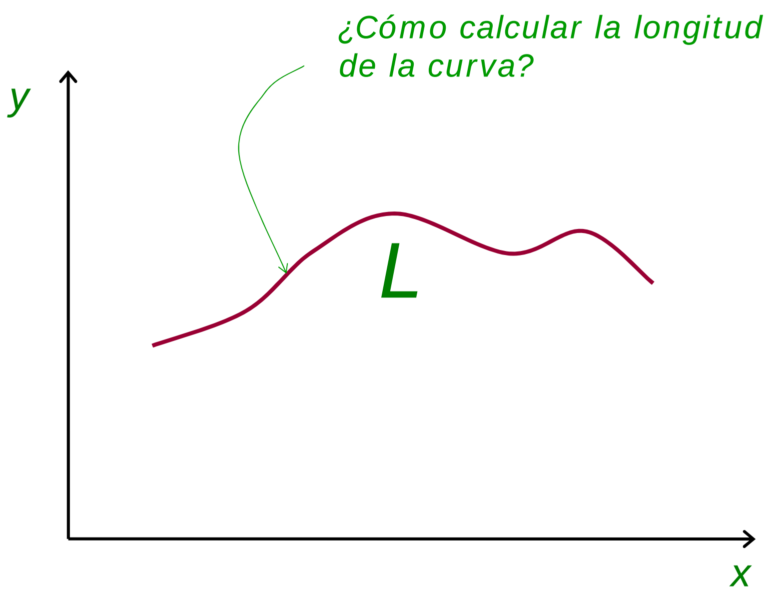
Integrales



$$A \approx A_1 + A_2 + A_3 + \ldots = \sum_i A_i = \sum_i y_i \Delta x$$

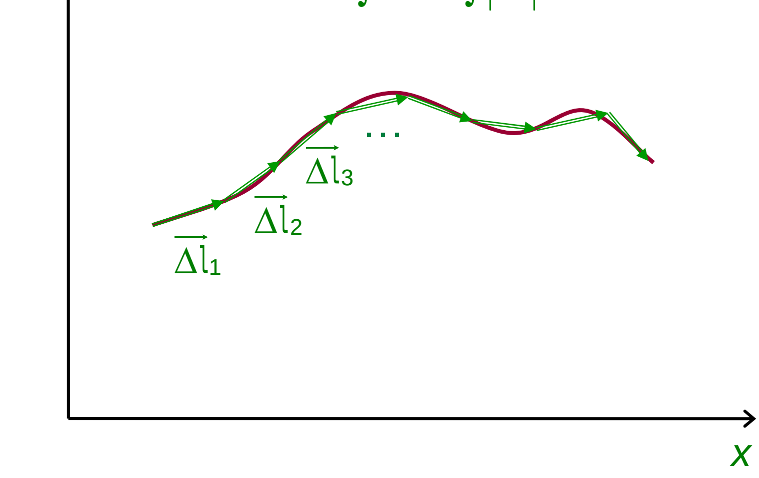
$$A = \int y dx$$

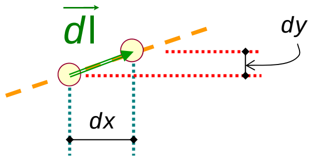




$$L \approx \Delta l_1 + \Delta l_2 + \Delta l_3 + \ldots = \sum_i \Delta l_i = \sum_i \left|\overrightarrow{\Delta l_i}\right|$$

$$L = \int dl = \int \left|\overrightarrow{dl}\right|$$



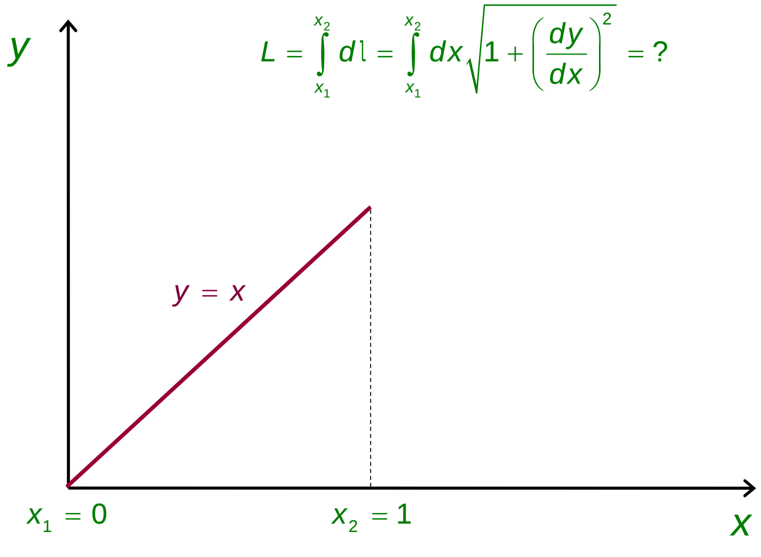


$$d\text{l} = \left|\overrightarrow{d\text{l}}\right| = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \frac{dy^2}{dx^2}} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$L = \int d\text{l} = \int dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

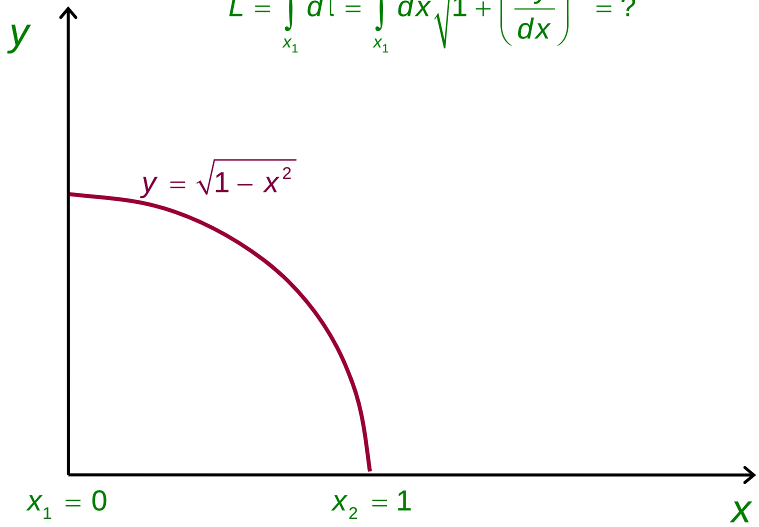
<u>Ejemplo</u>:

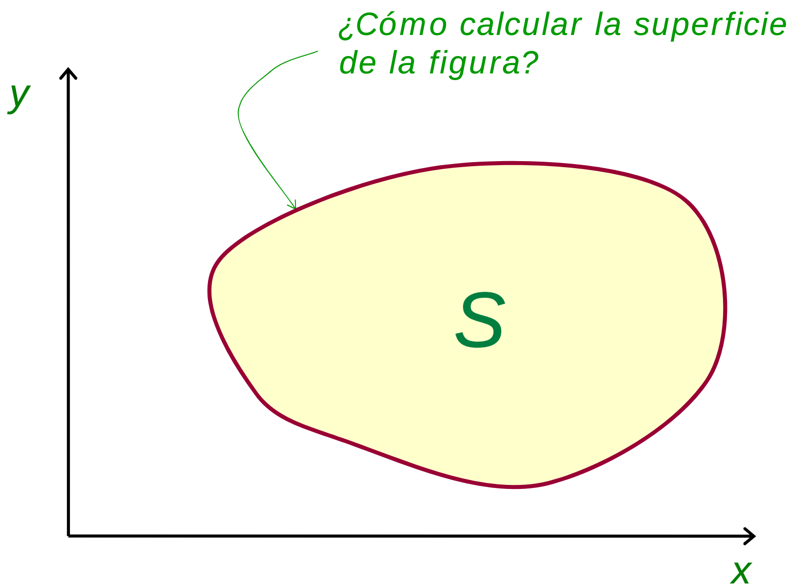
$$L = \int_{x_1}^{x_2} d\text{l} = \int_{x_1}^{x_2} dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = ?$$



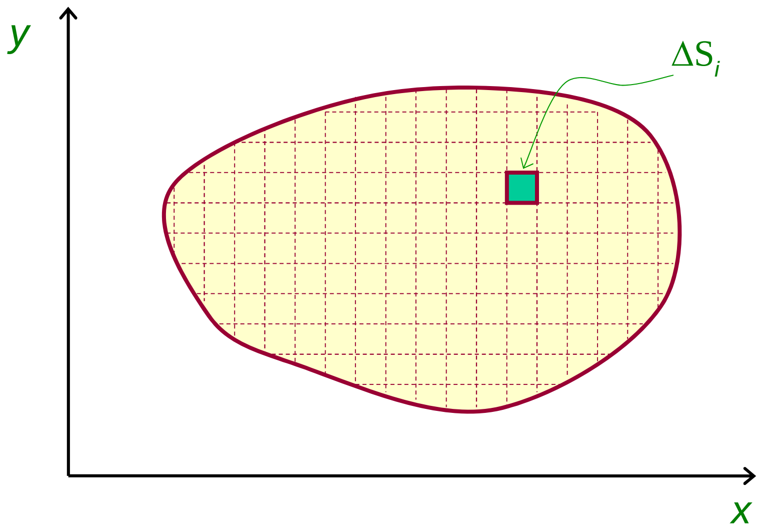
Ejemplo:

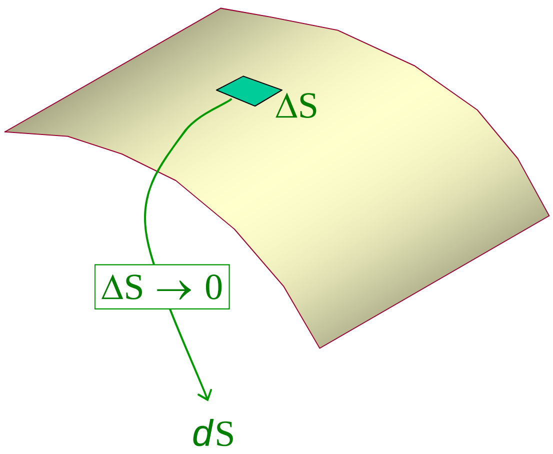
$$L = \int_{x_1}^{x_2} d\ell = \int_{x_1}^{x_2} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = ?$$



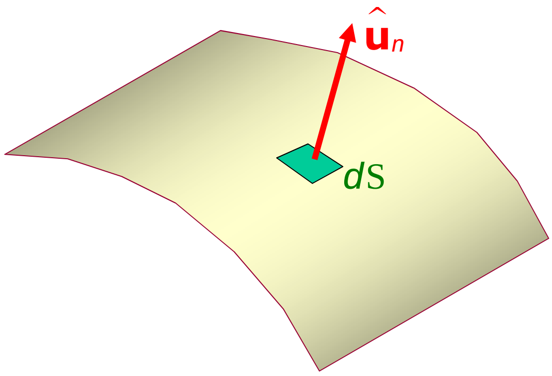


$$S \approx \Delta S_1 + \Delta S_2 + \Delta S_3 + \ldots = \sum_i \Delta S_i$$





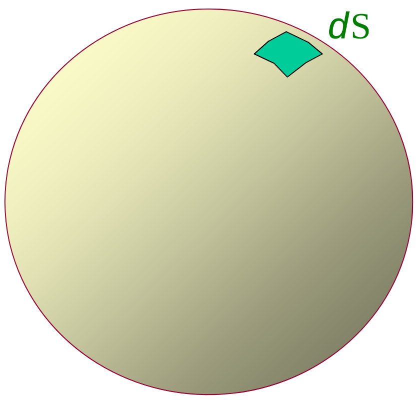
$$S = \iint dS$$



$$\overrightarrow{dS} = dS\hat{\mathbf{u}}_n$$

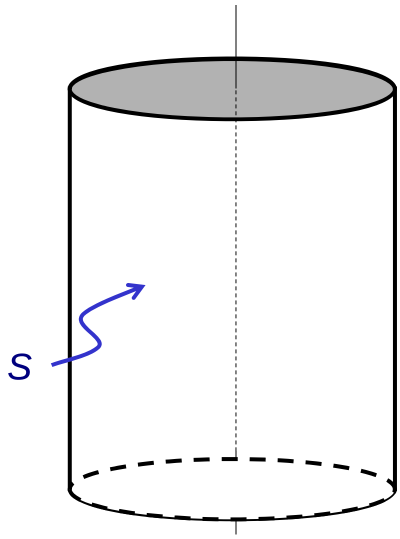
$$S = \iint dS = \iint \left|\overrightarrow{dS}\right|$$

Ejemplo: esfera



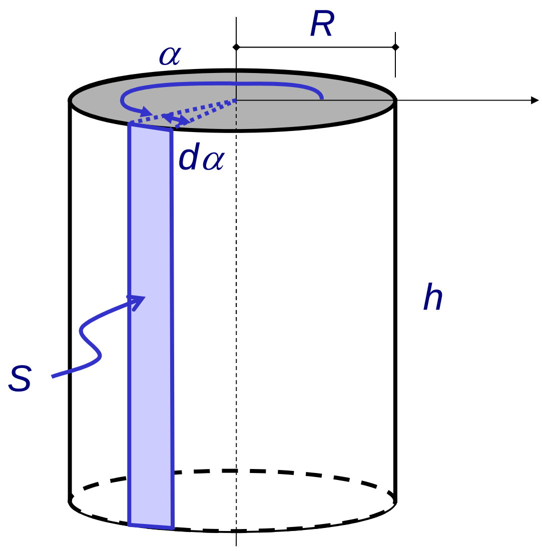
$$S = \iint dS = \iint \left|\overrightarrow{dS}\right| = 4\pi R^2$$

<u>Ejemplo</u>:
superficie lateral
de un cilindro



$$S = \iint dS = ?$$

<u>Ejemplo</u>:
superficie lateral
de un cilindro

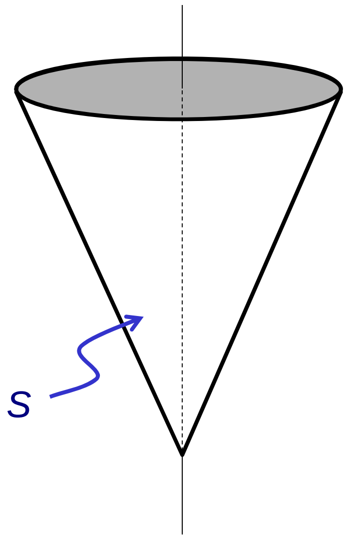


$$S = \iint dS = \int \left(hRd\alpha \right)$$

Ejemplo:

superficie lateral

de un cono

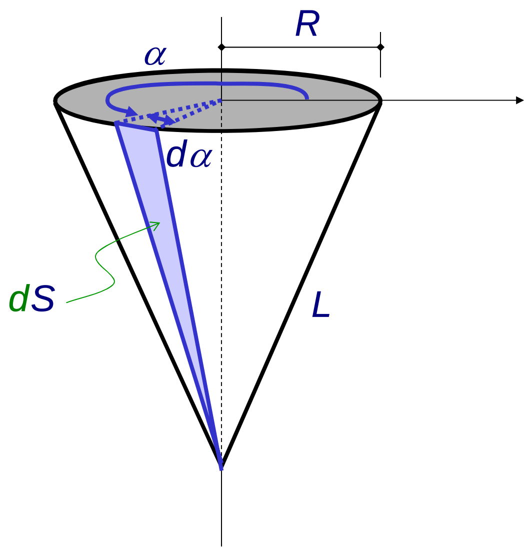


$$S = \iint dS = ?$$

Ejemplo:

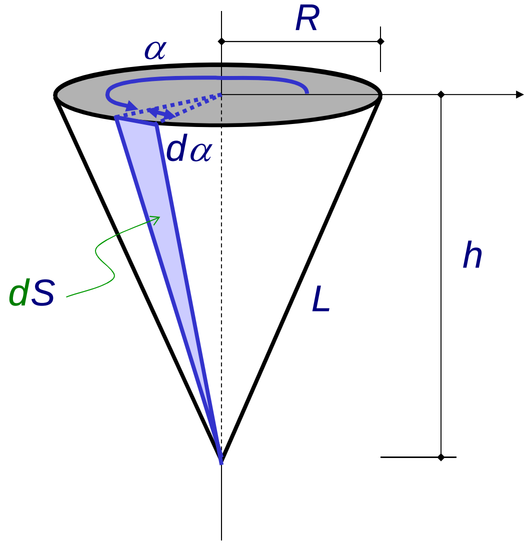
superficie lateral

de un cono



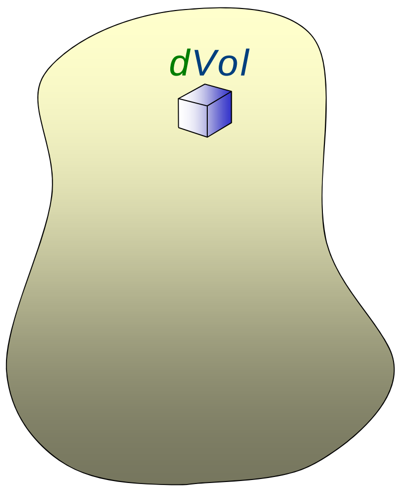
$$S = \iint dS = \int \left(\frac{1}{2} LR d\alpha \right) = \frac{1}{2} LR \int_{\alpha=0}^{\alpha=2\pi} d\alpha = \pi LR$$

Ejemplo:
superficie lateral
de un cono



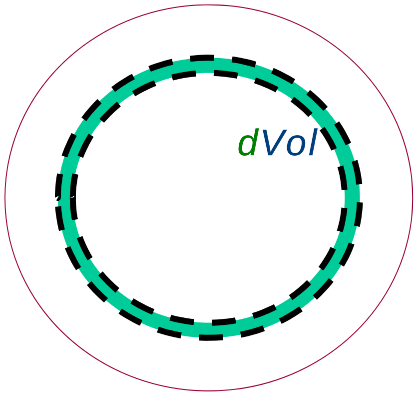
$$S = \pi L R = \pi R\sqrt{R^2 + h^2} = \pi R^2\sqrt{1+\left(\frac{h}{R}\right)^2}$$

Volumen de un cuerpo



$$Vol = \iiint dVol$$

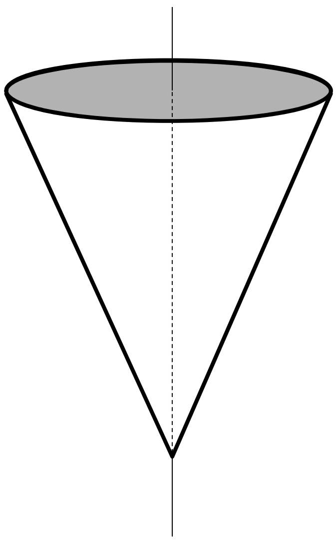
Ejemplo: esfera



$$Vol = \iiint dVol = \int (4\pi R^2 dR) = 4\pi \frac{R^3}{3} = \frac{4}{3}\pi R^3$$

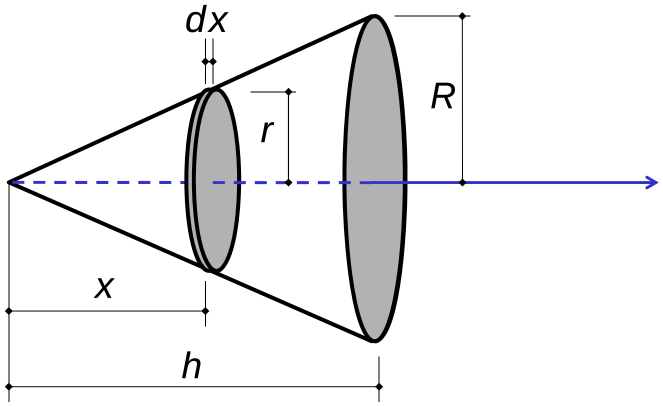
Ejemplo:

volumen de un cono



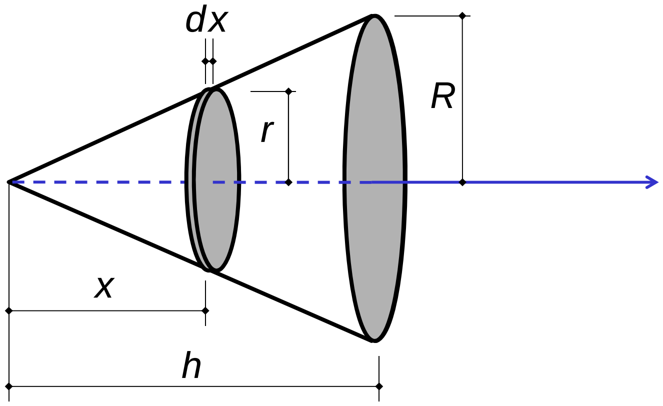
$$Vol = \iiint dVol = ?$$

Ejemplo: cono



$$Vol = \iiint dVol = ?$$

Ejemplo: cono



$$Vol = \int \pi r^2 dx = \int_{x=0}^{x=h} \pi \left(\frac{R}{h} x \right)^2 dx = \frac{\pi R^2}{h^2} \frac{h^3}{3} = \frac{1}{3} \pi R^2 h$$

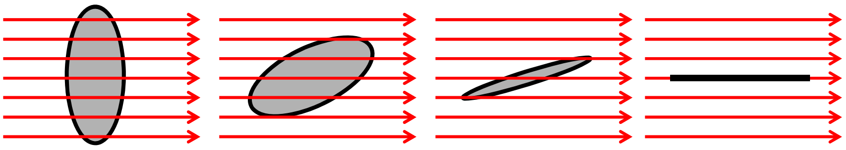
Flujo de un campo vectorial

Flujo de un campo vectorial a través de una superficie

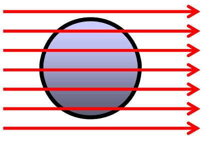
Sea **v** un campo de vectores y **S** una superficie. El flujo Φ del campo a través de la superficie S se escribe:

$$\Phi = \iint_S \vec{\mathbf{v}} \cdot \overrightarrow{\mathbf{dS}} = \iint_S \vec{\mathbf{v}} \cdot \hat{\mathbf{u}}_n dS$$

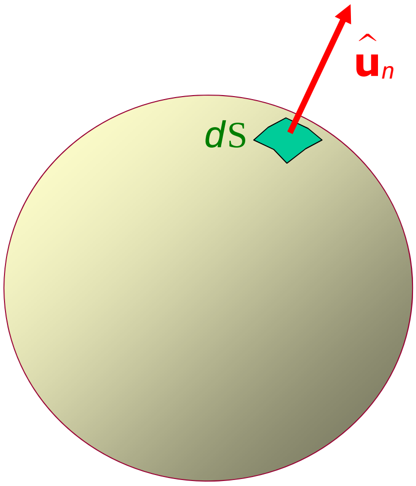
Sup. abierta



Sup. cerrada



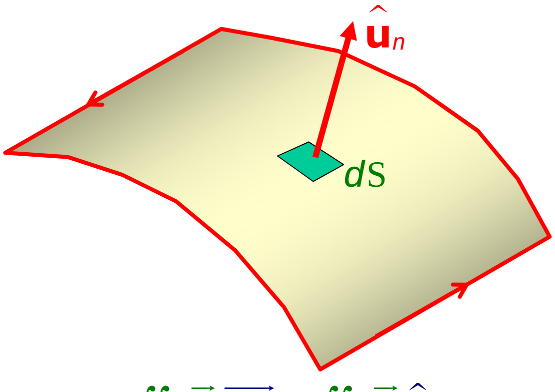
Flujo a través de una superficie cerrada:



$$\Phi = \oint\!\!\oint_S \vec{\mathbf{v}}\cdot\overrightarrow{\mathbf{dS}} = \oint\!\!\oint_S \vec{\mathbf{v}}\cdot\hat{\mathbf{u}}_n dS$$

La normal a la superficie está siempre orientada hacia el exterior.

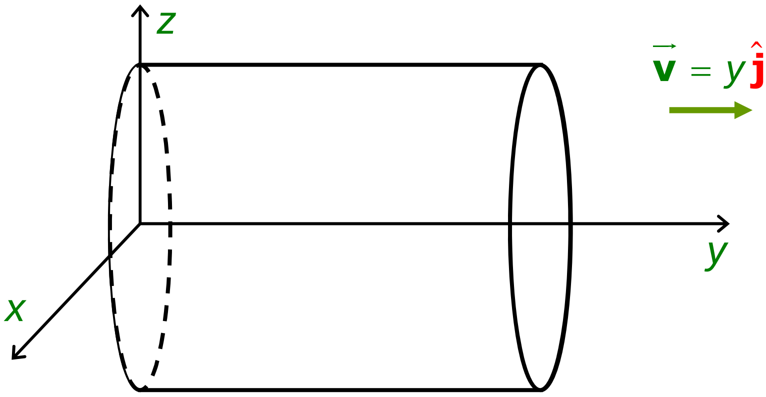
Flujo a través de una superficie abierta:



$$\Phi = \iint_S \vec{\mathbf{v}}\cdot\overrightarrow{\mathbf{dS}} = \iint_S \vec{\mathbf{v}}\cdot\hat{\mathbf{u}}_n dS$$

Se elige un sentido de recorrido de la línea sobre la que se apoya la superficie. El sentido de $\mathbf{u}_n$ sigue la “regla del sacacorchos”.

Ejemplo: flujo del campo **v**=y**j** a través de la superficie cerrada formada por el cilindro de la figura:



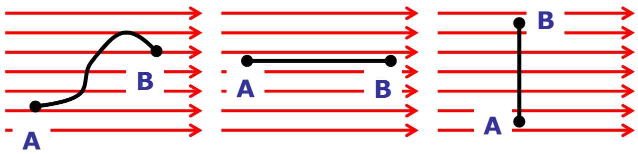
Circulación de un campo vectorial

Circulación de un campo vectorial a lo largo de una línea

Sea **v** un campo de vectores y Γ una línea. La circulación del campo a lo largo de la línea Γ se escribe:

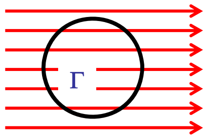
$$\int_{\Gamma} \vec{\mathbf{v}} \cdot \vec{\mathbf{d}l}$$

Línea abierta



$$\int_{A}^{B} \vec{\mathbf{v}} \cdot \vec{\mathbf{d}l}$$

Línea cerrada



$$\oint_{\Gamma} \vec{\mathbf{v}} \cdot \vec{\mathbf{d}l}$$

Recordatorio

Sistema internacional de unidades

Unidades básicas del SI

Magnitud	Nombre	Símbolo
Longitud	**metro**	**m**
Masa	**kilogramo**	**kg**
Tiempo	**segundo**	**s**
Corriente eléctrica	**amperio**	**A**
Temperatura termodinámica	**kelvin**	**K**
Intensidad luminosa	**candela**	**cd**
Cantidad de sustancia	**mol**	**mol**

Tabla de prefijos en el SI

factor	Prefijo	Símbolo
10^{24}	yotta	Y
10^{21}	zetta	Z
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10^{1}	deca	da / D
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	mili	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a
10^{-21}	zepto	z
10^{-24}	yocto	y