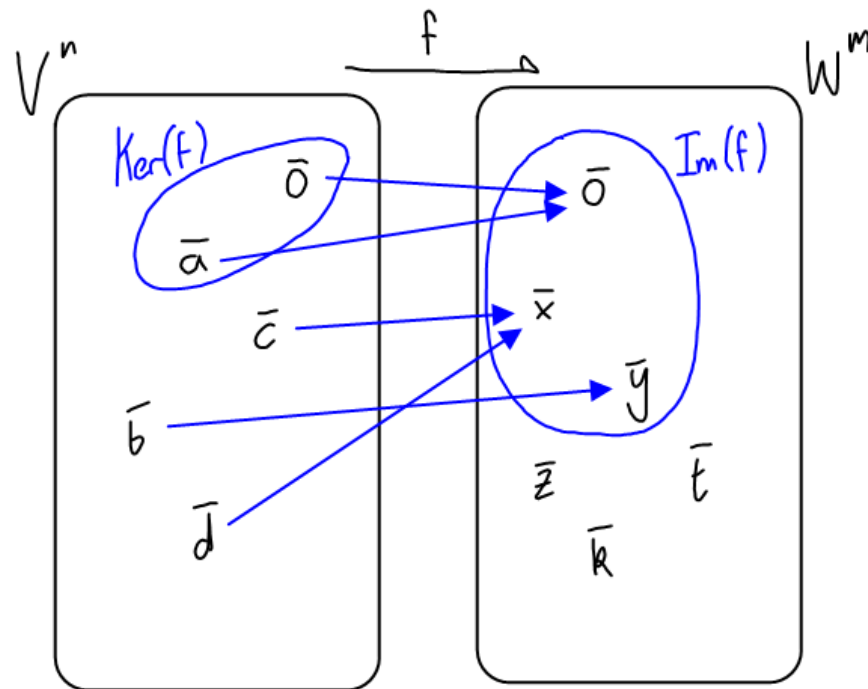


Homomorphisms in vector space

$$f: V^n \longrightarrow W^m \quad \left. \begin{array}{l} \bar{v} \in V^n \longrightarrow \bar{w} \in W^m \end{array} \right\} f(\bar{v}) = \bar{w}$$

An application f will be called a homomorphism between two Vector Spaces V^n and W^m if: $\forall \bar{u}, \bar{v} \in V^n \quad f(\lambda \bar{u} + \mu \bar{v}) = \lambda f(\bar{u}) + \mu f(\bar{v})$



$$\text{Ker}(f) = \left\{ \bar{x} \in V / f(\bar{x}) = \bar{0} \in W \right\}$$

$\text{Ker}(f)$ is a subspace of V

If $\dim(\text{Ker}(f)) = 0 \rightarrow f$ is INJECTIVE

$$\text{Im}(f) = \left\{ \bar{y} \in W / \exists \bar{x} \in V \quad f(\bar{x}) = \bar{y} \right\}$$

$\text{Im}(f)$ is a subspace of W

$$\dim(\text{Im}(f)) = \text{Rg}(f)$$

If $\dim(\text{Im}(f)) = \dim(W) \rightarrow f$ is SUPRAYECTIVE

$$\text{Grassman's Law: } \dim(\text{Ker}(f)) + \dim(\text{Im}(f)) = \dim(V)$$

The Matrix of a homomorphism

$$f: V^n \longrightarrow W^m$$

$$B_1 = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\} \text{ Base of } V$$

$$B_2 = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m\} \text{ Base of } W$$

$$\forall \bar{x} \in V^n \quad f(\bar{x}) = \bar{y} \in W^m$$

$$\text{if } \bar{x} = x^1 \bar{e}_1 + x^2 \bar{e}_2 + \dots + x^n \bar{e}_n = (x^1, x^2, \dots, x^n)_{B_1}$$

$$\text{if } \bar{y} = y^1 \bar{u}_1 + y^2 \bar{u}_2 + \dots + y^m \bar{u}_m = (y^1, y^2, \dots, y^m)_{B_2}$$

$$F_{B_1, B_2} \left(\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}_{B_1} \right) = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix}_{B_2} \iff f(\bar{x}) = \bar{y} \in W^m$$

$$\underbrace{(F_{B_1, B_2})}_{n \times m} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}_{B_1}}_{n \times 1} = \underbrace{\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix}_{B_2}}_{m \times 1}$$

$$f: V^3 \rightarrow V^3 \text{ ENDOMORPHISM}$$

$$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \quad \forall \bar{x} \in V^3 \quad \bar{x} = x^1 \bar{e}_1 + x^2 \bar{e}_2 + x^3 \bar{e}_3 = (x^1, x^2, x^3)_B$$

$$\text{If } f(\bar{x}) = \bar{y} \in V^3 \quad \bar{y} = y^1 \bar{e}_1 + y^2 \bar{e}_2 + y^3 \bar{e}_3 = (y^1, y^2, y^3)_B$$

$$\overbrace{\begin{pmatrix} \text{red oval} & \text{red oval} & \text{red oval} \end{pmatrix}}^{F_B} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_B$$

$f(\bar{e}_1) \quad f(\bar{e}_2) \quad f(\bar{e}_3)$
 in B
 3×3

3×1

3×1

Subspace operations

Given S_1, S_2 subspaces of V

$$S_1 \cap S_2 = \{ \bar{x} \in V / \bar{x} \in S_1 \overset{\text{AND}}{\wedge} \bar{x} \in S_2 \} \rightarrow S_1 \cap S_2 \text{ is also a SUBSPACE}$$

$$S_1 \cup S_2 = \{ \bar{x} \in V / \bar{x} \in S_1 \overset{\text{OR}}{\vee} \bar{x} \in S_2 \} \rightarrow S_1 \cup S_2 \text{ is NOT a SUBSPACE}$$

$$S_1 + S_2 = \{ \bar{x} \in V / \exists \bar{x}_1 \in S_1 \overset{\text{AND}}{\wedge} \exists \bar{x}_2 \in S_2 : \bar{x}_1 + \bar{x}_2 = \bar{x} \} \rightarrow S_1 + S_2 \text{ is also a SUBSPACE}$$

Direct Sum : $S_1 \oplus S_2$

$$S_1 \oplus S_2 \iff \begin{cases} S_1 \cap S_2 = \{ \vec{0} \} \\ S_1 + S_2 = V \end{cases}$$

$$\dim(S_1) + \dim(S_2) = \dim(S_1 + S_2) + \dim(S_1 \cap S_2)$$

Vector spaces 1

Given $\mathbb{R}^3(\mathbb{R})$ with a Base $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and an endomorphism f that works:

$$f(x^1, x^2, x^3) = (x^1 + x^2, x^1 + x^2, x^3)$$

1. Obtain $\text{Ker}(f)$ and $\text{Im}(f)$ with a base and a dimension for each.
2. If that same space has two subspaces defined as: $S_1 = \{\bar{x} \in \mathbb{R}^3 / x^1 + x^2 + x^3 = 0\}$ and $S_2 = \{\bar{x} \in \mathbb{R}^3 / x^1 - x^2 = 0\}$ obtain a base and a dimension for each as well as for $S_1 \cap S_2$.
3. Given another base $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ where $\bar{u}_1 = \bar{e}_1 + \bar{e}_2 + \bar{e}_3$, $\bar{u}_2 = \bar{e}_1 + \bar{e}_2$ and $\bar{u}_3 = \bar{e}_1$; obtain the calculated base of S_1 in this new base B' .

$$1. B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \quad f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$f(\bar{x}) = (x^1 + x^2, x^1 + x^2, x^3)$$

$$\begin{cases} \bar{e}_1 = (1, 0, 0)_B \\ \bar{e}_2 = (0, 1, 0)_B \\ \bar{e}_3 = (0, 0, 1)_B \end{cases}$$

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$f(\bar{e}_1) \quad f(\bar{e}_2) \quad f(\bar{e}_3)$
in B

$$f(\bar{e}_1) = (1+0, 1+0, 0) = (1, 1, 0)_B$$

$$f(\bar{e}_2) = (0+1, 0+1, 0) = (1, 1, 0)_B$$

$$f(\bar{e}_3) = (0+0, 0+0, 1) = (0, 0, 1)_B$$

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Kernel

$$\text{Ker}(f) = \{ \bar{x} \in \mathbb{R}^3 / f(\bar{x}) = \bar{0} \}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F_B} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_B \rightarrow \begin{cases} x^1 + x^2 = 0 \\ \cancel{x^1 + x^2 = 0} \\ x^3 = 0 \end{cases} \rightarrow \begin{cases} x^1 = \alpha \\ x^2 = -\alpha \\ x^3 = 0 \end{cases} \forall \alpha \in \mathbb{R}$$

$$\dim(\text{Ker}(f)) = 1 \quad B_K = \{ (1, -1, 0) \}$$

$$\forall \bar{x} \in \text{Ker} \quad \bar{x} = (\alpha, -\alpha, 0) \quad \forall \alpha \in \mathbb{R}$$

$$\bar{x} = \alpha (1, -1, 0)$$

Image

$$\text{Im}(f) = \{ \bar{y} \in \mathbb{R}^3 / \bar{x} \in \mathbb{R}^3 \quad f(\bar{x}) = \bar{y} \}$$

$$\dim(\text{Im}(f)) = \text{Rg}(F_B) = 2$$

$$F_B = \left(\begin{array}{c|c|c} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \hline f(\bar{e}_1) & f(\bar{e}_2) & f(\bar{e}_3) \end{array} \right) \quad B_{\text{Im}} = \{ (1, 1, 0), (0, 0, 1) \}$$

Alternate calculation of $\ker(f)$:

$$F_B = \begin{pmatrix} \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{pmatrix}$$

$f(\bar{e}_1) \quad f(\bar{e}_2) \quad f(\bar{e}_3)$

$$f(\bar{e}_1) - f(\bar{e}_2) = \bar{0} \rightarrow f(\bar{e}_1 - \bar{e}_2) = \bar{0} \rightarrow \bar{e}_1 - \bar{e}_2 \in \ker(f) \rightarrow B_K = \{(1, -1, 0)\}$$

If f is a homomorph.:

$$f(\lambda \bar{u} + \mu \bar{v}) = \lambda f(\bar{u}) + \mu f(\bar{v})$$

$$\ker(f) = \left\{ \bar{x} \in V / f(\bar{x}) = \bar{0} \in W \right\}$$

2.

$$S_1 = \{ \bar{x} \in \mathbb{R}^3 / x^1 + x^2 + x^3 = 0 \} \rightarrow x^1 + x^2 + x^3 = 0 \rightarrow \begin{cases} x^1 = \alpha \\ x^2 = \beta \\ x^3 = -\alpha - \beta \end{cases} \quad \forall \alpha, \beta \in \mathbb{R}$$

$$\dim(S_1) = 2$$

$$B_{S_1} = \left\{ (1, 0, -1)_B, (0, 1, -1)_B \right\}$$

$$\text{if } \bar{x} \in S_1, \quad \bar{x} = (\alpha, \beta, -\alpha - \beta) = \alpha(1, 0, -1) + \beta(0, 1, -1)$$

$$S_2 = \{ \bar{x} \in \mathbb{R}^3 / x^1 - x^2 = 0 \} \rightarrow x^1 - x^2 = 0 \rightarrow \begin{cases} x^1 = \gamma \\ x^2 = \gamma \\ x^3 = \delta \end{cases} \quad \forall \gamma, \delta \in \mathbb{R}$$

$$\dim(S_2) = 2$$

$$B_2 = \{(1, 1, 0), (0, 0, 1)\}$$

$$\text{if } \bar{x} \in S_2, \quad \bar{x} = (\gamma, \gamma, \delta) = \gamma(1, 1, 0) + \delta(0, 0, 1)$$

$$S_1 \cap S_2 = \begin{cases} x^1 + x^2 + x^3 = 0 \\ x^1 - x^2 = 0 \end{cases}$$

$$\begin{cases} x^1 = \lambda \\ x^2 = \lambda \\ x^3 = -2\lambda \end{cases} \quad \forall \lambda \in \mathbb{R} \longrightarrow \begin{cases} \dim(S_1 \cap S_2) = 1 \\ B_{S_1 \cap S_2} = \{(1, 1, -2)\} \end{cases}$$

3.

$$B_{S_1} = \left\{ \overbrace{(1, 0, -1)}^{\bar{a}_1}, \overbrace{(0, 1, -1)}^{\bar{a}_2} \right\}_B \longrightarrow \begin{aligned} \bar{a}_1 &= (1, 0, -1)_B = \bar{e}_1 - \bar{e}_3 \\ \bar{a}_2 &= (0, 1, -1)_B = \bar{e}_2 - \bar{e}_3 \end{aligned}$$

$$B \xrightleftharpoons[C^{-1}]{C} B'$$

$$|C| = -1$$

$$B' \begin{cases} \bar{u}_1 = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 = (1, 1, 1)_B \\ \bar{u}_2 = \bar{e}_1 + \bar{e}_2 = (1, 1, 0)_B \\ \bar{u}_3 = \bar{e}_1 = (1, 0, 0)_B \end{cases}$$

$$C = \begin{pmatrix} \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \end{pmatrix}_{\substack{\bar{u}_1, \bar{u}_2, \bar{u}_3 \\ \text{in } B}}$$

$$\text{Adj}(C) = \begin{pmatrix} +0 & -0 & +1 \\ -0 & +1 & -1 \\ +1 & -1 & +0 \end{pmatrix}$$

$$(C^{-1}) \begin{pmatrix} \overbrace{x^1}^{\bar{x}} \\ \overbrace{x^2}^{\bar{x}} \\ \overbrace{x^3}^{\bar{x}} \end{pmatrix}_B = \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \end{pmatrix}_{B'}$$

$$(C) \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \end{pmatrix}_{B'} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B$$

$$C^{-1} = \frac{\text{Adj}(C)^t}{|C|} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}_{\substack{\bar{e}_1, \bar{e}_2, \bar{e}_3 \\ \text{in } B'}}$$

$$C^{-1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_B = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}_{B'}$$

$$C^{-1} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}_B = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}_{B'}$$

$$\bar{a}_1 = (-1, 1, 1)_{B'} = -\bar{u}_1 + \bar{u}_2 + \bar{u}_3$$

$$\bar{a}_2 = (-1, 2, -1)_{B'} = -\bar{u}_1 + 2\bar{u}_2 - \bar{u}_3$$

Vector Spaces 2

$$\{ \bar{x} \in \mathbb{R}^3 / \bar{x} = (a, b, c) \forall a, b, c \in \mathbb{R} \}$$

$$\dim(\mathbb{P}_2) = 3$$

Given $\mathbb{P}_2(\mathbb{R}) = \{ p(x) \in \mathbb{P}_2 / p(x) = ax^2 + bx + c \forall a, b, c \in \mathbb{R} \}$ and given an endomorphism

with a matrix $F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ in base $B = \{x^2, x, 1\}$; calculate $\ker(f)$ and $\text{Im}(f)$

giving a base of each and obtain $F_{B'}$, the matrix in base $B' = \{x^2+x+1, x^2+x, x^2\}$.

$$\text{Rg}(F_B) = 1 = \dim(\text{Im}(f)) \quad B_{\text{Im}} = \{x^2+x+1\}$$

$$\dim(\text{Im}(f)) + \dim(\text{Ker}(f)) = \dim(\mathbb{P}_2) \rightarrow \dim(\text{Ker}(f)) = 2$$

If f is a homomorph.:

$$f(\lambda \bar{u} + \mu \bar{v}) = \lambda f(\bar{u}) + \mu f(\bar{v})$$

$$F_B = \begin{pmatrix} | & | & | \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$f(x^2) \quad f(x) \quad f(1)$
in B

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} a+b+c=0 \\ a+b+c=0 \\ a+b+c=0 \end{cases} \rightarrow \begin{cases} a=\alpha \\ b=\beta \\ c=-\alpha-\beta \end{cases}$$

$\forall \alpha, \beta \in \mathbb{R}$

$$B_{\text{Ker}} = \{x^2-1, x-1\}$$

$(1, 0, -1), (0, 1, -1)$

Alternate way:

$$\left. \begin{aligned} f(x^2) - f(1) = \bar{0} &\rightarrow f(x^2-1) = \bar{0} \rightarrow x^2-1 \in \text{Ker}(f) \\ f(x) - f(1) = \bar{0} &\rightarrow f(x-1) = \bar{0} \rightarrow x-1 \in \text{Ker}(f) \end{aligned} \right\}$$