$\mathbb{Z}_{(p)}$ $\begin{cases} (\mathbb{Z},+) \text{ Abelian Group} \\ (p) \equiv \text{Multiples of the number } p \in \mathbb{Z} \longrightarrow \text{Subgroup of } (\mathbb{Z},+) \end{cases}$

Example: $\mathbb{Z}_{(4)}$

$\bar{0} = \{\ldots,-12,-8,-4,0,4,8,12,16,\ldots\}$ $\quad (4)$

$\bar{1} = 1+(4) = \{\ldots,-7,-3,1,5,9,\ldots\}$

$\bar{2} = 2+(4) = \{\ldots,-6,-2,2,6,10,\ldots\}$

$\bar{3} = 3+(4) = \{\ldots,-5,-1,3,7,11,\ldots\}$

$\bar{4} = 4+(4) = \{\ldots,-4,0,4,8,12,\ldots\} = \bar{0}$

cryptography

$\mathbb{Z}_{(4)} = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$ $\qquad \bar{a}+\bar{b} = \overline{a+b}$

$(\mathbb{Z}_4,+)$

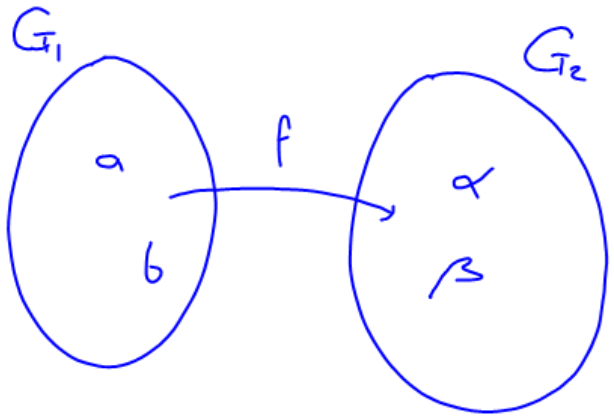
	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

$\bar{2}+\bar{3} = \overline{2+3} = \bar{5} = \bar{1}$

$\overline{37} = \bar{1}$

37 | 4
1 | 9

Homomorphisms Linear Aplications



$(G_1, *) \xrightarrow{f} (G_2, \Delta)$

$\forall a, b \in G_1$ $\quad$ $\forall \alpha, \beta \in G_2$

$a * b \in G_1$ $\quad$ $\alpha \Delta \beta \in G_2$

$e_1 \equiv$ Neutral of G_1 $\quad$ $e_2 \equiv$ Neutral of G_2

If $f(a) = \alpha \wedge f(b) = \beta$

f will be a homomorphism $\iff$ IF AND ONLY IF

$$\begin{cases} f(a*b) = f(a) \Delta f(b) \\ \qquad \wedge \longrightarrow \text{AND} \\ f(e_1) = e_2 \end{cases}$$

$f: G_1 \longrightarrow G_2$

Homomorphism

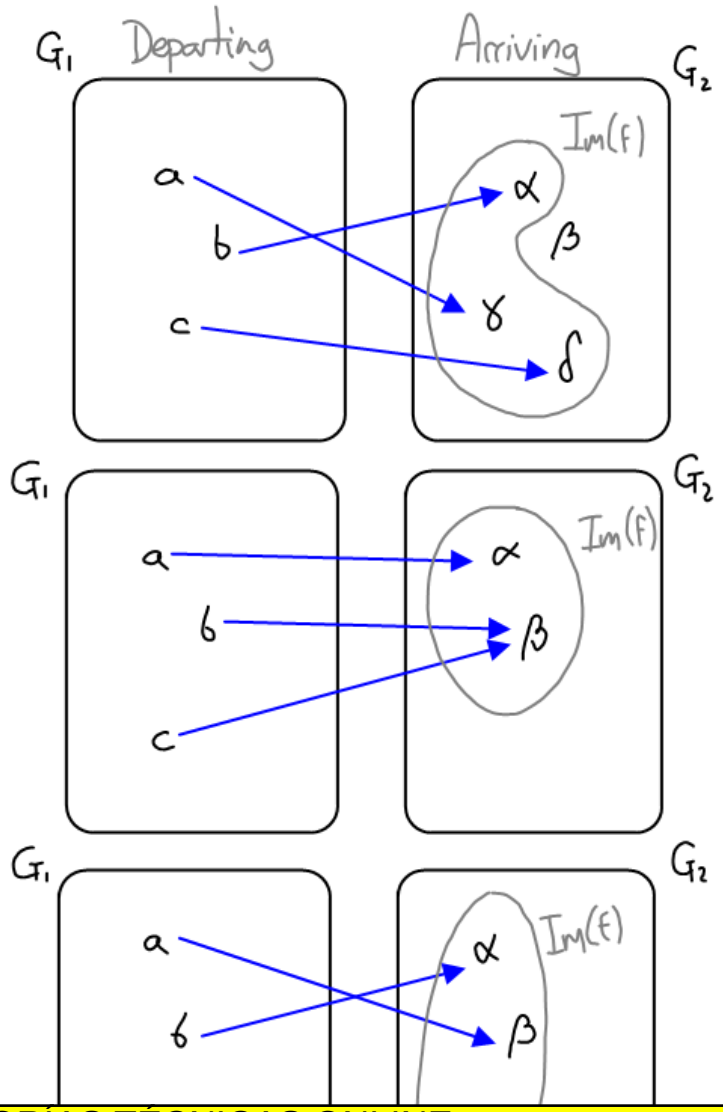
Injectivity ≡ Every element has an Image and that image is unique

$$\begin{cases} \text{If } f(a)=f(b) \longrightarrow a=b & \forall a,b \in G_1 \\ \text{If } a \neq b \longrightarrow f(a) \neq f(b) & \forall a,b \in G_1 \end{cases}$$

Suprayectivity ≡ Every element from the arriving group belongs to Im(f), but it doesn't need to be unique.

$$\forall y \in G_2 \quad \exists x \in G_1 \;/\; f(x)=y$$

Biyectivity ≡ Injective + Suprayective



$f: G_1 \longrightarrow G_2$

f	$G_1 \neq G_2$	$G_1 = G_2$
Not Biyective	Homomorphism	Endomorphism
Biyective	Isomorphism	Automorphism

Rings $(R, *, \Delta)$ — ILC's

R is a Ring $\iff$

- $(R, *)$ is an Abelian Group
  - Internal $\forall a,b \in R \quad a*b \in R$
  - Associative $\forall a,b,c \in R \quad a*(b*c) = (a*b)*c$
  - Neutral $\exists! e \in R \quad e*a = a*e = a \quad \forall a \in R$ → The 0 of the ring
  - Symmetrical $\forall a \in R \quad \exists a' \in R \quad a*a' = a'*a = e$
  - $\forall a,b \in R \quad a*b = b*a$
- (R, Δ) is a semigroup
  - Internal $\forall a,b \in R \quad a \Delta b \in R$
  - Associative $\forall a,b,c \in R \quad a \Delta (b \Delta c) = (a \Delta b) \Delta c$
- Distributives: $a*(b \Delta c) = (a*b) \Delta (a*c)$ ∧ $a \Delta (b*c) = (a \Delta b)*(a \Delta c)$

An Abelian Ring $\longrightarrow \forall a,b \in R \quad a \Delta b = b \Delta a$

A Unitary (or Unity) Ring $\longrightarrow \exists! 1_R \in R \longrightarrow a \Delta 1_R = 1_R \Delta a = a$ $\quad 1_A \to$ The 1 of the ring

We will say a Ring is Nondivisible by 0 $\longrightarrow \forall a,b \in R \quad a \Delta b \neq 0_R$

Example of divisors of 0:

$(\mathbb{Z}_{(6)}, +, \cdot)$ UNITARY ABELIAN RING

$\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$\bar{a} + \bar{b} = \overline{a+b}$

$\bar{a} \cdot \bar{b} = \overline{a \cdot b}$

$\bar{2} \cdot \bar{3} = \bar{0}$

Body $(B; *, \Delta)$

B is a BODY $\iff$ B is a UNITARY, ABELIAN, NONDIVISIBLE by 0 Ring

The multiplicative group of B is $(B - \{0_B\}, \Delta)$ Abelian Group

0_B: Neutral element with $*$

Vector Space

$V(B)$ is a vector space $\iff$

- V is an ABELIAN GROUP whose elements $\forall \bar{v} \in V$ are called vectors
- B is a BODY whose elements $\forall \lambda \in B$ are called scalars
- There is an ELC between V and B $\forall \lambda \in B, \forall \bar{v} \in V \to \lambda \cdot \bar{v} \in V$

External Law of Composition

- There is a DOUBLE DISTRIBUTIVE $\forall \lambda, \mu \in B \ \forall \bar{u}, \bar{v} \in V$
  - $\lambda \cdot (\bar{u} + \bar{v}) = \lambda \cdot \bar{u} + \lambda \cdot \bar{v}$
  - $(\lambda + \mu) \cdot \bar{u} = \lambda \cdot \bar{u} + \mu \cdot \bar{u}$
- There is an External Asociative: $(\lambda \cdot \mu) \cdot \bar{u} = \lambda \cdot (\mu \cdot \bar{u})$

ILC

- $(V, +)$ $\bar{u} + \bar{v} \in V$
- $(B, +, \cdot)$ $\lambda + \mu \in B$, $\lambda \cdot \mu \in B$

Vector subspace Given $V(\mathbb{R})$ a vector space and $S \subseteq V$

↳ $(\mathbb{R}, +, \cdot)$ Body

$S(\mathbb{R})$ is a subspace of $V(\mathbb{R}) \iff \forall \lambda, \mu \in \mathbb{R} \wedge \forall \bar{u}, \bar{v} \in S \rightarrow \lambda\bar{u} + \mu\bar{v} \in S$

Example: Given $\mathbb{M}_2(\mathbb{R})$ Vector space of order 2 regular matrixes

Prove that $\mathbb{S}_2(\mathbb{R})$ is a subspace of $\mathbb{M}_2$ (where $\mathbb{S}_2$ is the set of symmetrical matrixes in $\mathbb{M}_2$)

↳ $S = S^t$ when S is symmet.

$$\mathbb{M}_2(\mathbb{R}) = \left\{ M \in \mathbb{M}_2 \ / \ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \forall a,b,c,d \in \mathbb{R} \right\}$$

$$\mathbb{S}_2(\mathbb{R}) = \left\{ S \in \mathbb{S}_2 \ / \ S = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \forall \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$\forall \lambda, \mu \in \mathbb{R}$

$$\lambda S_1 + \mu S_2 = \lambda \begin{pmatrix} a & c \\ c & b \end{pmatrix} + \mu \begin{pmatrix} d & f \\ f & e \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda c \\ \lambda c & \lambda b \end{pmatrix} + \begin{pmatrix} \mu d & \mu f \\ \mu f & \mu e \end{pmatrix} =$$

Linear composition $\{\bar{u}_i\}$ is a set of vectors of a certain space $V(\mathbb{R})$

We say we have a linear composition of $\{\bar{u}_i\}$, $\mathcal{L}\{\bar{u}_i\}$, when:

$$\mathcal{L}\{\bar{u}_i\} = \lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + \dots + \lambda_n\bar{u}_n \quad \forall \lambda_i \in \mathbb{R}$$

Linear dependance We can say that the vector in $\{\bar{u}_i\}$ are linearly DEPENDANT when:

$$\mathcal{L}\{\bar{u}_i\} = \bar{0} \text{ and at least one of the } \lambda_i \text{ is } \underbrace{\lambda_k \neq 0}_{\text{one of the } \lambda_i}$$

Linear independance We can say that the vector in $\{\bar{u}_i\}$ are linearly INDEPENDANT when:

Example:

$\{\bar{u}_i\} = \{\underbrace{(1,1,1)}_{\bar{u}_1}, \underbrace{(1,1,0)}_{\bar{u}_2}, \underbrace{(0,0,1)}_{\bar{u}_3}\}$

$\mathcal{L}\{\bar{u}_i\} = \bar{0} \rightarrow \lambda_1(1,1,1) + \lambda_2(1,1,0) + \lambda_3(0,0,1) = (0,0,0)$

$(\lambda_1+\lambda_2,\ \lambda_1+\lambda_2,\ \lambda_1+\lambda_3) = (0,0,0) \rightarrow \begin{cases} \lambda_1+\lambda_2=0 \rightarrow \lambda_2=-\lambda_1 \\ \lambda_1+\lambda_2=0 \\ \lambda_1+\lambda_3=0 \rightarrow \lambda_3=-\lambda_1 \end{cases}$

so for example: if $\lambda_1=1 \rightarrow \begin{cases} \lambda_2=-1 \\ \lambda_3=-1 \end{cases}$

So I have $\lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \lambda_3 \neq 0$ that make $\mathcal{L}\{\bar{u}_i\} = \bar{0}$

$\{\bar{u}_i\}$ is Linearly Dependant

$\{\bar{v}_i\} = \{\underbrace{(1,1,1)}_{\bar{v}_1}, \underbrace{(1,1,0)}_{\bar{v}_2}, \underbrace{(1,0,0)}_{\bar{v}_3}\}$

$\mathcal{L}\{\bar{v}_i\} = \bar{0} \rightarrow \mu_1(1,1,1) + \mu_2(1,1,0) + \mu_3(1,0,0) = (0,0,0)$

$(\mu_1+\mu_2+\mu_3,\ \mu_1+\mu_2,\ \mu_1) = (0,0,0) \rightarrow \begin{cases} \mu_1+\mu_2+\mu_3=0 \rightarrow \mu_3=0 \\ \mu_1+\mu_2=0 \rightarrow \mu_2=0 \\ \mu_1=0 \end{cases}$

$\mu_1=\mu_2=\mu_3=0$