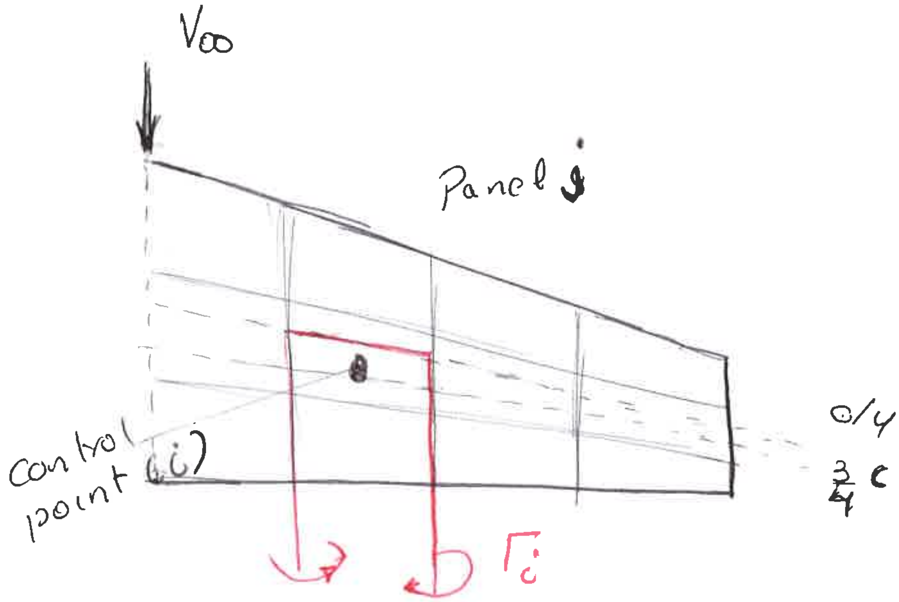
Vortex Lattice Method

Consider a incompressible, Inviscid flow, and the wing is modelled as a set of panel.



Bound vortex is located at panel $\frac{1}{4}$ chord position with two trailing vortex lines shed from each end.

The required strength of bound vortex on each panel will need to be calculated by applying a surface flow boundary conditions. The equation to be used is the condition of zero flow normal to the surface.

The normal velocity is made up of a freestream component and an induced flow component. We know that thés induced component is a function of strenghts of the all vortex panels on the wing.

$$V_n = 0 = V_\infty . \sin(\theta) + w_i$$

$$w_j = \sum_{j=1}^{N} A_{ij} . \Gamma_n$$

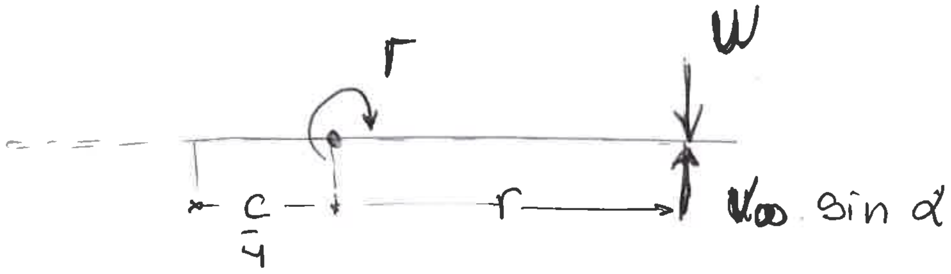
$$- V_\infty \sin(\theta) = \sum_{j=1}^{N} A_{ij} \Gamma_n$$

A_{ij} = Influence coefficient of induced flow on panel (i) due to the vortex on panel (j)

This influence ~~co~~efficient can be calculated as a relatively simple application of Biot-Savart Law along the three component vortex lines.

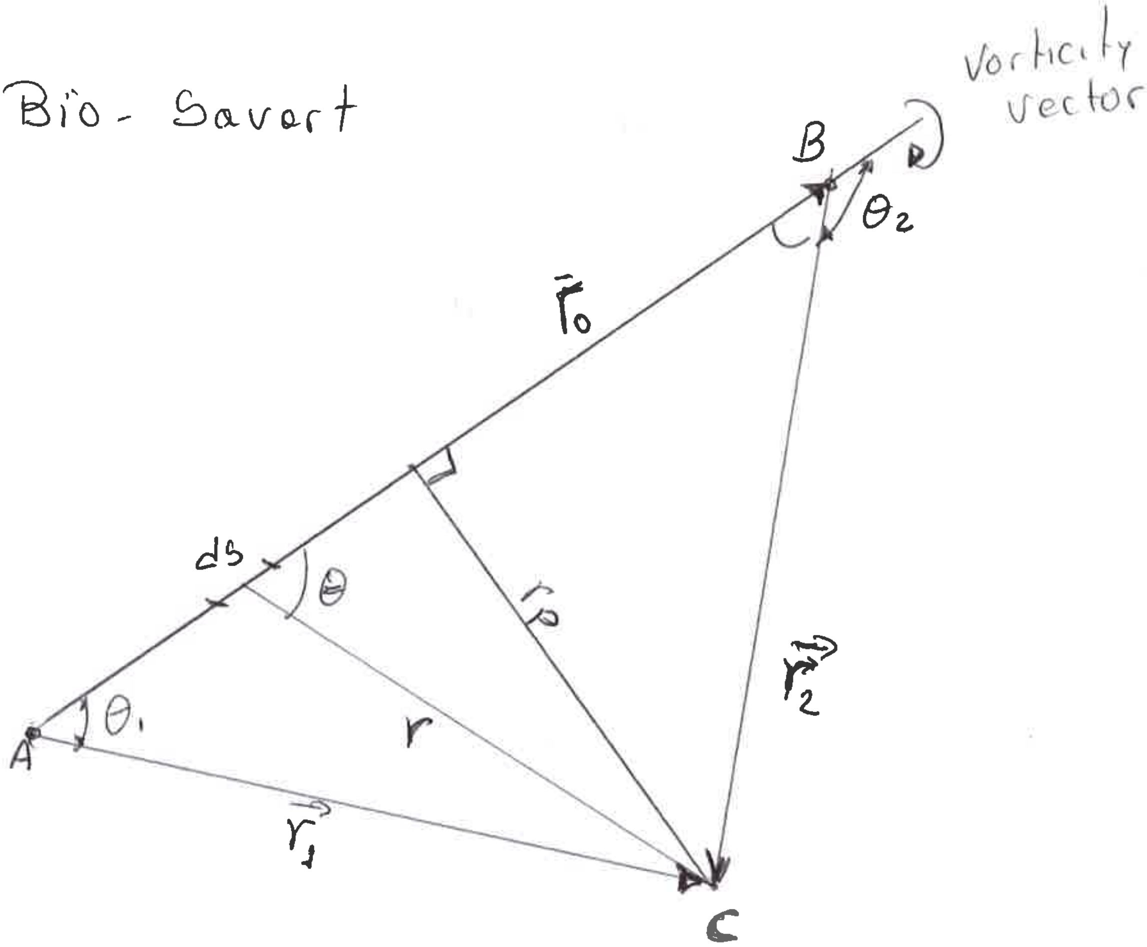
α : small then $\sin\alpha = \alpha$

$$[w_j = - V_\infty . \alpha]$$



Planar air foil section indicating location of control point where flow is parallel to the surface?

Bio - Savart



$$d\vec{V} = \frac{1}{4\pi r^3} \Gamma_n \left(d\vec{L} \times \vec{r} \right)$$

$$dV = \frac{\Gamma_n}{4\pi} \cdot \frac{\sin\theta \, dL}{r^2}$$

Applying to obtain the flow field induced by a horseshoe vortex which consists in three straight segments. Let AB be a segment, with the vorticity vector directed from A to B. Let C be a point in space whose normal distance from line AB is r_p.

$$V = \frac{\Gamma_n}{4\pi r_p} \int_{\theta_1}^{\theta_2} \sin\theta . \, d\theta = \frac{\Gamma_n}{4\pi r_p} . (\cos\theta_1 - \cos\theta_2)$$

For $\theta_1 = 0$ and $\theta = \pi$, the vortex filament extends to infinity. in both directions

$$L = \frac{r_p}{\tan\theta} \qquad r = \frac{r_p}{\sin\theta}$$

Then $r_p = \dfrac{|\vec{r}_1 \times \vec{r}_2|}{r_0}$

$$\cos\theta_1 = \frac{\vec{r}_0 \cdot \vec{r}_1}{r_0 . r_1}$$

$$\cos\theta_2 = \frac{\vec{r}_0 \cdot \vec{r}_2}{r_0 . r_2}$$

The direction of the induced velocity is given by the unit vector of the ~~cross product~~ of $\vec{r}_1$ and $\vec{r}_2$.

$$\frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

Unit vector direction of induced velocity vector.

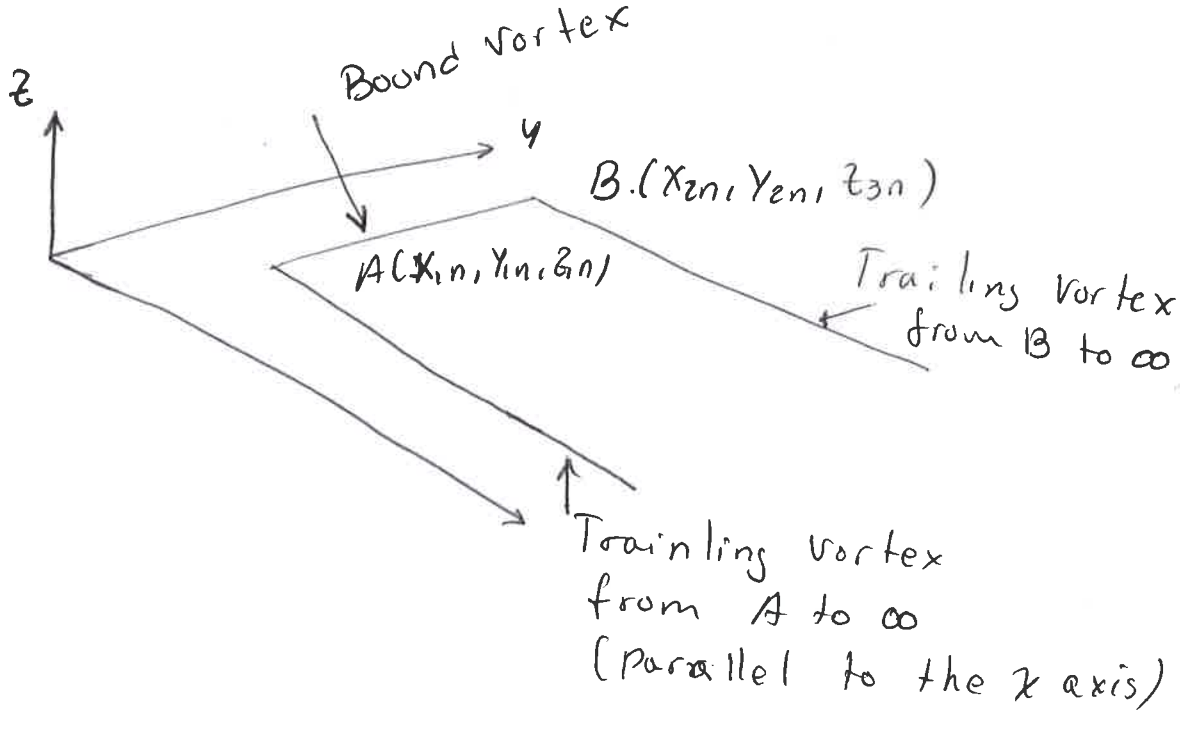
$$\vec{V} = \frac{\Gamma_n}{4\pi} \cdot \frac{(\vec{r}_1 \times \vec{r}_2)}{\dfrac{|\vec{r}_1 \times \vec{r}_2|}{\cancel{r_1}} \; \dfrac{|\vec{r}_1 \times \vec{r}_2|}{r_0}} \left[\frac{\vec{r}_0 \cdot \vec{r}_1}{r_0 \cdot r_1} - \frac{\vec{r}_0 \cdot \vec{r}_2}{r_0 r_2} \right]$$

$$\left[\vec{V} = \frac{\Gamma_n}{4\pi} \frac{(\vec{r}_1 \times \vec{r}_2)}{|\vec{r}_1 \times \vec{r}_2|^2} \left[\vec{r}_0 \left(\frac{\vec{r}_1}{r_1} - \frac{\vec{r}_2}{r_2} \right) \right] \right]$$

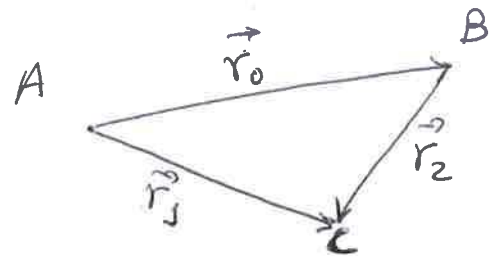
$$\vec{r}_0 = \vec{AB} = (X_{2n} - X_{1n})\hat{i} + (Y_{2n} - Y_{1n})\hat{j} + (Z_{2n} - Z_{1n})\hat{k}$$

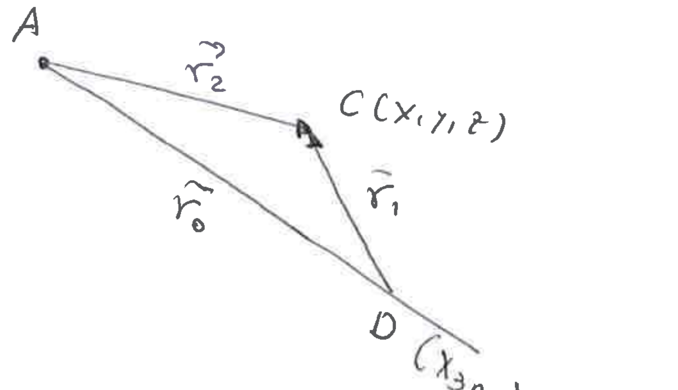
$$\vec{r}_1 = (X - X_{1n})\hat{i} + (Y - Y_{1n})\hat{j} + (Z - Z_{1n})\hat{k}$$

$$\vec{r}_2 = (X - X_{2n})\hat{i} + (Y - Y_{2n})\hat{j} + (Z - Z_{2n})\hat{k}$$



Using the above equation the velocity induced at point C (x, y, z). by the vortex filament AB.





Induced velocity by the filament that extends from A to ∞.

$$\vec{V}_{AB} = \frac{\Gamma_n}{4\pi}\{\mathrm{Fac1}_{AB}\}\{\mathrm{Fac2}_{AB}\}$$

$$\begin{aligned}
\{\mathrm{Fac1}_{AB}\} &= \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|^2} \\
&= \{[(y - y_{1n})(z - z_{2n}) - (y - y_{2n})(z - z_{1n})]\hat{i} \\
&\quad - [(x - x_{1n})(z - z_{2n}) - (x - x_{2n})(z - z_{1n})]\hat{j} \\
&\quad + [(x - x_{1n})(y - y_{2n}) - (x - x_{2n})(y - y_{1n})]\hat{k}\}/ \\
&\quad \{[(y - y_{1n})(z - z_{2n}) - (y - y_{2n})(z - z_{1n})]^2 \\
&\quad + [(x - x_{1n})(z - z_{2n}) - (x - x_{2n})(z - z_{1n})]^2 \\
&\quad + [(x - x_{1n})(y - y_{2n}) - (x - x_{2n})(y - y_{1n})]^2\}
\end{aligned}$$

and

$$\{\text{Fac2}_{AB}\} = \left(\vec{r}_0 \cdot \frac{\vec{r}_1}{r_1} - \vec{r}_0 \cdot \frac{\vec{r}_2}{r_2}\right)$$

$$= \{[(x_{2n} - x_{1n})(x - x_{1n}) + (y_{2n} - y_{1n})(y - y_{1n}) + (z_{2n} - z_{1n})(z - z_{1n})]/$$
$$\sqrt{(x - x_{1n})^2 + (y - y_{1n})^2 + (z - z_{1n})^2}$$
$$- [(x_{2n} - x_{1n})(x - x_{2n}) + (y_{2n} - y_{1n})(y - y_{2n}) + (z_{2n} - z_{1n})(z - z_{2n})]/$$
$$\sqrt{(x - x_{2n})^2 + (y - y_{2n})^2 + (z - z_{2n})^2}\}$$

For $\overrightarrow{DA}$

$$\vec{r}_0 = \overrightarrow{DA} = (x_{1n} - x_{3n})\hat{i}$$
$$\vec{r}_1 = (x - x_{3n})\hat{i} + (y - y_{1n})\hat{j} + (z - z_{1n})\hat{k}$$
$$\vec{r}_2 = (x - x_{1n})\hat{i} + (y - y_{1n})\hat{j} + (z - z_{1n})\hat{k}$$

$$V_{AD} = \frac{\Gamma_n}{4\pi}\left[(\text{Fac1}_{AB})(\text{Fac2}_{AD})\right]$$

where

$$\{\text{Fac1}_{AD}\} = \frac{(z - z_{1n})\hat{j} + (y_{1n} - y)\hat{k}}{[(z - z_{1n})^2 + (y_{1n} - y)^2](x_{3n} - x_{1n})}$$

and

$$\{\text{Fac2}_{AD}\} = (x_{3n} - x_{1n})\left\{\frac{x_{3n} - x}{\sqrt{(x - x_{3n})^2 + (y - y_{1n})^2 + (z - z_{1n})^2}} + \frac{x - x_{1n}}{\sqrt{(x - x_{1n})^2 + (y - y_{1n})^2 + (z - z_{1n})^2}}\right\}$$

Letting x_3 go to ∞, the first term of (Fac2_{AD}) goes to 1.0, the velocity induced by the vortex filament which extends ~~from~~ A to ∞ in a positive direction parallel to the x-axis is given.

$$\overrightarrow{V_{A\infty}} = \frac{\Gamma_n}{4\pi}\left\{\frac{(z - z_{1n})\hat{j} + (y_{1n} - y)\hat{k}}{[(z - z_{1n})^2 + (y_{1n} - y)^2]}\right\}$$

$$\left[1.0 + \frac{x - x_{1n}}{\sqrt{(x - x_{1n})^2 + (y - y_{1n})^2 + (z - z_{1n})^2}}\right]$$

Induced Velocity from B to ∞

$$\overrightarrow{V_{B\infty}} = -\frac{\Gamma_n}{4\pi}\left\{\frac{(z - z_{2n})\hat{j} + (y_{2n} - y)\hat{k}}{[(z - z_{2n})^2 + (y_{2n} - y)^2]}\right\}$$

$$\left[1.0 + \frac{x - x_{2n}}{\sqrt{(x - x_{2n})^2 + (y - y_{2n})^2 + (z - z_{2n})^2}}\right]$$

Then the Velocity induced at the mith. control point. by the vortex representing the jth. panel will be designated as W_{ij}. Examining equation.

$$\overrightarrow{W_{ij}} = \overrightarrow{A_{ij}} \cdot \Gamma_n$$

where $\overrightarrow{A_{ij}}$ depend on the geometry of jth horseshoe vortex and its distance from the control point ith. Since the governing equations is linear, the velocities induced by the 2N vortices are added together to obtain an expression for the total induced velocity at the jth control point.

$$\overrightarrow{W_j} = \sum_{n=1}^{2N} \overrightarrow{A_{jj}} \cdot \Gamma_n$$