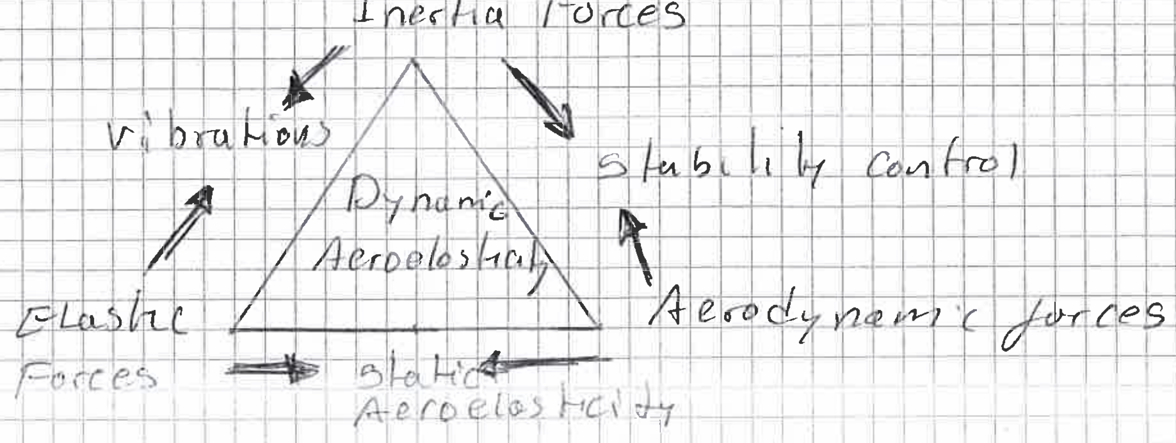
Aeroelasticity

<u>Static Aeroelasticity</u>: Consider non-oscillatory effects of aerodynamic forces acting on the flexible aircraft structure. The flexible nature of wing will influence the in-flight wing shape and hence the lift distribution in a steady manoeuvre or in the special case cruise.

Dynamic Aeroelasticity: This is concerded with the oscillatory effects of the aeroelastic interactions, and the main area of interest is the potentially catastrophic phenomenon Flutter. This instability involves two or more modes of vibration and arises from the unfavourable coupling of aerodynamic, inertial and elastic forces, it means that the structure can effectively extract energy from the airstream.

Aeroelasticity is the subject that describes the interaction of aerodynamic inertia and elastic forces for a flexible structure and the phenomena that can result.

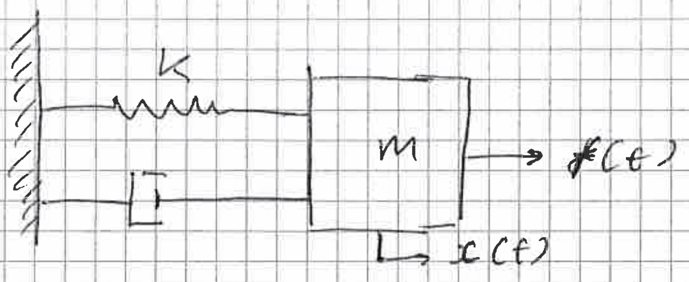


Collar's aeroelastic triangle.

Background Material

Vibrations of single degree of freedom systems.

setting up equations of motion for single DOF systems.



Applying the Lagrange's equation for an single Degree of freedom (DOF) system with a displacement coordinate x may be writtem as

$$① \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial \Upsilon}{\partial \dot{x}} + \frac{\partial U}{\partial x} = Q_x = \frac{\partial}{\partial}\left(\frac{\delta W}{\delta x}\right)$$

Where T: is the kinetic energy
U: is the potential energy (strain)
Υ: Dissipative function
Q: General force
W: Work quantity.

$$T = \frac{1}{2} m \dot{x}^2 \quad \text{(kinetic energy)}$$

$$U = \frac{1}{2} K x^2 \quad \text{(Strain Energy)}$$

$$\Upsilon = \frac{1}{2} C \dot{x}^2 \quad \text{(Damper contribution)}$$

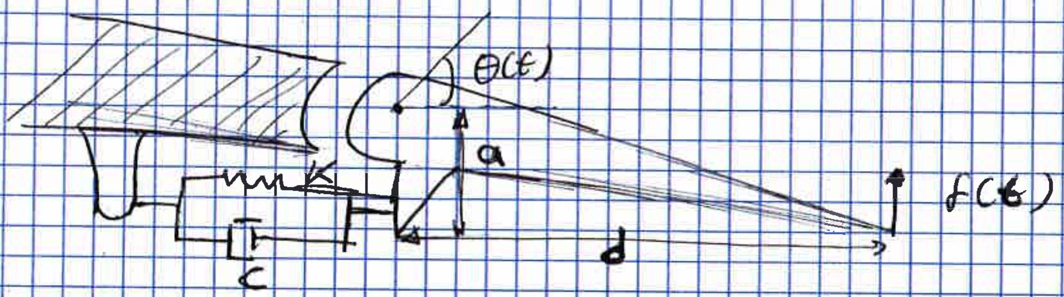
$\delta W = f\,\delta x$ (Effect of the force)

Finally the Lagrange's equation is as following

$$\left[m\ddot{x} + c\dot{x} + Kx = f(t) \right] \quad (2)$$

Ordinary second-order differencial equation (SDOF)

Example: Aircraft control surface



J: moment of inertia about the hinge

θ: rotation of the control surface

a: actuator lever arm

f(t): force applied to control surface

We know that

$$T = \frac{1}{2} J \dot{\theta}^2$$

$$U = \frac{1}{2} k(\theta a)^2$$

$$\mathcal{D} = \frac{1}{2} c\,(a\dot{\theta})^2$$

$$\delta W = (fd)\,\delta\theta$$

Applying the Lagrange's equation we get

$$\left[J\ddot{\theta} + a^2 c\dot{\theta} + K a^2 \theta = d\,f(t) \right] \quad (3)$$

In free vibration, initial condition is imposed and motion then occurs in the absence of any external force. The oscillatory decay corresponds to the low values of damping.

Assuming a form of motion given by

$$x(t) = X e^{\lambda t}$$

X : Amplitude

λ : Characteristic exponent defining decay.

Substituting into (2).

$$- \; m\ddot{x} + c\dot{x} + Kx = 0$$

$$- \; mX\lambda^2 e^{\lambda t} + cX\lambda e^{\lambda t} + KXe^{\lambda t} = 0$$

$$- \; \lambda^2 m + \lambda c + K = 0$$

$$\lambda_{1,2} = -\frac{c}{2m} \pm i\sqrt{\frac{K}{m} - \left(\frac{c}{2m}\right)^2}$$

If $\omega_n = \sqrt{\frac{K}{m}}$ $\qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\zeta = \frac{c}{2m\omega_n}$$

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$\left[\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d \right]$$

ω_n: (Undamped) Natural frequency (frequency in rad/s of free vibration in absence of damping)

ω_d: is the damped natural frequency

$\mathcal{L}$: The damping ratio

The solution is

$$x(t) = X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t}$$

$$x(t) = e^{\mathcal{L}\omega_n t}\left[(X_1 + X_2)\cos\omega_d t + i(X_1 - X_2)\sin\omega_d t\right]$$

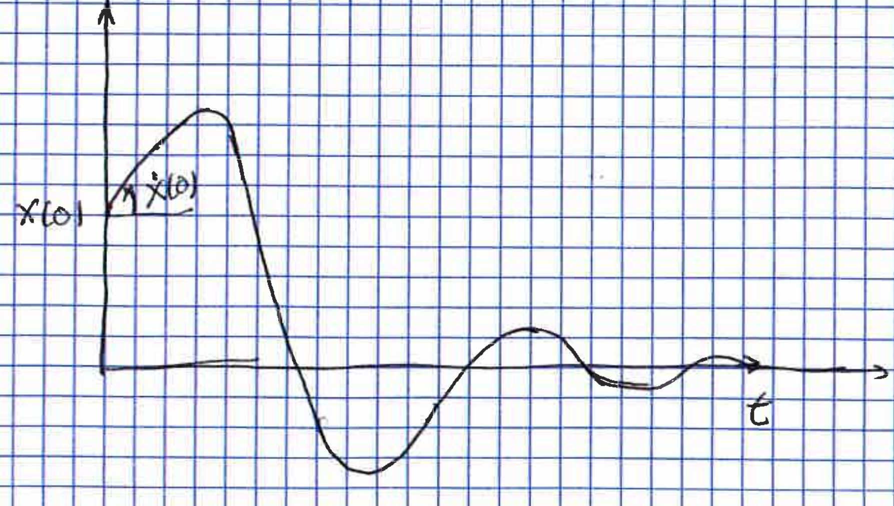
The displacement must be real quantity then X_1 and X_2, must be complex conjugate pairs.

$$x(t) = e^{-\mathcal{L}\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t)$$

Aircraft actuator control surface

The damping ratio $\mathcal{L}: \dfrac{Ca}{2\sqrt{KJ}}$

Undamped natural frequency: $\omega_n = \sqrt{\dfrac{Ka^2}{J}}$



Free vibration response for an underdamped single degree of freedom system.

→ Forced vibration of single DoF systems

Consideration: the aircraft response to a number of different types for forcing functions needs to be considered.

1) Harmonic excitation: Is primarily concerned with excitation at a single frecuency (For engine, rotor out of balance.

2). Non harmonic deterministic excitation: Includes the "1-cosine" input (for discrete gusts or runway bumps) and various shaped inputs (For flight manoeuvers) this forcing functions often have clearly defined analytical forms and are of short duration, often called Transient.

3). Random excitation: continuos turbulence and runway profiles, the latter required for taxing.

The aircraft dynamics are sometimes nonlinear (Double the input does mean to double the response).

HARMONIC FORCED VIBRATION

When a harmonic force is applied, there is a initial transient response, followed by a steady-state phase where the response will also be sinusoidal at the same frequency as the excitation but lagging it in phase. We only consider the steady state response.

the excitation input is defined by

$$f(t) = F \sin(\omega t)$$

and the steady state response is given by

$$x(t) = X \sin(\omega t - \phi).$$

F, X are the amplitudes and ϕ is the amount 'lags' the excitation in phase

In one approach, the steady-state response may be determined by substituting these expressions into the equation of motion and the equating sine and cosine terms using trigonometric expansion. However, an alternative approach uses complex algebra. This is more powerful and commonly used.

$$f(t) = Fe^{i\omega t} = F.\cos(\omega t) + iF\sin(\omega t)$$

$$x(t) = Xe^{i(\omega t - \phi)} = X^{-i\phi}(e^{i\omega t})$$

$$= \tilde{X}(\cos(\omega t) + \tilde{x} i \sin(\omega t))$$

when $\tilde{x} = x^{-i\phi}$ complex amplitude quantity

Only the imaginary part is used for sine excitation.

$$\left\{\begin{array}{l} \dot{x} = i\tilde{X}\omega e^{i\omega t} \\ \ddot{x} = -\tilde{X}\omega^2 e^{i\omega t} \end{array}\right\}$$

Replacing these equations into the ordinary second-order differential equation of motion

$$-m\tilde{X}\omega^2 e^{i\omega t} + \tilde{x} c i\omega e^{i\omega t} + k\tilde{x}e^{i\omega t} = Fe^{i\omega t}$$

$$(-m\omega^2 + i\omega c + k)\tilde{x} = F$$

$$\tilde{x} = Xe^{-i\phi} = \frac{F}{k - \omega^2 m - i\omega c}$$

$$X = \frac{F}{\sqrt{(k - \omega^2 m)^2 + \omega c^2}} \qquad \phi = \tan^{-1}\left(\frac{\omega c}{k - \omega^2 m}\right)$$

Hence, the time response may be calculated using X, ϕ from this equation.

Alternative way of writting the complex response amplitude is

$$H_D(\omega) = \frac{\tilde{X}}{F} = \frac{1}{k - \omega^2 m + i\omega c}$$

where $H_D(\omega)$ is the displacement or receptance

$$H_D(\omega) = \frac{1/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\,2\zeta\,(\omega/\omega_n)}$$

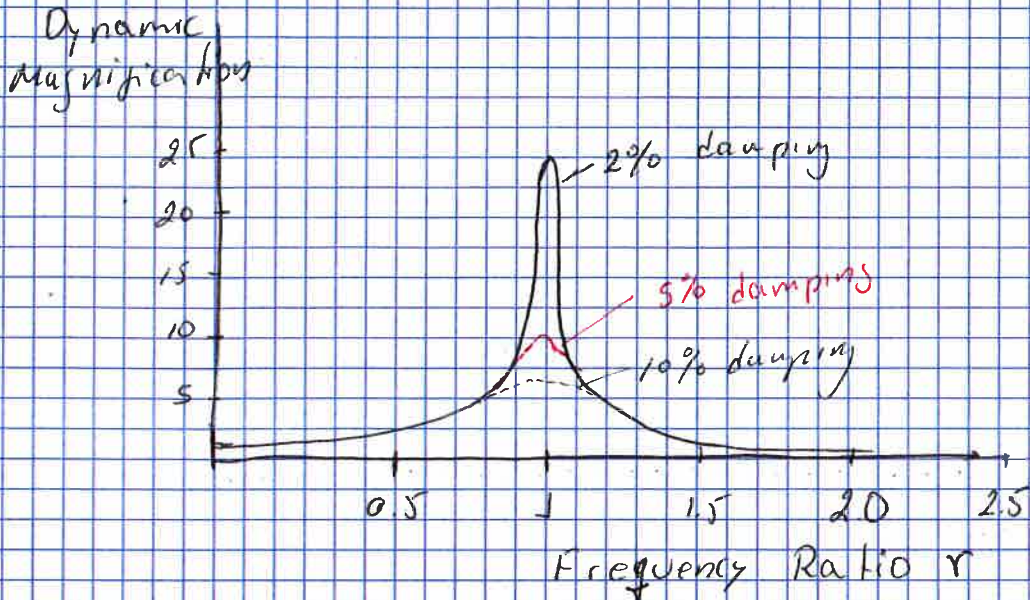
$$= \frac{1/k}{1 - r^2 + i\,2\zeta\,(\omega/\omega_n)}$$

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n} \quad \text{and } r = \frac{\omega}{\omega_n}$$

$H_D(\omega)$ is the frequency response function (FRF) this dictates how the system behaves under harmonic excitation.

H_V (equivalent velocity) $= i\omega H_D$

H_A (equivalent acceleration) $= -\omega^2 H_D$



K Hp(ω) is a nondimensional expression, or dynamic magnification, relating the dynamic amplitude to the static deformation for several damping values. The amplitude peak that occurs when the excitation frequency (ω). is at resonance frequency, close in value to the undamped natural frequency (ω_n), the phase changes rapidly in this region, passing through 90° at resonance. See that the resonant peak increases in amplitude as the Damping ratio reduces and that the dynamic magnification $\frac{1}{2\xi}$ can be extremely large.

It is common practice to combine the damping and stiffness properties of a system having hysteretic damping into a complex stiffnes

$$K^* = K(1 + ig)$$

g: is the loss factor or structural damping coefficient.

The SDOF equation of motion to employ hysteretic or structural Damping may then be written as

$$m\ddot{x} + k(1 + ig)x = f(t).$$

It is not possible to solve this equation in this form. However, it is feasible to write the equation in the time domain as

$$m\ddot{x} + C_{eq}\cdot\dot{x} + kx = f(t)$$

Wher $C_{eq} = \frac{gk}{\omega}$ is the equivalent viscous damping.

The equivalent ratio expression may be solven to be

$$\zeta = \frac{g}{2}\left(\frac{\omega_n}{\omega}\right)$$

Or, if the system is actually vibrating at the natural frequency, then

$$\zeta = g/2$$

The factor of 2 is often seen when comparing flutter damping plots.

Another way of considering hysteretic damping is to convert $\frac{\tilde{X}}{\tilde{F}} = H_D(\omega) = \frac{1}{k(1+ig) - \omega^2 m}$

and now we can see the complex stiffness takes a more suitable form.

When a transient random excitation is present, the time response may be calculated in one of three ways

- Analytical Approach

If the excitation is deterministic, having a relatively simple mathematical form, the analytical method suitable for ordinary differential equations may be used

For example, a unit step force applied to the system initially at rest may be shown to give rise to the response

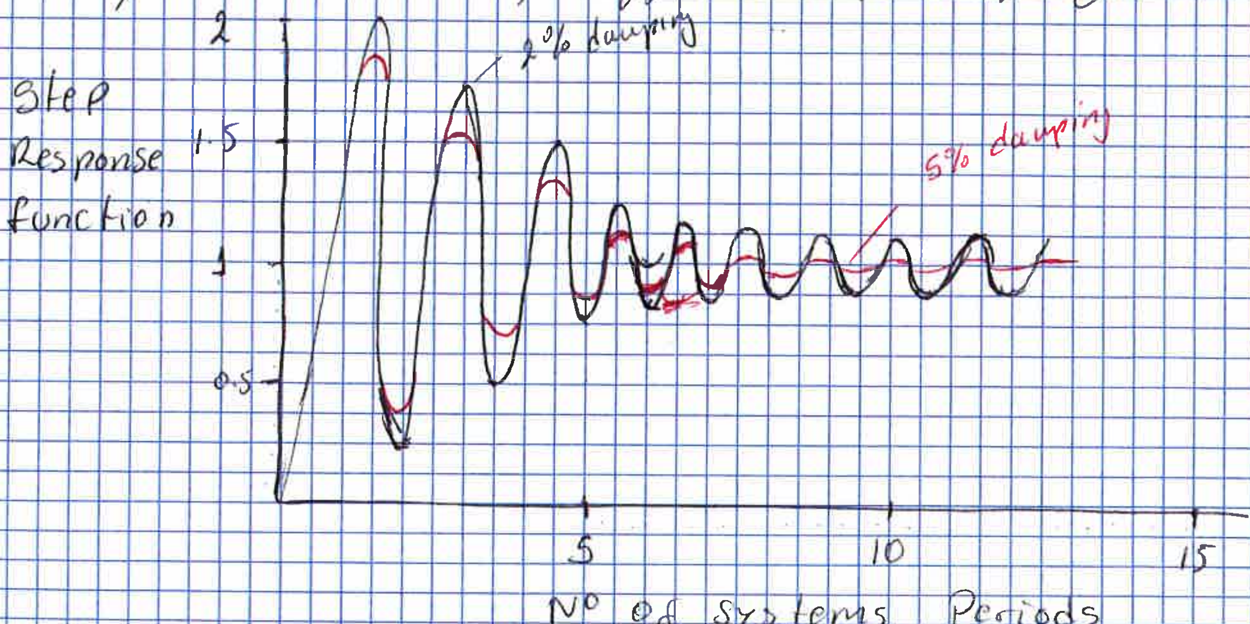
$$S(t) \quad X_{SRF}(t)$$

SRF = Step response function where

$$X_{SRF}(t) = \frac{1}{k}\left[1 - \frac{e^{-(\zeta \omega_n t)}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \varphi)\right]$$

with $\tan \varphi = \frac{\sqrt{1+\zeta^2}}{\zeta}$

The term in brackets is the ratio of the dynamic-to-static response. Note that there is a tendency of the transient response to "overshoot" the steady state value, but the initial peak response is hardly affected by damping.



Thus it may be shown that the response to a unit impulse or Impulse response function $h(t)$ is

$$h(t) = X_{IRF}(t) = \frac{1}{m\omega_d} \cdot e^{-\zeta \omega_n t} \cdot \sin(\omega_d t)$$

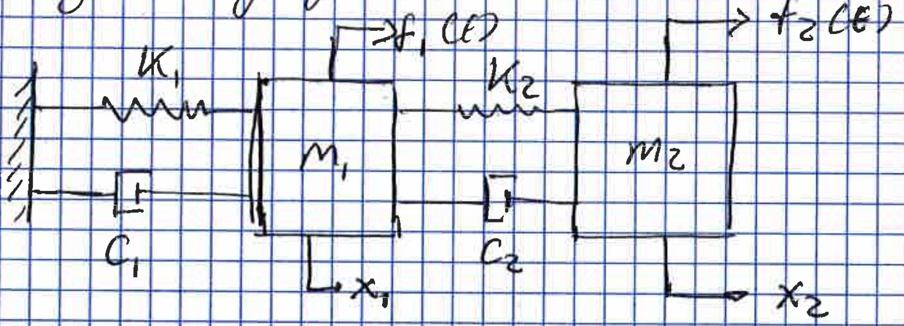
Random forced vibration is experienced by aircraft when flying through continuos turbulence and taxing on a runway with non smooth profile.

For continuos turbulence, it's normal practice to use spectral approach based on linearized model of the aircraft. However for taxing, the solution would be carried out in the time domain using numerical integration of the equations of motion. When Random excitation is considered, then statistical approach is normally employed by defining power spectral density (PSD) of the excitation and response.

Vibrations of Multiple Degree of Freedom Systems.

Lagrange's energy equations will be employed. Two examples will be considered: a classical "chain-like" and a rigid aircraft capable of heave and picth motion while supported on its landing gears.

Suppose we get the following system of two degree of freedom.



The kinetic energy is given by

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 \dot{x}_2^2$$

The strain energy in the springs depends upon the relative extension/compression of each and is given by

$$U = \frac{1}{2} K_1 X_1^2 + \frac{1}{2} K_2 (X_2 - X_1)^2$$

The dissipative term for the dampers depends on the relatives velocitie

$$\Upsilon = \frac{1}{2} C_1 \dot{x}_1^2 + \frac{1}{2} C_2 (\dot{x}_2 - \dot{x}_1)^2$$

The forces included in Lagrages equation by considering the incremental work done δW by two forces.

$$\delta W = f_1 \delta x_1 + f_2 \delta x_2$$

Now the Lagrange for multiple degree of freedom N. may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} + \frac{\partial T}{\partial \dot{x}_j} + \frac{\partial U}{\partial x_j} = Q_j = \frac{\partial(\delta W)}{\partial(\delta x_j)}$$

for $j = 1, 2, 3 \ldots N.$

In our case $N = 2$

$$f_1(t) = m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2$$

$$f_2(t) = m \ddot{x}_2 - c_2 \dot{x}_1 + c_2 \dot{x}_2 - k_2 x_1 + k_2 x_2$$

We can express on Matrix form.

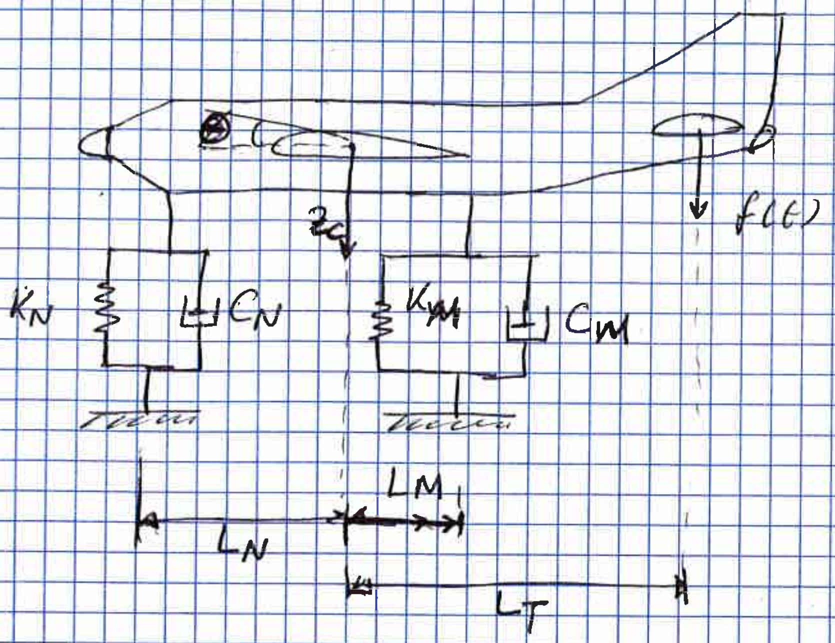
$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

M	**C**	**K**
Uncoupled diagonal Matrix	Coupled diagonal Matrix	Coupled diagonal Matrix

$$\left[M\ddot{X} + C\dot{X} + KX = f(t) \right]$$

Example: Two DOF Rigid Aircraft

This example involves both translational and rotational coordinates,



$$T = \frac{1}{2} m \dot{z}_c^2 + \frac{1}{2} I_y \dot{\theta}^2$$

$$U = \frac{1}{2} K_N (z_c - L_N \theta)^2 + \frac{1}{2} K_M (z_c + L_M \theta)^2$$

$$\Psi = \frac{1}{2} C_N (\dot{z}_c - L_N \dot{\theta})^2 + \frac{1}{2} C_M (\dot{z}_c + L_M \theta)^2$$

$$\delta W = f(\delta z_c + L_t \delta\theta)$$

Applying Lagrange's equations with physical coordinates z_c and θ. Then we get

$$\begin{Bmatrix} f(t) \\ L_T f(t) \end{Bmatrix} = \begin{bmatrix} m & 0 \\ 0 & I_y \end{bmatrix} \begin{Bmatrix} \ddot{z}_c \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} C_N + C_M & -L_N C_N + L_M \cdot C_M \\ -L_N C_N + L_M C_M & L_N^2 C_N + L_M^2 C_M \end{bmatrix} \begin{Bmatrix} \dot{z}_c \\ \dot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} K_N + K_M & -L_N K_N + L_M K_M \\ -L_N K_N + L_M K_M & L_N^2 K_N + L_M^2 \cdot K_M \end{bmatrix} \begin{Bmatrix} z_c \\ \theta \end{Bmatrix}$$

Consider on aircraft with 4000 kg of mass and $I_y = 12000\ kg\ m^2$, $L_N = 4m$ and $L_M = 1m$, $K_N = 40000\ N/m$ and $K_M = 120000\ N/m$. Damping and force value are set to zero for the determination of Natural frequency and mode shape.

$$M = \begin{bmatrix} \underset{M_{11}}{4000} & \underset{M_{12}}{0} \\ \underset{M_{21}}{0} & \underset{M_{22}}{12000} \end{bmatrix} \qquad K = \begin{bmatrix} \underset{K_{11}}{16000} & \underset{K_{12}}{-40000} \\ \underset{K_{21}}{-40000} & \underset{K_{22}}{760000} \end{bmatrix}$$

For each natural frequency W_j, the response may be characterized by the vector X_j, given by the solution of

$$\left[K - W_j^2 M\right] X_j = 0 \quad \text{for } j = 1, 2, \ldots N$$

The determinant of $|K - W^2 M|$ must be set to zero

$$\begin{vmatrix} K_{11} - W^2 M_{11} & K_{12} - W^2 M_{12} \\ K_{21} - W^2 M_{21} & K_{22} - W^2 M_{22} \end{vmatrix} = 0$$

$$\begin{vmatrix} 160000 - W^2 4000 & -40000 \\ -40000 & 760000 - 12000\, W^2 \end{vmatrix}$$

$$48 \times 10^6\, W^4 - 4960 \times 10^6\, W^2 + 120000 \times 10^6 = 0$$

$$48 W^4 - 4960 W^2 + 120000 = 0$$

The root $W_1^2 = 38.65$ $\quad W_2^2 = 64.68$ rad/s

So the undamped natural frecuency are

$\omega = 2\pi f \Rightarrow \frac{\omega}{2\pi} = f \qquad f_1 = 0.989\,Hz \quad f_2 = 1.280\,Hz$

and solving the above equation for the mode shape vector.

for $j = 1$

$$\begin{bmatrix} 5400 & -40000 \\ -40000 & 296.200 \end{bmatrix} \begin{bmatrix} X_1 \\ X_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5400\,X_1 - 40000\,X_1 = 0 \qquad X_1 = 1$$

$$-40000\,X_1 + 296200\,X_1 = 0 \qquad X_1 = 0.135$$

$j = 2$

$$\begin{bmatrix} -98720 & -40000 \\ -40000 & -16160 \end{bmatrix} \cdot \begin{bmatrix} X_2 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$+98720\,X_2 - 40000\,X_2 = 0 \qquad X_2 = -0.405$$

$$-40000\,X_2 - 16160\,X_2 = 0 \qquad X_2 = 1$$

and then the modal Matrix is

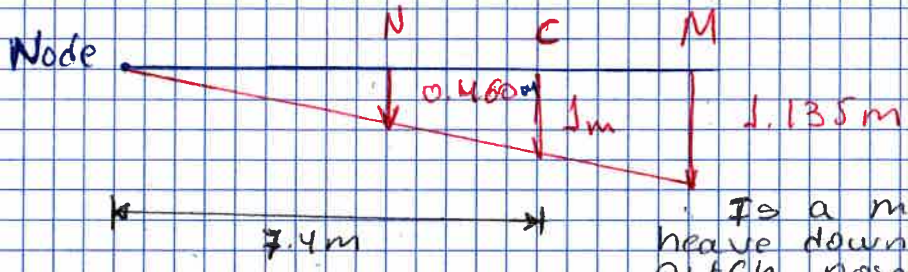
$$\Phi = \begin{bmatrix} 1 & -0.405 \\ 0.135 & 1 \end{bmatrix}$$

- The mode shape vectors need to be interpreted physically since the two values in the vector refer to the motion of the centre of mass (downwards positive) and the pitch angle (nose up positive)

In the example, these values must be obtained by using $(z_C - L_N\theta)$ and $(z_C + L_M\theta)$.

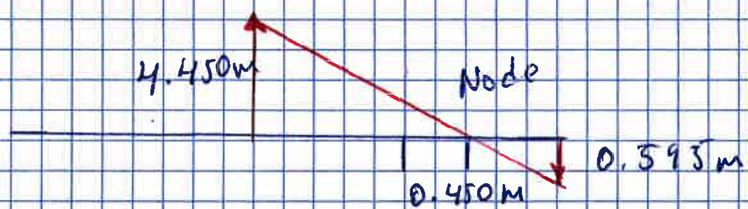
For $X_1 = 1$ and $X_1 = 0.135$ we obtained the corresponding nose and main gear displacement in the mode shape are 0.460 and 1.135, whereas $X_2 = -0.405$ and $X_2 = 1$ imply nose and main gear modal displacements of 4.405 and 0.595.

Mode 1



Is a motion of heave down/up and pitch nose up/down with a stationary point at 7.407m in front of the center of mass.

Mode 2



Is primarily a pitching motion with node point 0.405m behind the center of mass

Vibration of Continuos Systems.

There are several ways of modelling "Continuos" systems, namely:

(a) exact approach using the partial differential equations of the systems to achieve exact modes.

(b) approximate approach using a series assumed shapes to represent the deformation

(c) Approximate approach using some form of spatial 'discretization'

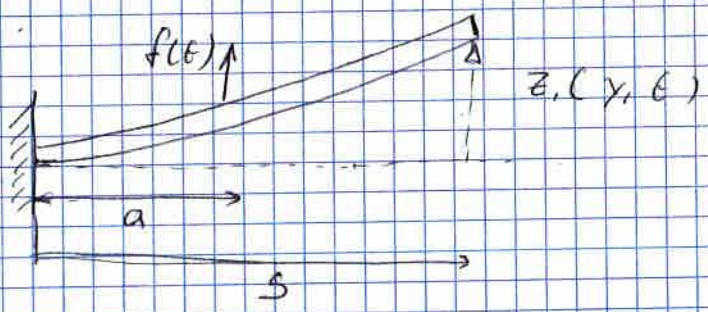
The Rayleigh - Ritz approach is used to represent the deformation of the system by a finite series of known assumed deformation shapes multiplied by unknown coefficients.

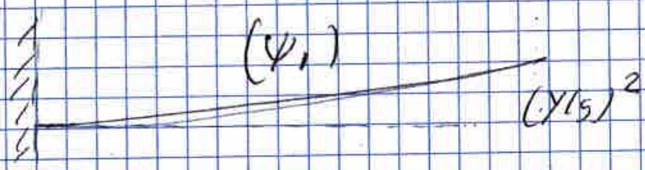
One - dimensional analysis
For a system where the deformation varies in only one dimension, the bending deformation $z(y,t)$. Can be represent like this series

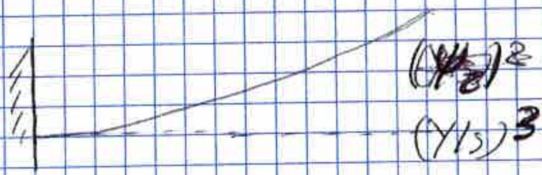
$$z(y,t) = \sum_{j=1}^{N} \psi_j(y) \cdot q_j(t)$$

Where $\psi_j(y)$ is the jth assumed deformation shape and $q_j(t)$ is the jth unknown coefficient, which is a function of time. and N is the number of terms in the series.

Suppose, the following figure has a deformation of a slender member in bending
$N = 2$, $z(y,t) = \psi_1 q_1 + \psi_2 q_2$







The principle of assumed shapes is somewhat akin to using Fourier series to represent a time signal by the summation of a series of sinusoids of different amplitude and phase.

Built-in member in bending - single assumed shape, the above figure the member has length s, mass per length μ, material Young's Modulus E and section second moment of area for vertical bending I. The product EI is known as the flexural rigidity. A force $f(t)$ is applied at position $y = a$. No damping is included. Firstly, only one term in the series will be chosen and the polynomial will be a simple quatratic function,

$$z(y,t) = \psi(y) \cdot q(t) = \left(\frac{y}{s}\right)^2 \cdot q(t)$$

The use of Lagrange's equations, requires various energy and work terms to be determined for discrete system, but in this continuos case the quantities need to be found by integration over the member.

$$dT = \frac{1}{2}(\mu\, dy)\, \dot{z}^2$$

$$T = \frac{1}{2}\int_0^S (\mu\, dy).\, \dot{z}^2$$

We know that $\dot{z} = \left[\frac{y^2}{S^2} \cdot \dot{z}(t)\right]$

$$T = \frac{1}{2}\int_0^S \mu\, dy \left[\frac{y^2}{S^2}\right]^2 \cdot \dot{q}^2$$

$$\left[\ T = \frac{\mu}{10} \times S\, \dot{q}^2 = \frac{\mu S}{10}\dot{q}^2\ \right] \text{ Kinetic Energy}$$

The strain energy in bending depends upon the curvature and flexural rigidity

$$U = \int_0^S \frac{1}{2} EI \left(\frac{d^2 z}{dy^2}\right)^2 . dy$$

$$U = \frac{1}{2}\int_0^S EI \left(\frac{\partial^2 z}{\partial y^2}\right)^2 dy$$

$$U = \frac{1}{2}\int_0^S EI \left(\frac{2}{S^2}\, q\right)^2 dy$$

$$\left[\ U = 2\frac{EI}{S^3}\, q^2\ \right]$$

Finally the work done by applied force moving through an incremental displacement δz at $y = a$ will be.

$$\delta W = f(t) . \delta z(a,t) = f(t) . \left(\frac{a}{s}\right)^2 \delta q$$

Taking Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} + \frac{\partial U}{\partial q_j} = Q_j = \frac{\partial(\delta W)}{\partial(\delta q_j)}$$

For $j = 1, 2 \ldots N$,

$$\frac{\partial T}{\partial \dot{q}} = \frac{\mu . s}{10} . 2\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{\mu . s}{5} \ddot{q} \quad \text{and} \quad \frac{\partial U}{\partial q} = \frac{4EI}{s^3} . q$$

Then we obtain

$$\frac{\mu . s}{5} . \ddot{q} + \frac{4EI}{s^3} . q = \left(\frac{a}{s}\right)^2 . f(t)$$

By inspection of this SDOF equation the undamped natural frecuency is given by

$$\omega = 4.47 \sqrt{\frac{EI}{\mu s^4}}$$

This overestimate by 27% on the exact solution value of $3.516 \sqrt{\frac{EI}{\mu s^4}}$

Built-in Member in Bending - Two Assumed Shapes

We get the following for two assumed shapes

$$z(y,t) = \psi_1(y)\, q_1(t) + \psi_2(y)\, q_2(t)$$

$$z(y,t) = \left(\frac{y}{S}\right)^2 q_1 + \left(\frac{y}{S}\right)^3 q_2$$

The energy (kinetic and strain) are followed

$$T = \frac{1}{2}\int_0^S \mu \left[\left(\frac{y}{S}\right)^2 \dot{q}_1 + \left(\frac{y}{S}\right)^3 \dot{q}_2\right]^2 dy$$

$$= \frac{\mu S}{10}\dot{q}_1^2 + \frac{\mu S}{14}\dot{q}_2 + \frac{\mu}{6}S\, \dot{q}_1 \dot{q}_2$$

and

$$U = \frac{1}{2}\int_0^S EI\left(\frac{2}{S^2} q_1 + \frac{6y}{S^3} q_2\right)^2 dy$$

$$U = \frac{2EI}{S^3} q_1^2 + \frac{6EI}{S^3} q_2^2 + \frac{6EI}{S^3} q_1 q_2$$

and work done term is

$$\delta W = f(t)\, \delta z(a,t) = f(t)\left[\left(\frac{a}{S}\right)^2 \delta q_1 + \left(\frac{a}{S}\right)^3 \delta q_2\right]$$

Applying Lagrange's equation for generalized coordinates q_1 and q_2 $(N=2)$

$$\frac{\mu S}{5}\ddot{q}_1 + \frac{\mu S}{6}\ddot{q}_2 + \frac{4EI}{S^3} q_1 + 6\frac{EI}{S^3} q_2 = \left(\frac{a^2}{S^2}\right) f(t)$$

$$\frac{\mu S}{6}\ddot{q}_1 + \frac{\mu S}{7}\ddot{q}_2 + 6\frac{EI}{S^3} q_1 + \frac{12EI}{S^3} q_2 = \left(\frac{a}{S}\right)^3 f(t)$$

by Matrix representation

$$\begin{bmatrix} \frac{\mu S}{5} & \frac{\mu S}{6} \\ \frac{\mu S}{6} & \frac{\mu S}{7} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \frac{4EI}{S^3} & \frac{6EI}{S^3} \\ \frac{6EI}{S^3} & \frac{12EI}{S^3} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} \left(\frac{a}{S}\right)^2 \\ \left(\frac{a}{S}\right)^3 \end{bmatrix} f(t)$$

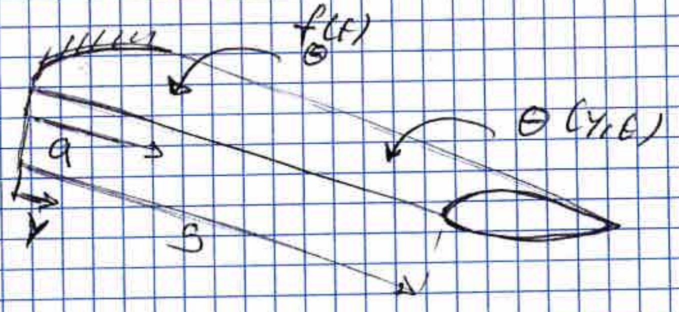
The natural frequencies and undamped mode shapes is obtained for MDOF.

$$\omega_1 = 3.533 \sqrt{\frac{EI}{\mu S^4}} \qquad \omega_2 = 34.81 \sqrt{\frac{EI}{\mu S^4}}$$

The exact value of

$$\omega_{1\,exact} = 3.516 \sqrt{\frac{EI}{\mu S^4}} \qquad \omega_{2\,exact} = 22.03 \sqrt{\frac{EI}{\mu S^4}}$$

Built in Member in torsion - One assumed shape



$$\theta(y,t) = \gamma(y)\, q(t) = \left(\frac{y}{S}\right) \cdot q$$

The uniform member built in at one, now has a moment of inertia in twist per unit length of χ, and a torsional rigidity GJ

G is the material shear modulus

J: section torsion constant, which is not equal to the polar second moment of area, as is sometimes incorrectly stated except for the special case of a circular section

$$T = \frac{\chi s}{6} \dot{q}^2 \qquad U = \frac{\sigma J}{2s} q^2$$

$$\delta W = f_\theta(t)\, \delta\theta(a) = f_\theta(t) \cdot \frac{a}{s} \delta q$$

$$\frac{\chi s}{3} \ddot{q} + \frac{\sigma J}{s} q = \frac{a}{s} f_\theta(t)$$

The Estimate natural is therefore $1.73 \sqrt{\frac{\sigma I}{\chi s^2}}$ and this value is a 10% overeshimate to the exact value of $1.57 \sqrt{\frac{\sigma I}{\chi s^2}}$

The General Equations of Motion - Matrix Approach

$$z(y, t) = \sum_{j=1}^{N} \psi_j(y) \cdot q_j(t) = \psi^T q \text{ or } q^T \psi$$

$$\psi(y) = \{\psi_1(y)\ \psi_2(y) \cdots \psi_N\}^T$$

$$q(t) = \{q_1, q_2 \cdots q_N\}^T$$

$$T = \frac{1}{2} \int_0^s \mu (\dot{q}^T \psi) \cdot (\psi^T \dot{q})\, dy = \frac{1}{2} \dot{q}^T \left[\int_0^s \mu (\psi \psi^T) dy\right] \dot{q}$$

$$T = \frac{1}{2} \dot{q}^T M_q \dot{q}$$

$$U = \frac{1}{2} \int_0^s EI \left(\frac{\partial^2 z}{\partial y^2}\right)^2 dy = \frac{1}{2} q^T \left[\int EI (\psi'' (\psi'')^T\right] q$$

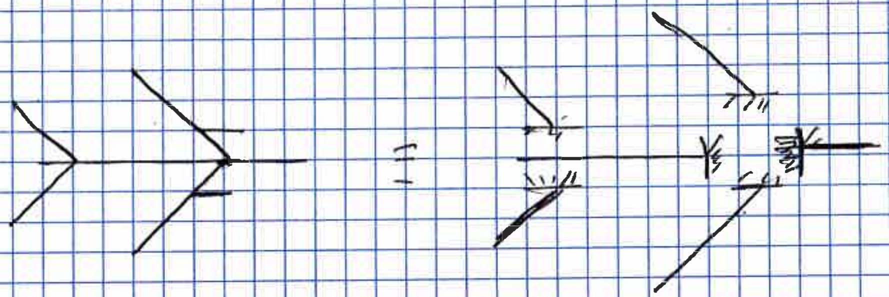
$$U = \frac{1}{2} q^T K_q q$$

$$M_q \cdot \ddot{q} + K_q q = \psi(a) \cdot f(t)$$

Example:

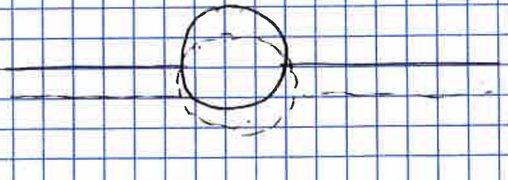
Consider an aircraft of two uniform flexible wings of mass per length μ_w, and flexural rigidity EI, plus a rigid fuselage mass m_f. The aircraft of mass m is assumed not to undergo any pitch motion of the wings only bend and the fuselage only heaves.

Assume that the first two exact normal 'branch' mode shapes, for a single wing constrained/built in at its root, are known and given by the function ψ_{b_1} and ψ_{b_2}

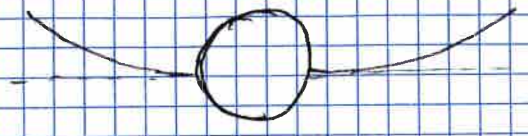


In order to free, the aircraft so that it behaves as a free-free structure and so be able to determine the equivalent free-free flexible modes.

This can be achieved by assuming that the displacement of the aircraft is a combination of the exact flexible branch $\psi_{b_{1,2}}$ and a rigid body heave assumed shape.



ψ_b



ψ_{b_1}

ψ

Thus the assumed total displacement along the wing ($y \geq 0$) is given by

$$z(y,t) = \psi_b \cdot q_b + \psi_{b_1} q_1 + \psi_{b_2} \cdot q_2 .$$

Seeing that the two wings move-in-phase (i.e. only symmetric modes are required) and that the fuselage width is ignored in the integrals, the total energy is

$$T_{Aircraft} = T_{wings} + T_{fuselage}$$

$$T_{wings} = 2\left(\frac{1}{2}\int_0^s \mu_w \dot{z}^2 \, dy\right)$$

$$T_{fuselage} = \frac{1}{2} m_f \dot{z}(0)^2 = \frac{1}{2} m_f (\psi_b \dot{q}_h)^2$$

Since the strain energy is only present in wing bending for this simple system, then

$$U = 2\left(\frac{1}{2}\int_0^s EI \cdot z''^2 \, dy\right).$$

Since the additional rigid shape has no elastic deformation, then $\phi'' = 0$, which simply the final expression.

$$0 = \begin{bmatrix} m_b & 2m_{bb_1} & 2m_{bb_2} \\ 2m_{bb_1} & 2m_{b_1} & 0 \\ 2m_{bb_2} & 0 & 2m_{b_2} \end{bmatrix} \begin{Bmatrix} \ddot{q}_b \\ \ddot{q}_{b_1} \\ \ddot{q}_{b_2} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2K_{b_1} & 0 \\ 0 & 0 & 2K_{b_2} \end{bmatrix} \begin{Bmatrix} q_b \\ q_{b_1} \\ q_{b_2} \end{Bmatrix}$$

$M_b = M = M_f + 2\mu_w S$ $\quad M_{bj} = \int_0^S \mu_w \cdot \psi_{bj}^2 \cdot dy$

$M_{bbi} = \int_0^S \mu_w \psi_{bj} \cdot dy$

$K_{bj} = \int_0^S EI \psi_{bj}''^2 \, dy \qquad j = 1, 2$

Consider $M_F = 1200\,Kg$ $\quad \mu_w = 50\,Kg/m$

$S = 6\,m$ and $EI = 500000\,N/m^2$

It has been shown that for the built-in member of length S and mass per length μ_w, built in one end, the jth mode natural frequency is given by

$$\omega_{bj} = (\beta_j S)^2 \sqrt{\frac{EI}{\mu_w S^4}}$$

$\beta_1 S = 1.875$ $\beta_2 S = 4.694$; the frequency (Natural) values are 1.55 and 9.74 Hz

The corresponding mode shapes are given by

$$\psi_{bj}(y) = (\cosh \beta_j y - \cos \beta_j y) - \sigma_j (\sinh \beta_j y - \sin \beta_j y)$$

where $\sigma_j = \cos \beta_j S + \cosh \beta_j S / \sin \beta_j S + \sinh \beta_j S$

The modal mass values for these mode shapes are given as $M_1 = M_2 = \mu_w \cdot S$, the mass coupling terms may be shown to equal $M_{bb_1} = 0.734\,\mu_w S$ $\quad M_{bb_2} = 1.018\,\mu_w \cdot S$

Solving the above equation, the rigid body heave mode has a frequency 0 Hz and generalized mode shape $\{1, 0, 0\}$

The natural frecuencies of the two free-free elastic modes of the aircraft are 1.74 and 10.15 Hz and generalized mode shapes are.

$$\begin{Bmatrix} -0.261 \\ 1 \\ -0.004 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} -0.183 \\ 0.147 \\ 1 \end{Bmatrix}$$

These solutions is by using the generating aircraft Free-Free Modes from 'Branch' modes

Whole aircraft 'Free-Free' Modes

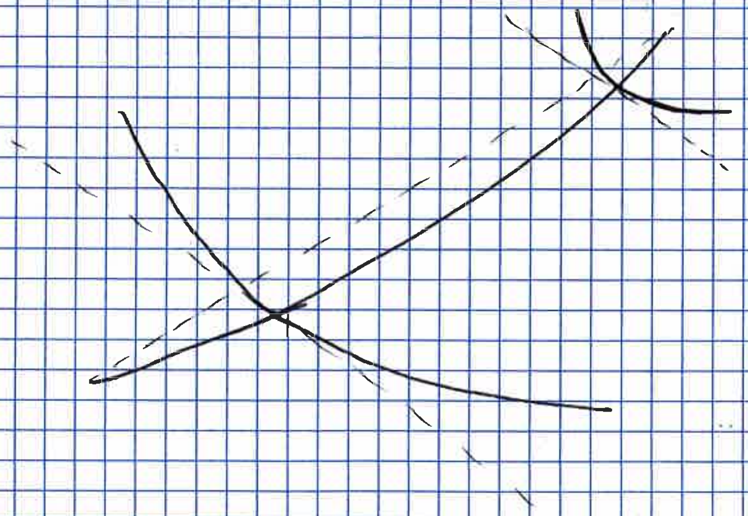
$$z(x, y, t) = \psi_r^T q_r + \psi_e^T q_e = q\,\psi^T$$

r: Stand for rigid body

e: elastic and flexible body

ψ: normal mode Shapes

q: generalized / modal Coordinate

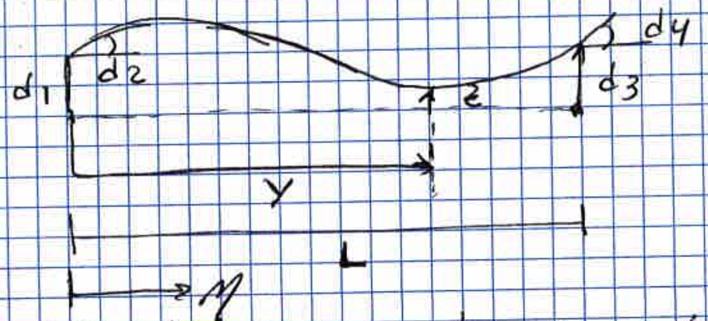


Flexible aircraft with free-free symetric modes.

Vibration of continuos systems - Discretization Approach.

For simplicity, bending in only one plane, with no shear deformation, is considered

The bending deformation will again be denoted using the symbol z;



The nodal displacement are denoted by the vector $d = \{d_1, d_2, d_3, d_4\}$

The variation of the displacement $z(Y)$ along the beam element is expressed as a cubic polynomial.

$$z = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3$$

$a_0 \dots a_3$ are unknown coefficients

$Y = 0$ displacement $a_0 = d_1$

$Y = 0$ slope $a_1 = d_2$

$Y = L$ displacement $d_3 = a_0 + a_1 L + a_2 L^2 + a_3 L^3$

$Y = L$ slope $d_4 = a_1 + 2a_2 L + 3a_3 L^2$

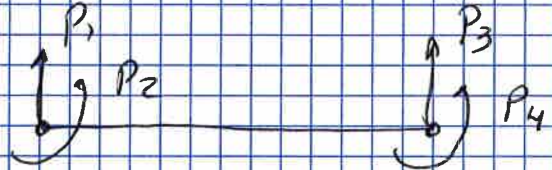
Equation must be solved to yield expressions for the coefficients $a_0, \dots a_3$ in term of the displacements.

$$z = N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4 = \mathbf{N}^T \mathbf{d}$$

where N is column vector of the shape functions: $N_1 \dots N_4$ each being a cubic polynomial in y.

Element equation of the motion.

In FE (Finite Element) representation, forces and moments may only be applied to the element at the nodes as shown in the figure, these are termed nodal forces $P = (P_1 \; P_2 \; P_3 \; P_4)^T$. The element equation relating nodal forces, displacement and acceleration will then be sought.



We apply the Lagrange's equation of Motion

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 z}{\partial y^2}\right)^2 dy \Rightarrow U = \frac{1}{2} d^T \left[\int_0^L EI(N'' N''^T)\,dy\right] \mathbf{d}$$

$$'' = \frac{\partial}{\partial y} \quad \text{(shorthand notion)}$$

$$T = \frac{1}{2}\int_0^L \mu^2 \dot{z}^2 dy \Rightarrow T = \frac{1}{2} \dot{d}^T \left[\int_0^L \mu (N.N^T)\,dy\right] \dot{d}$$

$$\delta w = P^T \delta d$$

then the equation is

$$m\ddot{d} + Kd = P$$

$$m = \left[\int_0^L \mu (N N^T)\, dy\right] \qquad K = \left[\int_0^L EJ (N'' N''^T)\, dy\right]$$

If N_1, N_2, N_3, and N_4 (relevant shape function polynomials)

$$m = \frac{\mu . L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & 6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

This mass representation is known as a "consistent" mass matrix. The more simple "lumped" mass model, where for two-node beam element

$$m_{Lumped-NO-Rotaty-Inertia} = \frac{\mu L}{24}\, diag(12\ 0\ 12\ 0)$$

$$m_{Lumped-Rotary-Inertia} = \frac{\mu L}{24}\, diag(12\ L^2\ 12\ L^2)$$

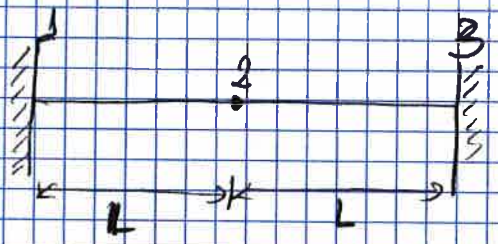
$$N_1 = \frac{1}{L^3}(2y^3 - 3y^2L + L^3)$$

$$N_2 = \frac{1}{L^3}\left(y^3L - 2y^2L^2 + yL^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2y^3 + 3y^2L\right)$$

$$N_4 = \frac{1}{L^3}\left(x^3L - y^2L^2\right)$$

Example 1 Determine the natural frequency of vibration for a beam fixed both ends. The beam density ρ, modulus of elasticity E Cross-sectional area A. area moment of inertia I and length $2L$. Distretized in 2 elements



To obtain the natural frequency is by solving the ~~integra~~ matrix.

$$\left[K - \omega^2 M\right] = 0$$

Boundary conditions

$d_1 = 0 \qquad d_3 = 0$

$d_2 = 0 \qquad d_4 = 0$

$$K = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \qquad m = \frac{\mu L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left| \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} - \omega^2 \mu L \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right| = 0$$

$$24\frac{EI}{L^3} - \omega^2 \mu L = 0$$

$$\omega^2 = \frac{24 EI}{\mu L^4}$$

$$\left[\omega = \frac{4.90}{L^2}\left(\frac{EI}{\mu}\right)^{1/2} \right]$$

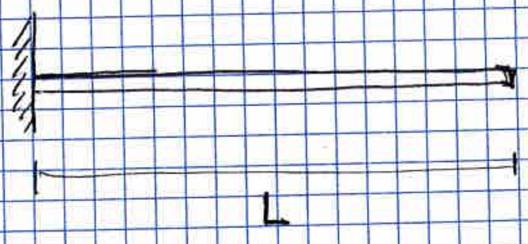
The exact solution for the first natural frequency, from simple beam theory is given by

$$\omega = \frac{5.59}{L^2}\left(\frac{EI}{\mu}\right)^{1/2}.$$

Example 2.

Determine the first natural frequency of vibration of the cantilever beam. The wind of aircraft can be analyzed by a cantilever beam.

Characteristic $\left\{ \begin{array}{l} L,\ E,\ I,\ A,\ \rho \text{ and} \\ \nu \text{ poisson's ratio} \end{array} \right.$



Doing the finite element analysis for the first frequency (natural) is given by

$$\omega = \frac{3.148}{L^2}\left(\frac{EI}{\mu}\right)^{1/2}$$

The exact solution according to beam theory.

$$\omega = \frac{3.516}{L^2}\left(\frac{EI}{\mu}\right)^{1/2}$$

According to vibration theory for a clamped-free beam, we can relate the second and third natural frequencies to the first natural frequency by

$$\frac{\omega_2}{\omega_1} = 6.2669 \qquad \frac{\omega_3}{\omega_1} = 17.5475$$

$c = 1m$

$L = 5m$

$E = 112 \times 10^6 \frac{N}{m^2}$

$I = 0.000074 \, m^4 \cong K_I \cdot c^4 \left(\frac{t}{c}\right)\left(\left(\frac{t}{c^2}\right)^2 + \left(\frac{h}{c^2}\right)^2\right)$

K_I = proportionality coefficient = 0.036

(t/c) = The thickness and chord relation

(h/c) = Maximum camber and chord relation

Supposing the Naca 0012 is the airfoil of the wing then $t/c = 0.12$ and $\frac{h}{c} = 0$

$A = K_A \cdot c \, t = 0.6 \, c^2 \left(\frac{t}{c}\right) = 0.072 \, m^2$

K_A = Coefficient of proportionality = 0.6

$\rho = 2700 \, kg/m^3$

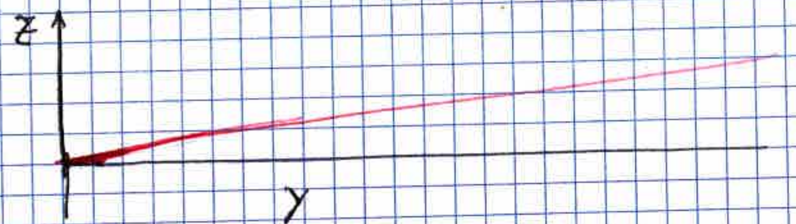
$\mu = \rho A =$ [illegible] kg/m 38 kg/m

Exact solution is for the natural frequencies

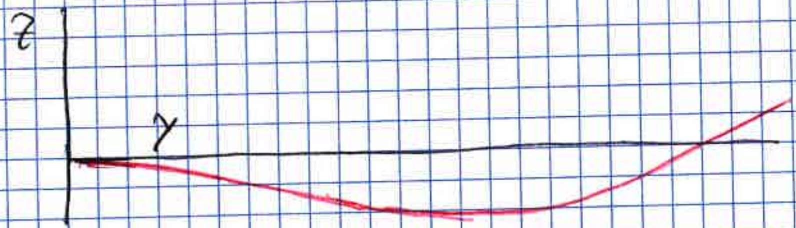
$\omega_1 = 2.0760 \, rad/s \qquad \omega_2 = 12.93 \, rad/s \qquad \omega_3 = 33. \, rad/s$

Using 60 elements. we obtain $\omega_1 = \frac{3.514}{L^2}\sqrt{\frac{EI}{\mu}}$

$W_1 = 2.077 \ \frac{rad}{s}$ $\quad \omega_2 = 13.020 \ \frac{rad}{s}$



First mode



Second mode

Third mod

Static Aeroelasticity.

We know that the static aeroelasticity is the study of the deflection of flexible aircraft structures under aerodynamic loads, where the forces and motion are considered to be independent of time.

Lift and moment acting upon a wing to depend upon the incidence of the chordwise. These loads cause the wing to bend and twist, so changing the incidence and consequently the aerodynamic flow, which in turn changes the loads acting on the wing and the deflections and so on until an equilibrium conditions is usually reached.

Through the elimination of time-dependent forces and motion, inertial forces can be ignored in the equilibrium equations as these are dependent upon acceleration.

The divergence phenomena is happening when the moments due to aerodynamic forces overcome the restoring moments due to the structural stiffness, so resulting in structural failure. The most common type is that of wing torsional divergence.

In general, for aeroelasticity considerations the stiffness is of much greater importance than the strength.

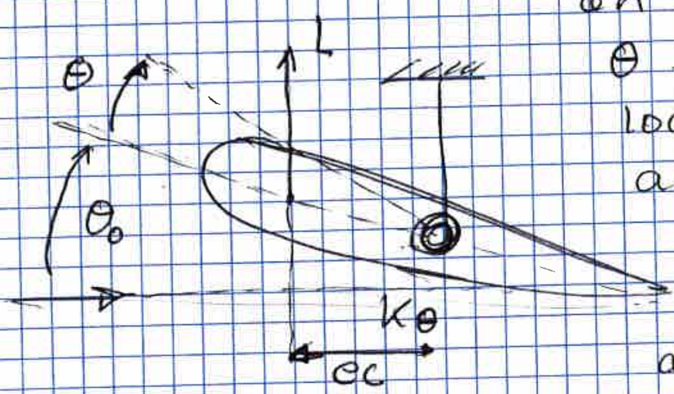
In modern Aircraft, the flutter speed is usually reached before divergence speed so divergence is not normally a problem.

However, the divergence speed is a useful measure of the general stiffness of the aircraft structure and must be considered as part of the certification process.

Static Aeroelastic behaviour of two-dimen. Rigid aerofoil with spring attachment.

Interative analysis

Supposing an airfoil as shown in below figure. The airfoil has an initial incidence θ_0 on twists through angle θ due to aerodynamic loading. The lift acting on the airfoil at air speed V (True air speed, TAS) and initial θ_0 causes an pitching moment given by



$$\frac{dC_L}{d\alpha} = a_1$$

$$M = \left[\frac{1}{2} \rho V^2 c a_1 \theta_0\right] ec = q\, ec^2 a_1 \theta_0$$

Applying Lagrange's equations on the airfoil, and only, we are considering static effects, kinetic energy can be ignored. Then the potential strain is given by

$$U = \frac{1}{2} K_\theta \theta^2$$

The incremental work done by the pitching moment acting through the incremental angle $\delta\theta$

$$Q_\theta = \frac{\partial(\delta W)}{\partial(\delta\theta)} = \frac{\partial(q ec^2 a_1 \theta_0 \delta\theta)}{\partial(\delta\theta)} = q ec^2 a_1 \theta_0$$

Then application of Langrange's equations for coordinates θ gives

$$K_\theta \theta = q e c^2 a_1 \theta_0$$

$$\theta = \frac{q e c^2 a_1 \theta_0}{K_\theta} = q R \theta_0$$

$R: e a_1 c^2 / K_\theta$

- Initial Iteration

The incidence of the airfoil now includes the intial incidence and the estimate of twist, so the revised pitching moment becomes.

$$M = q e c^2 a_1 (\theta_0 + qR\theta_0)$$

Application of Langrange's equation for the revised elastic twist angle of

$$\theta = q e c^2 a_1 \left(\frac{1 + qR}{K_\theta}\right) \theta_0$$

$$\theta = qR(1 + qR)\theta_0$$

- Further iterations

Repeating the above process continues by using the updated elastic twist value in the pitching moment and work expressions, these leading to an infinite series expansion for elastic twist.

$$\theta = qR\left[1 + qR + (qR)^2 + (qR)^3 + \cdots\right]\theta_0$$

Revising the binomial series

$$(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots \quad \text{with } |x| \leq 1$$

The series of elastic expansions convergence to $\left(\frac{1}{1-qR}\right)$ then we obtain

$$\theta = \frac{qR}{(1-qR)} \cdot \theta_0$$

Direct (single step) Analysis

Same two-dimensional airfoil as above, but the incidence angle includes the unknown aeroelastic twist θ.

Moment is $M = q e c^2 a_1 (\theta_0 + \theta)$

The strain energy is the same

$$U = \frac{1}{2} K_\theta \theta^2 = \frac{1}{2} K_\theta \theta^2$$

The work done is

$$Q_\theta = q e c^2 a_1 (\theta_0 + \theta)$$

Application Lagrange's equation of motion

$$K_\theta \theta = q e c^2 a_1 (\theta_0 + \theta)$$

$$(K_\theta - q e c^2 a_1)\theta = q e c^2 a_1 \theta_0$$

$$\theta = \frac{q e c^2 a_1}{K_\theta - q e c^2 a_1} \theta_0$$

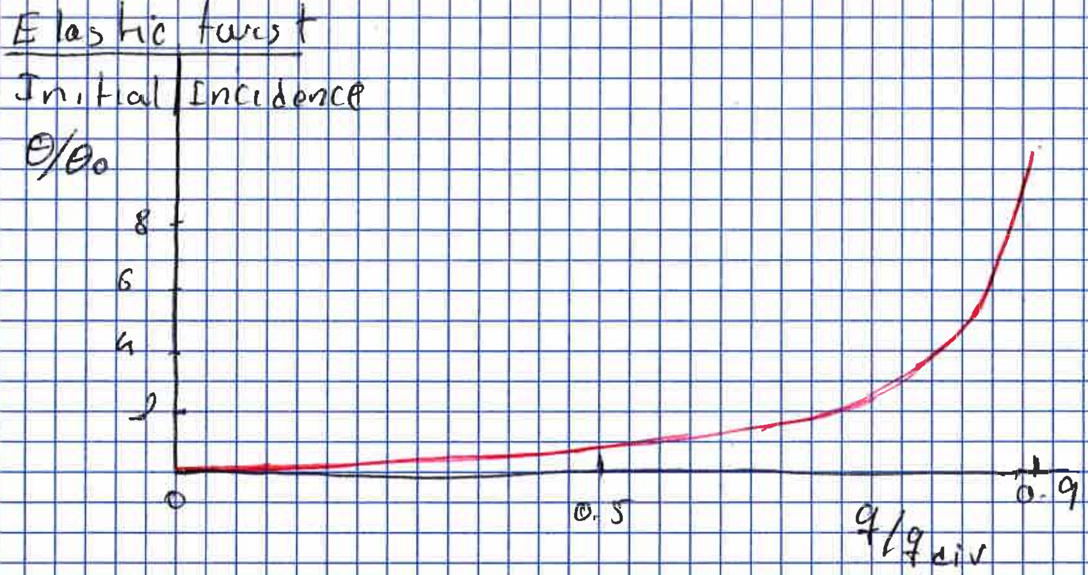
$$\theta = \frac{qR}{\ldots} \theta_0$$

the elastic twist becomes infinite as q approaches $1/R$ and this defines the so called divergence speed.

$$q_{div} = \frac{1}{R} = \frac{K_\Theta}{ec^2 a_1}$$

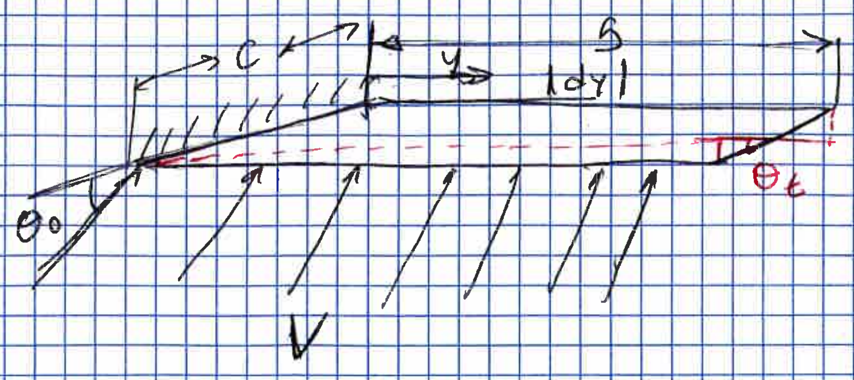
Then we obtain

$$\Theta = \left(\frac{q/q_{div}}{1 - q/q_{div}} \right) \Theta_0$$



Typical twist behaviour for a two-dimensional airfoil with torsion spring.

A more realistic example of static aeroelastic behaviour is now examined for a flexible wing fixed at the root



For simplicity, assume that the wing twist behaviour is characterized by the the idealized linear relationship.

$$\theta = (y/s)\ \theta_T$$

The lift is taken as acting at the aerodynamic center, because the section is symmetric. then the lift is

$$dL = q c a_w \left(\theta_0 + \frac{y}{s} \theta_T \right) \theta_T$$

$$L = q c a_w \left(s\theta_0 + \frac{s}{2} \theta_T \right)$$

there is no motion of the wing; kinetic energy $T=0$, then the strain energy is given by

$$U = \frac{1}{2} \int_0^s GJ \left(\frac{d\theta}{dy} \right)^2 dy$$

$$U = \frac{GJ}{2s} \theta_T^2$$

The incremental twist angle being expressed in terms of incremental generalized coordinate

$$\delta\theta = \frac{y}{s} \delta\theta_T$$

The Incremental Work done is given by

$$\delta W = \int_0^s dL\ ec\ \delta\theta$$

$$\delta W = q e c^2 a_w \left(\frac{s\theta_0}{2} + \frac{s\theta_T}{3} \right) \delta\theta_T$$

Lagrange's equations yield

$$\frac{GJ}{s} = q e c^2 a_w \left(\frac{s\theta_0}{2} + \frac{s\theta_T}{3} \right)$$

$$\left(\frac{GJ}{s} - q e c^2 a_w \frac{s}{3} \right) \theta_T = q e c^2 a_w \frac{s\theta_0}{2}$$

$$\theta_T = \left(\frac{3 q e c^2 s^2 a_w}{6 GJ - 2 q e c^2 s^2 a_w} \right) \theta_0$$

For this fixed root wing, the dynamic pressure at divergence q_W is found as

$$q_W = \frac{3\, GJ}{e c^2 s^2 a_w}$$

* The smaller the distance between the aerodinamic center and the flexural axis and/or the greater the flexural rigid GJ, the greater the divergence speed becomes.

* If the flexural axis lies on the axis of aerodynamic center there is no twist due to aerodynamic loading and divergence will not occur.

* Should the flexural axis actually lie forward of the aerodynamic center, the applied aerodynamic moment becomes negative; so the tip twist is nose downwards and divergence cannot occur.

these last two generally are not possible so divergence must be considered for aeroelastic design and adequate torsional stiffness.

Dynamic Aeroelasticity - Flutter

Flutter is the most important of all the aeroelastic phenomena and is the most difficult to predict. It is an unstable self-excited vibration in which the structure extracts energy from the air stream and often results in catastrophic structural failure.

The clasical binary flutter occurs when the aerodynamic forces associated with motion in two modes of vibrations cause the modes to couple in an unfavorable manner

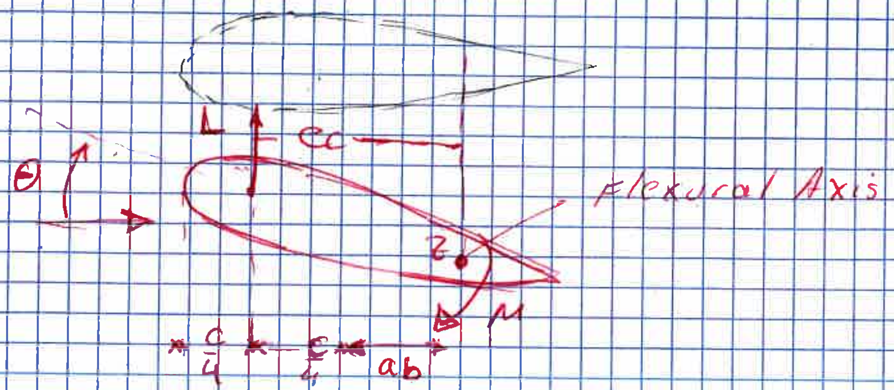
At some critical speed, known as the flutter speed, the structure sustains oscillations following some initial disturbance. Below this speed the oscillations are damped whereas above it one of the modes becomes negatively damped and unstable oscillations occur, unless some form of nonlinearity bounds the motion.

The flutter involves different pairs of interacting modes. wing bending / torsion wing torsion / Control surface, wing / engine

Simplified unsteady aerodynamic model

Consider, once, again the two-dimensional airfoil with flexural axis positional a distance (ec) aft of the aerodynamic center and (ab) aft of the mean chord.

where $ec = \frac{c}{4} + ab = \frac{c}{4} + \frac{ac}{2}$



Lift and moment per unit span for an airfoil may be expressed for a particular reduced frequency.

$$L = \rho V^2 \left(L_z z + L_{\dot{z}} \frac{b\dot{z}}{V} + L_\theta b\theta + L_{\dot{\theta}} \frac{b^2\dot{\theta}}{V} \right)$$

$$M = \rho V^2 \left(M_z bz + M_{\dot{z}} \frac{b^2\dot{z}}{V} + M_\theta b^2\theta + M_{\dot{\theta}} \frac{b^3\dot{\theta}}{V} \right)$$

V is the true air speed and ρ the density

For all of the aerodynamic derivatives the quasi-steady assumption leads to a lift and moment per unit span about the flexural axis becomes

$$L = \frac{1}{2} \rho V^2 c a_1 \left(\theta + \frac{\dot{z}}{V} \right)$$

$$M = \frac{1}{2} \rho V^2 e c^2 a_1 \left(\theta + \frac{\dot{z}}{V} \right)$$

Comparing with the static aeroelastic case, there is now an extra term due to the effective incidence associated with the airfoil moving downwards with constant heave velocity $\dot{z}$, causing an effective 'upwash'

Quasi-steady assumption implies that the aerodynamic loads acting on an airfoil undergoing variable heave and pitch motions are equal, at any moment in time, to the characteristics of the same airfoil with constant position and velocity value.

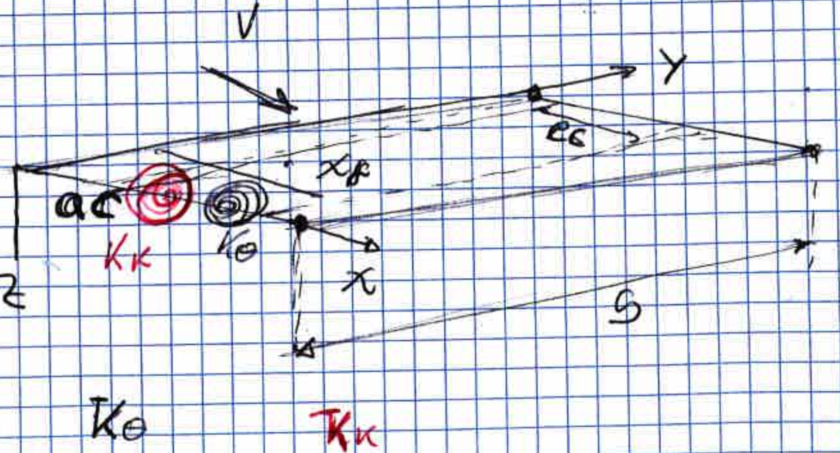
BINARY AEROELASTIC MODEL

Suppose, that we have a rectangular wing of span s and chord c is rigid but has two rotational springs at the root to provide flap (κ) and pitch (θ) degrees of freedom. The springs are attached at a distance ec behind aerodynamic center

The wing is assumed to heave a uniform mass distribution and thus the mass axis lies on the mid-chord.

The displacement z (downwards +ve) of a general point on the wing is

$$z(x,y,t) = y\kappa(t) + (x - x_f)\,\theta(t) = \phi_\kappa \cdot \kappa + \phi_\theta \cdot \theta$$



Where κ and θ are generalized coordinates and ϕ_κ and ϕ_θ are simple assumed shape.

Equation of motion can be found to be by using Lagrange's equation.

Kinetic energy $T = \frac{m}{2}\int_0^s \int_0^c (y\dot{\kappa} + (x - x_f)\dot{\theta})^2 \, dx\, dy$

The potential or strain energy $U = \frac{1}{2} K_\kappa \kappa^2 + \frac{1}{2} K_\theta \cdot \theta^2$

Whereas for general bending and torsional vibrations of a flexible wing it would take the form

$$U = \frac{1}{2}\int EI\left(\frac{d^2 z}{dy^2}\right)^2 dy + \frac{1}{2}\int GJ\left(\frac{d\theta}{dy}\right)^2 dy$$

Then we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \kappa}\right) = m\left[\frac{s^3}{3} \cdot c\, \ddot{\kappa} + \frac{s^2}{2}\left(\frac{c^2}{2} - x_f \cdot c\right)\ddot{\theta}\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \theta}\right) = m\left[\frac{s^2}{2}\left(\frac{c^2}{2} - x_f \cdot c\right)\ddot{\kappa} + s\left(\frac{c^3}{3} - c^2 x_f + x_f^2 c\right)\ddot{\theta}\right]$$

and

$$\frac{\partial U}{\partial \kappa} = K_\kappa \cdot \kappa \quad \text{and} \quad \frac{\partial U}{\partial \theta} = K_\theta \cdot \theta.$$

We obtain the equation of motion for the wing, without aerodynamic forces acting.

$$\begin{bmatrix} m s^3 c & \frac{m}{2} s^2 \left[\frac{c^2}{2} - c x_f\right] \\ \frac{m}{2} s^2 \left[\frac{c^2}{2} - c x_f\right] & m s \left[\frac{c^3}{3} - c^2 x_f + c x_f^2\right] \end{bmatrix} \cdot \begin{Bmatrix} \ddot{\kappa} \\ \ddot{\theta} \end{Bmatrix} +$$

$$+ \begin{bmatrix} K_\kappa & 0 \\ 0 & K_\theta \end{bmatrix} \begin{Bmatrix} \kappa \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The inertia matrix takes the form

$$\begin{bmatrix} I_\kappa & I_{\kappa\theta} \\ I_{\kappa\theta} & I_\theta \end{bmatrix}$$

$$I_\kappa = \int_0^s y^2 dm \qquad I_\theta = \int_0^c (x - x_f)^2 dm$$

$$I_{\kappa\theta} = \int_0^s \int_0^c (x - x_f)\, y \, dm.$$

If there is no inertia coupling $I_{\kappa\theta} = 0$

$x_f = \frac{c}{2}$ then the flap and pitch natural frequencies are

$$\omega_\kappa = \sqrt{\frac{K_\kappa}{I_\kappa}} \qquad \omega_\theta = \sqrt{\frac{K_\theta}{I_\theta}}$$

However, the presence of a nonzero value of $I_{\kappa\theta}$ couples the two motions in the mode shapes and the natural frequencies differ.

Generalized forces Q_K and Q_θ act on the systems in the form of unsteady aerodynamic force. for an oscillatory motion they may be written in terms of the aerodynamic derivatives for a particular reduced frequency $K = \frac{\omega c}{2V}$,

These forces are complex ~~and~~ can be expressed in terms of displacements and velocities.

Applying strip theory, together with the simplified unsteady aerodynamics representation leads to expressions for lift and pitching moment (about the flexural axis) for each elemental strip

$$dL = \frac{1}{2} \rho V^2 c\, dy\, a_w \left(\frac{y \dot{K}}{V} + \theta \right)$$

$$dM = \frac{1}{2} \rho V^2 c^2 dy \left(e a_w \left(\frac{y \dot{K}}{V} + \theta \right) + M_{\dot\theta} \frac{\dot\theta c}{4V} \right)$$

The incremental work done

$$\delta w = \int_{wing} \left[dL(-y\,\delta K) + dM\,\delta\theta \right]$$

Generalized forces are

$$Q_K = -\frac{1}{2} \rho V^2 c s^2 a_w \left(\frac{\dot{K} s}{3V} + \frac{\theta}{2} \right)$$

$$Q_\theta = \frac{1}{2} \rho V^2 c^2 s \left[e a_w \left(\frac{\dot{K} s}{2V} + \theta \right) + M_{\dot\theta} \frac{\dot\theta c}{4V} \right]$$

Thus, the full aeroelastic equation of motion becomes

$$\begin{bmatrix} J_\kappa & J_{\kappa\theta} \\ J_{\kappa\theta} & J_\theta \end{bmatrix} \begin{Bmatrix} \ddot{\kappa} \\ \ddot{\theta} \end{Bmatrix} + \rho V \begin{bmatrix} c s^3 a_w & 0 \\ -\dfrac{e c^2 s^2 a_w}{4} & -\dfrac{c^3 s M_{\dot\theta}}{8} \end{bmatrix} \begin{Bmatrix} \dot{\kappa} \\ \dot{\theta} \end{Bmatrix}$$

$$+ \left[\rho V^2 \begin{bmatrix} 0 & c s^2 a_w \\ 0 & -\dfrac{e c^2 s a_w}{2} \end{bmatrix} + \begin{bmatrix} K_\kappa & 0 \\ 0 & K_\theta \end{bmatrix} \right] \begin{bmatrix} \kappa \\ \theta \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Types of flutter

- Binary + wing torsion - wing bending flutter
- Complex couplings between:
  + Wing - engine
  + Tailplane - fin
  + Wing - Tailplane - fuselage - fin
- Control surface flutter
  + Coupling of control surfaces with wing, tail
  + Tab coupled with control surface.

Aeroelastic Design

- After the general aircraft configuration
- Developing of an aeroelastic mathematical model
- The model is a combination of a structural model with an aerodynamic model
  "CFD - CSD method

- Wind tunnel testing
- Ground Vibration testing
- Flight flutter testing