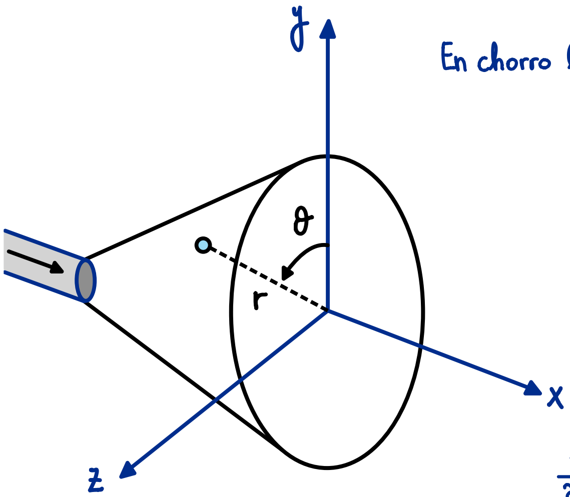
Chorro Axilsimétrico Laminar



En chorro lejano $\left(\frac{x}{R} >> 1\right)$: $\dfrac{\partial \blacksquare}{\partial \theta} = 0$

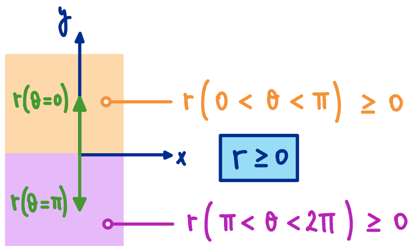
Continuidad : $\dfrac{\partial}{\partial x}(r\cdot u) + \dfrac{\partial}{\partial r}(r\cdot v) = 0 \quad (\mathrm{I})$

ECdM$_x$: $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(\nu r \dfrac{\partial u}{\partial r}\right) \quad (\mathrm{II})$

$$\Big\downarrow \; u\cdot(\mathrm{I}) + r\cdot(\mathrm{II})$$

$$\frac{\partial}{\partial x}(r\cdot u\cdot u) + \frac{\partial}{\partial r}(r\cdot u\cdot v) = \frac{\partial}{\partial r}\left(\nu r\frac{\partial u}{\partial r}\right)$$

FORMA CONSERVATIVA
(podemos integrar transversalmente)



Integramos transversalmente :

$$\frac{d}{dx}\int_0^\infty r u^2 dr + \underbrace{r\cdot u\cdot v\Big|_0^\infty}_{0} = \underbrace{\nu r \frac{\partial u}{\partial r}\Big|_0^\infty}_{0} \longrightarrow \boxed{\int_0^\infty u^2 r\,dr = I\,(\text{cte } \forall x)}$$

$r=0 \quad v=0$ $\qquad$ $r=0 \quad \dfrac{\partial u}{\partial r}=0$

$r\to\infty \begin{cases} u\to 0 \\ v\to 0\end{cases}$ $\qquad$ $r\to\infty \quad \dfrac{\partial u}{\partial r}\to 0$

Eliminamos $\nu \longrightarrow x, \dfrac{r}{\sqrt{\nu}}, u, \dfrac{v}{\sqrt{\nu}}$

Reescribiendo, el problema queda :

$$\frac{\partial}{\partial x}\left(\frac{r}{\sqrt{\nu}}u\right) + \frac{\partial}{\partial\left(\frac{r}{\sqrt{\nu}}\right)}\left(\frac{r}{\sqrt{\nu}}\frac{v}{\sqrt{\nu}}\right) = 0$$

$$\frac{r}{\sqrt{\nu}} = 0 : \frac{v}{\sqrt{\nu}} = 0 \; ; \; \frac{\partial u}{\partial\left(\frac{r}{\sqrt{\nu}}\right)} = 0$$

Introducimos la función de corriente ψ:

$$ru = \underbrace{\frac{\partial \psi}{\partial r}} \longrightarrow u = \frac{1}{r}\frac{\partial \psi}{\partial r} \longrightarrow u = \frac{1}{\frac{r}{\sqrt{\nu}}}\frac{\partial\left(\frac{\psi}{\nu}\right)}{\partial\left(\frac{r}{\sqrt{\nu}}\right)}$$

SE ADIMENSIONALIZA CON ν Y NO CON $\sqrt{\nu}$
COMO SE HACÍA EN CHORRO PLANO

$$r v = -\frac{\partial \psi}{\partial x} \longrightarrow v = -\frac{1}{r}\frac{\partial \psi}{\partial x} \longrightarrow \frac{v}{\sqrt{\nu}} = -\frac{1}{\frac{r}{\sqrt{\nu}}}\frac{\partial\left(\frac{\psi}{\nu}\right)}{\partial x}$$

$$\longrightarrow \frac{\psi}{\nu} = F\left(x, \frac{r}{\sqrt{\nu}}, \frac{I}{\nu}\right)$$

DUPLA DE ADIMENSIONALIZACIÓN

Ecuaciones de dimensiones:

$$[x] = [r] = L$$

$$\left[\frac{r}{\sqrt{\nu}}\right] = \frac{L}{(L^2T^{-1})^{1/2}} = T^{1/2}$$

$$\left[\frac{\psi}{\nu}\right] = \frac{L^3T^{-1}}{L^2T^{-1}} = L$$

$$\left[\frac{I}{\nu}\right] = \frac{L^2T^{-2}L^2}{L^2T^{-1}} = L^2T^{-1}$$

Adimensionalizamos $\frac{\psi}{\nu}$ y $\frac{r}{\nu}$:

$$\frac{\frac{\psi}{\nu}}{x^\alpha\left(\frac{I}{\nu}\right)^\beta} \longrightarrow \frac{L}{L^\alpha L^{2\beta}T^{-\beta}} \longrightarrow \begin{cases} 1 = \alpha + 2\beta \\ -\beta = 0 \end{cases} \longrightarrow \begin{cases} \alpha = 1 \\ \beta = 0 \end{cases} \longrightarrow \frac{\psi}{\nu x}$$

$$\frac{\frac{r}{\sqrt{\nu}}}{x^\alpha\left(\frac{I}{\sqrt{\nu}}\right)^\beta} \longrightarrow \frac{T^{1/2}}{L^\alpha L^{2\beta}T^{-\beta}} \longrightarrow \begin{cases} 0 = \alpha + 2\beta \\ \frac{1}{2} = -\beta \end{cases} \longrightarrow \begin{cases} \alpha = 1 \\ \beta = -\frac{1}{2} \end{cases} \longrightarrow \frac{r\sqrt{I}}{\nu x}$$

Por tanto:

que la ECdM$_x$ debe pasar de ser EDP(x, r) a ser EDO(η).

$$\psi = \nu x\, f(\eta) \; ; \; \eta = \frac{r\sqrt{I}}{\nu x} \longrightarrow \frac{\partial \eta}{\partial x} = -\frac{\eta}{x} \; ; \; \frac{\partial \eta}{\partial r} = \frac{\sqrt{I}}{\nu x}$$

Sustituimos en la ECdMx : $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial r} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(\nu r \dfrac{\partial u}{\partial r}\right)$

$\boxed{u}$

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[\nu x f(\eta)\right] = \frac{1}{r}\nu x \underbrace{\frac{df}{d\eta}}_{f'} \frac{\partial \eta}{\partial r} = \frac{\nu x}{r} f' \frac{\sqrt{I}}{\nu x} = \frac{\sqrt{I}}{r} f' \longrightarrow \boxed{u = \frac{I}{\nu x}\frac{1}{\eta} f'}$$

$\boxed{v}$

$$v = -\frac{1}{r}\frac{\partial \psi}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial x}\left[\nu x f(\eta)\right] = -\frac{1}{r}\nu f - \frac{\nu x}{r} f' \frac{\partial \eta}{\partial x} = -\frac{\nu}{r} f + \frac{\nu x}{r} f' \frac{\eta}{x} = -\frac{\nu}{r}\left(f - \eta f'\right) =$$

$$= -\frac{\sqrt{I}}{\eta x}\left(\frac{f}{\eta} - f'\right) \longrightarrow \boxed{v = \frac{\sqrt{I}}{x}\left(f' - \frac{f}{\eta}\right)}$$

$\boxed{\dfrac{\partial u}{\partial x}}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{I}{\nu x}\frac{1}{\eta} f'\right) = \cancel{-\frac{I}{\nu x^2}\frac{1}{\eta} f'} + \cancel{\frac{I}{\nu x}\frac{1}{\eta^2}\frac{\eta}{x} f'} - \frac{I}{\nu x}\frac{1}{\eta} f''\frac{\eta}{x} \longrightarrow \boxed{\frac{\partial u}{\partial x} = -\frac{I}{\nu x^2} f''}$$

$\boxed{\dfrac{\partial u}{\partial r}}$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(\frac{I}{\nu x}\frac{1}{\eta} f'\right) = -\frac{I}{\nu x}\frac{1}{\eta^2}\frac{\sqrt{I}}{\nu x} f' + \frac{I}{\nu x}\frac{1}{\eta} f''\frac{\sqrt{I}}{\nu x} \longrightarrow \boxed{\frac{\partial u}{\partial r} = \frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left(\eta f'' - f'\right)}$$

$\boxed{\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)}$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} = \frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left(\eta f'' - f'\right) + \frac{\eta \nu x}{\sqrt{I}}\left\{ -\frac{I\sqrt{I}}{(\nu x)^2}\frac{2}{\eta^3}\frac{\sqrt{I}}{\nu x}\left(\eta f'' - f'\right) + \right.$$

Sustituyendo :

$$\frac{\nu}{r} = \frac{\sqrt{I}}{\eta x}$$

$$-\frac{I}{\sqrt{x}}\frac{1}{\eta} f' \frac{I}{\nu x^2} f'' + \frac{\sqrt{I}}{x}\frac{1}{\eta}\left(f' - \frac{f}{\eta}\right)\frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left(\eta f'' - f'\right) = \frac{\sqrt{I}}{\nu x}\frac{I\sqrt{I}}{\eta^3(\nu x)^2}\left(\eta f' - \eta^2 f'' + \eta^3 f'''\right)$$

$$-\cancel{\frac{I^2}{\eta\nu^2x^3}} f'f'' + \cancel{\frac{I^2}{\eta\nu^2x^3}} f'f'' - \frac{I^2}{\eta\nu^2x^3}(f')^2 - \frac{I^2}{\eta\nu^2x^3} ff'' + \frac{I^2}{\eta\nu^2x^3} ff' = \frac{I^2}{\eta\nu^2x^3} f' - \frac{I^2}{\eta\nu^2x^3} f'' + \frac{I^2}{\eta\nu^2x^3} f'''$$

$$-\eta(f')^2 - \eta f'f'' + ff' = f' - \eta f'' + \eta^2 f'''$$

$$\boxed{\frac{1}{\eta^2}\left[ff' - \eta(f')^2 - \eta f'f''\right] = \left(f'' - \frac{1}{\eta}f'\right)'}$$

Condiciones de contorno :

$$\eta = 0 : \; f = 0 \; ; \; f' = 0$$

$$\cancel{\eta \to \infty : \; f' \to 0}$$

Incluida en esta

$$\int_0^\infty \frac{1}{\eta}(f')^2 d\eta = 1$$

Por tanto, el problema queda :

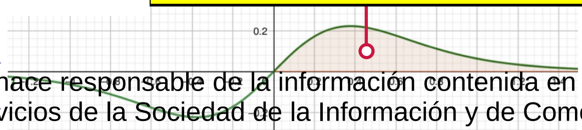
$$\boxed{\begin{gathered}\frac{1}{\eta^2}\left[ff' - \eta(f')^2 - \eta f'f''\right] = \left(f'' - \frac{1}{\eta}f'\right)' \\ \eta = 0 : \; f = 0 \; ; \; f' = 0 \\ \int_0^\infty \frac{1}{\eta}(f')^2 d\eta = 1\end{gathered}}$$

La solución al problema es :

$$f = \frac{4(\alpha\eta)^2}{1 + (\alpha\eta)^2} \qquad f' = \frac{8\alpha^2\eta\left[1 + (\alpha\eta)^2\right] - 4(\alpha\eta)^2 2\alpha^2\eta}{\left[1 + (\alpha\eta)^2\right]^2} = \frac{8\alpha^2\eta}{\left[1 + (\alpha\eta)^2\right]^2}$$

El valor de "α" se obtiene de aplicar la condición de contorno que falta :

$\to 0.17 = 1/6$



$$\alpha = \frac{1}{4}\sqrt{\frac{3}{2}}$$

Por tanto:

$$\boxed{f = \frac{12\,\eta^2}{32 + 3\eta^2} \qquad f' = \frac{768\,\eta}{\left(32 + 3\eta^2\right)^2}} \xrightarrow{u = \frac{I}{\sqrt{x}}\frac{1}{\eta} f'} \boxed{u = \frac{768\, I}{\sqrt{x}}\left(32 + 3\eta^2\right)^{-2}}$$

Velocidad en el centro del chorro: $\boxed{u_{máx} = u\left(\eta = 0\right) = \frac{3I}{4\sqrt{x}}}$

Flujo a través de una sección "x" del chorro:

$$Q = \int_{x=cte} \underbrace{\vec{v}\cdot\vec{n}}_{u}\, dA = 2\pi \int_0^{\infty} u r dr = 2\pi \int_0^{\infty} \frac{768\, I}{\sqrt{x}}\left(32 + 3\eta^2\right)^{-2} \frac{\sqrt{x}\,\eta}{\sqrt{I}}\, d\left(\frac{\sqrt{x}\,\eta}{\sqrt{I}}\right) = 1536\, \pi \sqrt{x} \int_0^{\infty} \frac{\eta\, d\eta}{\left(32 + 3\eta^2\right)^2} =$$

$$= \frac{3}{2}\pi\sqrt{x} \int_0^{\infty} \frac{\eta\, d\eta}{\left(1 + \frac{3}{2}\eta^2\right)^2} = \frac{3}{2}\pi\sqrt{x}\,\frac{32}{3} \underbrace{\int_0^{\infty} \frac{\sqrt{\frac{3}{32}}\,\eta\, d\left(\sqrt{\frac{3}{32}}\,\eta\right)}{\left[1 + \left(\sqrt{\frac{3}{32}}\,\eta\right)^2\right]^2}}_{\frac{1}{2}} \longrightarrow \boxed{Q = 8\pi\sqrt{x}}$$