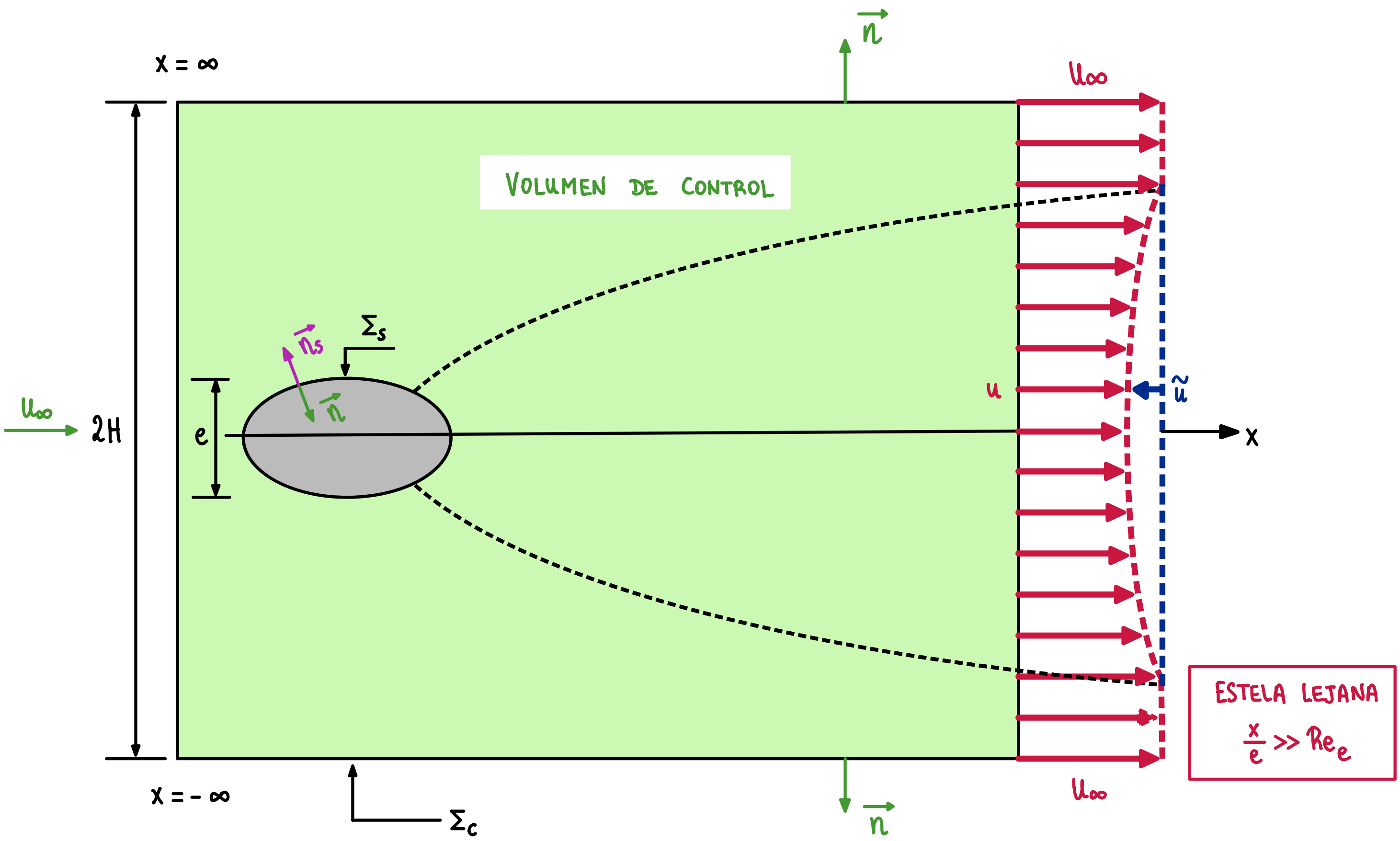
Estela Axilsimétrica Laminar



$$\boxed{u(x,r) = U_\infty - \tilde u(x,r)}$$

Estela lejana: $Re_e = \frac{U_\infty e}{\nu}$

$$\boxed{\frac{U_\infty - u}{U_\infty} = \frac{\tilde u}{U_\infty} \ll 1 \quad \left(\frac{x}{e} \gg Re_e\right)}$$

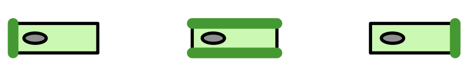
$$\frac{\partial \tilde u}{\partial \blacksquare} = -\frac{\partial u}{\partial \blacksquare} \longrightarrow$$

Ecuación de continuidad: $\underbrace{\frac{\partial}{\partial x}(r\tilde u)}_{\sim \frac{\tilde u_c}{x}\delta} - \underbrace{\frac{\partial}{\partial r}(r v)}_{\sim v_c} = 0 \longrightarrow \boxed{\frac{v_c}{\tilde u_c} \sim \frac{\delta}{x} \ll 1}$

$$\boxed{v_c \ll \tilde u_c \ll U_\infty}$$

Ecuación de continuidad aplicada al volumen de control:

$$\int_{\Sigma_c} \vec v\, \vec n\, dA = 0 \longrightarrow -U_\infty \frac{\pi (2H)^2}{4} + 2\int_{-\infty}^{x} v \pi H\, dx + 2\int_{0}^{H} u \pi r\, dr = 0$$



ECdM$_x$ aplicada al volumen de control:

$$\int_{\Sigma_c} \rho \vec v (\vec v\, \vec n)\, dA\, \vec i = \underbrace{-\int_{\Sigma_s} (p - p_\infty)\, \vec n\, dA\, \vec i + \int_{\Sigma_c} \overline{\overline{\tau}}_\mu \cdot \vec n\, dA\, \vec i}_{-D}$$

$$-U_\infty^2\,\frac{\pi(2H)^2}{4} + 2U_\infty\int_{-\infty}^{x} v\pi H\,dx + 2\int_0^H u^2\pi r\,dy = -\frac{D}{\rho}$$

Combinando ambas ecuaciones :

$$\left.\begin{aligned} -U_\infty\,\frac{\pi(2H)^2}{4} + 2\int_{-\infty}^{x} v\pi H\,dx + 2\int_0^H u\pi r\,dr &= 0 \\ -U_\infty^2\,\frac{\pi(2H)^2}{4} + 2U_\infty\int_{-\infty}^{x} v\pi H\,dx + 2\int_0^H u^2\pi r\,dr &= -\frac{D}{\rho} \end{aligned}\right\} \longrightarrow \int_0^H\left(u - \frac{u^2}{U_\infty}\right)\pi r\,dy = \frac{D}{2\rho U_\infty} \longrightarrow$$

$$\longrightarrow \int_0^H u\left(1 - \frac{u}{U_\infty}\right)\pi r\,dr = \frac{D}{2\rho U_\infty} \longrightarrow \int_0^H u\,(U_\infty - u)\,\pi r\,dr = \frac{D}{2\rho} \longrightarrow$$

$$\longrightarrow \boxed{\frac{D}{\rho} = 2\int_0^H u\,(U_\infty - u)\,\pi r\,dr = 2\int_0^H u\,\tilde{u}\,\pi r\,dr}$$

En el exterior de la estela $(U_\infty - u) \to 0$, de modo que no contribuye a la integral anterior $(H \to \infty)$:

$$\forall\, x \in \text{estela lejana}: \int_0^\infty u\,\tilde{u}\,\pi r\,dr = \int_0^\infty \left(U_\infty - \cancel{\tilde{u}}\right)\tilde{u}\,\pi r\,dr \simeq \int_0^\infty U_\infty\,\tilde{u}\,\pi r\,dr = \frac{D}{2\rho} \longrightarrow \boxed{\frac{D}{\rho U_\infty} = 2\int_0^\infty \tilde{u}\,\pi r\,dr}$$

Como consecuencia de esta disparidad de escalas podemos linealizar la ECdM_x :

$$\underbrace{u\,\frac{\partial \tilde{u}}{\partial x}}_{\sim U_\infty \frac{\tilde{u}_c}{x}} + \underbrace{\cancel{v\,\frac{\partial \tilde{u}}{\partial r}}}_{\sim v_c \frac{\tilde{u}_c}{\delta}} = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(\nu r\,\frac{\partial \tilde{u}}{\partial r}\right)}_{\sim \nu \frac{\tilde{u}_c}{\delta^2}}$$

$$v_c\,\frac{\tilde{u}_c}{\delta} \sim \frac{\tilde{u}_c^2}{x}$$

$$\sim \frac{\nu x}{U_\infty}\left(\frac{x}{\delta}\right)^2 \sim Re_x^{-1}\left(\frac{\delta}{x}\right)^{-2}$$

$$\sim \frac{\tilde{u}_c^2}{x}\,\frac{x}{U_\infty\,\tilde{u}_c} \sim \frac{\tilde{u}_c}{U_\infty} \ll 1$$

$$1$$

$$\boxed{\frac{\delta}{x} \sim Re_x^{-1/2} \ll 1}$$

La ecuación queda :

$$u\,\frac{\partial \tilde{u}}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu r\,\frac{\partial \tilde{u}}{\partial r}\right) \longrightarrow U_\infty\left(1 - \cancel{\frac{\tilde{u}}{U_\infty}}\right)\frac{\partial \tilde{u}}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu r\,\frac{\partial \tilde{u}}{\partial r}\right) \longrightarrow \boxed{U_\infty\,\frac{\partial \tilde{u}}{\partial x} \approx \frac{1}{r}\frac{\partial}{\partial r}\left(\nu r\,\frac{\partial \tilde{u}}{\partial r}\right)}$$

EDP LINEAL

En cuanto a las condiciones de contorno:

- Simetría: $\begin{cases} \tilde{u}(r) = \tilde{u}(-r) \\ \left.\dfrac{\partial \tilde{u}}{\partial r}\right|_{r=0} = \lim\limits_{\Delta r \to 0} \dfrac{\tilde{u}(\Delta r/2) - \tilde{u}(-\Delta r/2)}{\Delta r} = 0 \longrightarrow r=0: \dfrac{\partial \tilde{u}}{\partial r} = 0 \end{cases}$

- En $r \to \infty$ la estela no ejerce ninguna influencia: $r \to \infty : u \to U_\infty \longrightarrow r \to \infty : \tilde{u} \to 0$

- Además sabemos que $\forall x \in$ estela lejana: $\dfrac{D}{2\rho U_\infty} = \displaystyle\int_0^\infty \tilde{u}\pi r\,dr$

El problema queda:

$$\boxed{\begin{aligned} & U_\infty \frac{\partial \tilde{u}}{\partial x} \approx \frac{\nu}{r}\frac{\partial}{\partial r}\left(\nu r \frac{\partial \tilde{u}}{\partial r}\right) \\ & r=0: \frac{\partial \tilde{u}}{\partial r} = 0 \\ & r \to \infty : \tilde{u} \to 0 \\ & \forall x : \frac{D}{2\rho U_\infty} = \int_0^\infty \tilde{u}\pi r\,dr \end{aligned}} \xrightarrow[x,\ \sqrt{\frac{U_\infty}{\nu}}r,\ \tilde{u}]{\text{Eliminamos } \nu;\ U_\infty}$$

$$\boxed{\begin{aligned} & \frac{\partial \tilde{u}}{\partial x} \approx \frac{\nu}{\sqrt{\frac{U_\infty}{\nu}}r}\frac{\partial}{\partial\left(\sqrt{\frac{U_\infty}{\nu}}r\right)}\left[\sqrt{\frac{U_\infty}{\nu}}r\frac{\partial \tilde{u}}{\partial\left(\sqrt{\frac{U_\infty}{\nu}}r\right)}\right] \\ & \sqrt{\frac{U_\infty}{\nu}}r = 0 : \frac{\partial \tilde{u}}{\partial\left(\sqrt{\frac{U_\infty}{\nu}}r\right)} = 0 \\ & \sqrt{\frac{U_\infty}{\nu}}r \to \infty : \tilde{u} \to 0 \\ & \forall x : \frac{D}{2\rho\nu} = I = \int_0^\infty \tilde{u}\pi\sqrt{\frac{U_\infty}{\nu}}r\,d\left(\sqrt{\frac{U_\infty}{\nu}}r\right) \end{aligned}}$$

$$\downarrow$$

$$\tilde{u} = \tilde{u}\left(x, \sqrt{\frac{U_\infty}{\nu}}r, I\right)$$

Ecuaciones de dimensiones:

$[x] = L$

$\left[\sqrt{\frac{U_\infty}{\nu}}r\right] = \left(\frac{LT^{-1}}{L^2T^{-1}}\right)^{1/2} L = L^{1/2}$

$[\tilde{u}] = LT^{-1}$

$[I] = \left[\frac{D}{2\rho\nu}\right] = \left[\frac{U_\infty^2 r^2}{\nu}\right] = L^2T^{-1}$

2 magnitudes dimensionalmente independientes

$\boxed{x, I}$

Adimensionalizamos :

$$\tilde{u} \longrightarrow \frac{\tilde{u}}{x^{\alpha} I^{\beta}} \longrightarrow \frac{LT^{-1}}{L^{\alpha} L^{2\beta} T^{-\beta}} \longrightarrow \begin{cases} \beta = 1 \\ 1 = \alpha + 2\beta \end{cases} \longrightarrow \begin{cases} \alpha = -1 \\ \beta = 1 \end{cases} \longrightarrow \frac{\tilde{u}\, x}{I}$$

$\sqrt{\frac{U_\infty}{\nu}}\, r \longrightarrow \left[\sqrt{\frac{U_\infty}{\nu}}\, r\right] = L^{1/2} \longrightarrow$ Se adimensionaliza con $x^{1/2}$ $\longrightarrow \sqrt{\frac{U_\infty}{\nu x}}\, r = \eta$

Entonces :

$$\boxed{\frac{\tilde{u}\, x}{I} = f(\eta)\ , \text{ con } \eta = \sqrt{\frac{U_\infty}{\nu x}}\, r}$$

$\longrightarrow \tilde{u} = \frac{I}{x} f(\eta)$

$\longrightarrow \boxed{\frac{\partial \eta}{\partial x} = -\frac{\eta}{2x}} \quad \boxed{\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}}$

Si recordamos, la ECDM$_x$ quedaba reducida a : $U_\infty \frac{\partial \tilde{u}}{\partial x} \approx \frac{\nu}{r}\frac{\partial}{\partial r}\left(\nu r \frac{\partial \tilde{u}}{\partial r}\right)$

$\boxed{\frac{\partial \tilde{u}}{\partial x}}$

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{I}{x} f(\eta)\right] = -\frac{I}{x^2} f - \frac{I}{x} f' \frac{\eta}{2x} \longrightarrow \boxed{\frac{\partial \tilde{u}}{\partial x} = -\frac{I}{x^2}\left(f + \frac{1}{2}\eta f'\right)}$$

$\boxed{\frac{\partial \tilde{u}}{\partial r}}$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial}{\partial r}\left[\frac{I}{x} f(\eta)\right] = \frac{I}{x} f' \sqrt{\frac{U_\infty}{\nu x}} = \frac{I}{x}\sqrt{\frac{U_\infty}{\nu}}\, f'$$

$\boxed{\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{u}}{\partial r}\right)}$

$$\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{u}}{\partial r}\right) = \frac{\partial \tilde{u}}{\partial r} + r\frac{\partial^2 \tilde{u}}{\partial r^2} = \frac{I}{x}\sqrt{\frac{U_\infty}{\nu x}}\, f' + \cancel{\sqrt{\frac{\nu x}{U_\infty}}}\, \eta\, \frac{I}{x}\cancel{\sqrt{\frac{U_\infty}{\nu x}}}\, f'' \sqrt{\frac{U_\infty}{\nu x}} \longrightarrow$$

$$\longrightarrow \boxed{\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{u}}{\partial r}\right) = \frac{I}{x}\sqrt{\frac{U_\infty}{\nu x}}\left(f' + \eta f''\right)}$$

Sustituyendo:

$$-\frac{I}{x^2}\left(f+\frac{1}{2}\eta f'\right) \approx \frac{\nu}{r}\frac{I}{x}\sqrt{\frac{U_\infty}{\nu x}}\left(f'+\eta f''\right)$$

$$-\frac{I}{x^2}\left(f+\frac{1}{2}\eta f'\right) \approx \cancel{\nu}\sqrt{\frac{U_\infty}{\cancel{\nu} x}}\frac{1}{\eta}\frac{I}{x}\sqrt{\frac{U_\infty}{\cancel{\nu} x}}\left(f'+\eta f''\right)$$

$$-\frac{U_\infty I}{x^2}\eta\left(f+\frac{1}{2}\eta f'\right) \approx \frac{U_\infty I}{x^2}\left(f'+\eta f''\right)$$

$$-\eta f-\frac{\eta^2}{2}f' \approx f'+\eta f''$$

$$\eta f''+\left(1+\frac{\eta^2}{2}\right)f'+\eta f \approx 0$$

$$\underbrace{f''+\frac{1}{\eta}f'}_{\frac{1}{\eta}(\eta f)'}+\frac{\eta}{2}f'+f \approx 0$$

$$\boxed{\frac{1}{\eta}(\eta f)'+\frac{\eta}{2}f'+f \approx 0}$$

En cuanto a las condiciones de contorno:

$$r=0:\ \frac{\partial\tilde{u}}{\partial r}=0 \longrightarrow \eta=0:\ f'=0 \quad (1)$$

$$r\to\infty:\ \tilde{u}\to 0 \longrightarrow \eta\to\infty:\ f=0 \quad (2)$$

Si necesitamos que $\int_0^\infty f\,d\eta$ esté acotada, se debe satisfacer $\eta\to\infty:\ f\to 0$, colapsando las condiciones (2) y (3)

$$\forall x:\ \frac{D}{2\rho U_\infty}=\int_0^\infty \tilde{u}\pi r\,dr \longrightarrow I\frac{\nu}{U_\infty}=\int_0^\infty \frac{I}{x}f(\eta)\pi\sqrt{\frac{\nu x}{U_\infty}}\eta\, d\left(\sqrt{\frac{\nu x}{U_\infty}}\eta\right) \longrightarrow \boxed{\int_0^\infty \eta f(\eta)\,d\eta=\frac{1}{\pi}}$$

(3)

DESHACE LA HOMOGENEIDAD "D" RESPONSABLE DE LA APARICIÓN DE LA ESTELA

Por tanto, el problema queda:

$$\boxed{\frac{1}{\eta}(\eta f)'+\frac{\eta}{2}f'+f \approx 0}$$

$$\boxed{\begin{array}{l}\eta=0:\ f'=0\\ \forall x:\ \int_0^\infty \eta f(\eta)\,d\eta=\frac{1}{\pi}\end{array}}$$

Integración:

$$\frac{1}{\eta}\left(\eta f' + \frac{\eta^2}{2} f\right)' \approx 0$$

$$\Big\downarrow \int$$

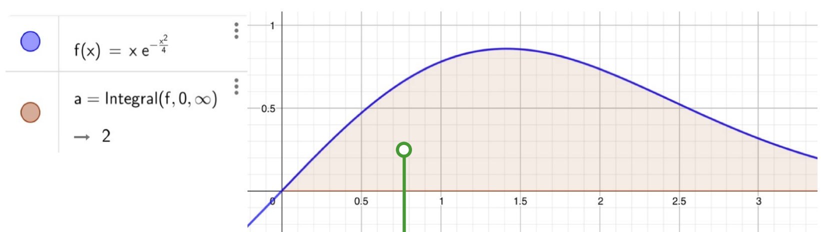
$$\eta f' + \frac{\eta^2}{2} f \approx C_1 \xrightarrow{\ \eta = 0\ :\ f' = 0\ } C_1 \approx 0$$

$$\Big\downarrow$$

$$\eta f' + \frac{\eta^2}{2} f \approx 0$$

$$f' + \frac{\eta}{2} f \approx 0$$

$$\Big\downarrow \int$$



$$\int_0^\infty \eta \exp\left(-\frac{\eta^2}{4}\right) d\eta = 2$$

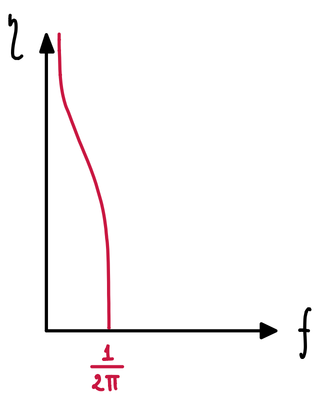
$$\int \frac{f'}{f}\, df \approx -\frac{1}{2}\int \eta\, d\eta$$

$$\ln f + K_1 \approx -\frac{\eta^2}{4} + K_2$$

$$f \approx C_2 \exp\left(-\frac{\eta^2}{4}\right) \xrightarrow{\ \int_0^\infty \eta f\, d\eta = \frac{1}{\pi}\ } C_2 \approx \frac{1/\pi}{\underbrace{\boxed{\int_0^\infty \eta \exp\left(-\frac{\eta^2}{4}\right) d\eta}}_{=2}} = \frac{1}{2\pi}$$

$$\Big\downarrow$$

$$\boxed{f(\eta) \approx \frac{1}{2\pi} \exp\left(-\frac{\eta^2}{4}\right)}$$



En cuanto a "u":

$$u = U_\infty - \tilde{u} = U_\infty - \frac{D}{2\rho\nu x}\frac{1}{2\pi}\exp\left(-\frac{U_\infty}{4\nu x}r^2\right)$$

$$\boxed{u = U_\infty - \frac{\frac{D}{\rho}}{4\pi\nu x}\exp\left(-\frac{U_\infty}{4\nu x}r^2\right)}$$

Nota:

$$\boxed{\tilde{u}\Big|_{r=0} \sim x^{-1}}$$

El perfil de velocidades se va uniformizando al alejarnos del obstáculo, pero más rápidamente que en estela plana ($\sim x^{-1/2}$). Es decir, el efecto del obstáculo abarca una menor distancia en 3D.