Electrical Systems

Lecture 4: Alternating current (AC) circuits II

Arnau Dòria-Cerezo
arnau.doria@upc.edu

Dept. Electrical Engineering, and
Inst. of Industrial and Control Engineering

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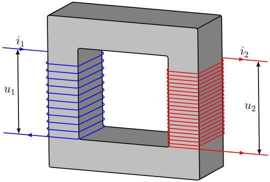
Outline

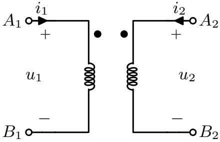
1. A two-port element: magnetic coupling
2. Analysis of mixed-frequency AC circuits
3. Electric resonance
4. Exercises and solutions

A two-port element: magnetic coupling

A magnetic coupling is a two-port element that relates the voltages and currents of ports $1, 2$ connected through a magnetic circuit.



Magnetic coupling



Circuit representation

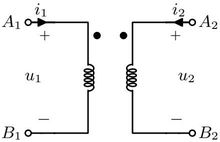
A two-port element: magnetic coupling

Using the Faraday's Law

$$u_j(t) = \frac{\mathrm{d}\Psi_j}{\mathrm{d}t}$$

and the relationships among the flux linkage, Ψ, number of coils, N_j, and magnetic flux, ϕ_j, given by

$$\Psi_j(t) = N_j\phi_j(t), \quad \phi_j(t) = \phi(t) + \phi_{dj}(t),$$
$$\phi_{dj}(t) = \frac{N_j}{\mathcal{R}_{dj}} i_j(t), \quad \phi(t) = \frac{N_1 i_1(t) + N_2 i_2(t)}{\mathcal{R}}$$



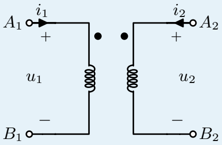
Magnetic coupling equations

$$u_1(t) = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + M\frac{\mathrm{d}i_2}{\mathrm{d}t}$$
$$u_2(t) = M\frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t}$$

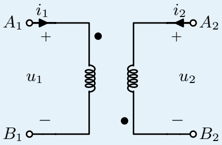
where $L_j = \frac{N_j^2}{\mathcal{R}} + \frac{N_j^2}{\mathcal{R}_{dj}}$ and $M = \frac{N_1 N_2}{\mathcal{R}}$.

A two-port element: magnetic coupling

Dots in the circuit representation determine how the magnetic flux is related with the currents (wiring direction of coils)



Magnetic fluxes with the same directions



Magnetic fluxes with opposite directions

$$u_1(t) = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t}$$
$$u_2(t) = M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

$$u_1(t) = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} - M \frac{\mathrm{d}i_2}{\mathrm{d}t}$$
$$u_2(t) = -M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

A two-port element: magnetic coupling

From the instantaneous power definition, $p(t) = u(t)i(t)$, we get

$$\begin{aligned} p(t) &= p_1(t) + p_2(t) = \\ &= \left(L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t} \right) i_1(t) + \left(M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \right) i_2(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2 \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} w(t) \end{aligned}$$

where $w(t)$ is the stored energy in the magnetic coupling

$$w(t) = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2.$$

The physical constraint $w(t) > 0$ implies that

$$L_1 L_2 - M^2 \geq 0.$$

The case

$$L_1 L_2 = M^2$$

is known as a perfect coupling.

A two-port element: magnetic coupling

In steady-state, a magnetic coupling can be written with phasors as

$$\underline{U}_1 = j\omega L_1 \underline{I}_1 + j\omega M \underline{I}_2$$
$$\underline{U}_2 = j\omega M \underline{I}_1 + j\omega L_2 \underline{I}_2$$

In a matrix form,

$$\begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$$

or, using the reactance definitions

Magnetic coupling equation

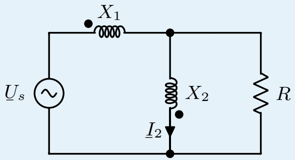
$$\begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \end{bmatrix} = \begin{bmatrix} jX_1 & jX_M \\ jX_M & jX_2 \end{bmatrix} \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$$

A two-port element: magnetic coupling

Exercise 1

Calculate the current flowing through inductor X_2, $\underline{I}_2$, for the following cases:

1. *No magnetic coupling ($X_M = 0$)*
2. *Magnetic coupling ($X_1X_2 - X_M^2 \geq 0$) and short-circuit ($R = 0$)*
3. *Perfect coupling ($X_M^2 = X_1X_2$)*
4. *Perfect coupling and $R = 0$*
5. *General case*



Outline

Analysis of mixed-frequency AC circuits

Linearity

An electrical circuit is linear if satisfies both additivity and homogeneity properties:

- Additivity: $f(x+y) = f(x) + f(y)$.
- Homogeneity: $f(kx) = kf(x)$, for all k.

An electrical circuit linear if only if is only composed by linear elements. Its analysis results in a set of linear equations.

Superposition principle

From the additivity property of a circuit the superposition principle can be derived.

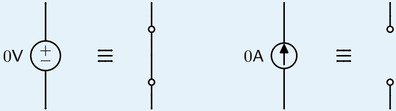
The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltage across (or currents through) that element due to each independent source acting alone.

The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately.

Analysis of mixed-frequency AC circuits

Analysis of a circuit using the superposition principle. Steps:

1. Turn off all independent sources except one source (independent voltage sources are replaced by short circuit and independent current sources are replaced by open circuit).



2. Find the circuit voltages or currents due to that active source using nodal or mesh analysis.
3. Repeat step 1 for each of the other independent sources.
4. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Analysis of mixed-frequency AC circuits

The analysis of mixed-frequency AC circuits can be done by using the Superposition Principle.

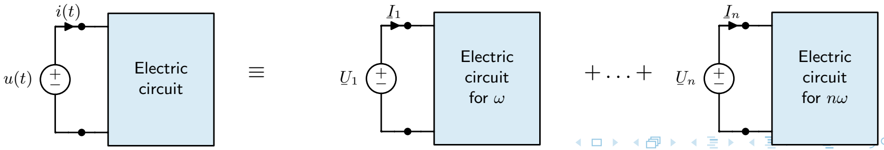
For example, consider a voltage source with harmonic content

$$u(t) = \sqrt{2}\sum_{k=1}^{n} U_k \cos(k\omega t + \phi_k),$$

connected to a circuit with linear elements. The current flowing through the source, $i(t)$, can be obtained as

$$i(t) = \sum_{k=1}^{n} i_k(t),$$

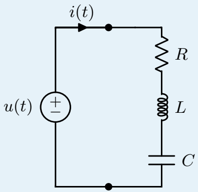
where $i_k(t)$ is the current due to the k-th term in $u(t)$.



Analysis of mixed-frequency AC circuits

Exercise 2

Consider the circuit below, where $R = 1\Omega$, $L = 1/2H$, $C = 2/3F$ and the voltage source contain a third harmonic, and takes the form $u(t) = 100cos(t) + 100cos(3t)$. Calculate the current absorbed by the RLC load, $i(t)$.

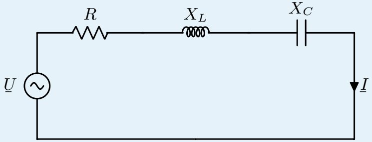


Outline

Electric resonance

Exercise 3

Motivating example: *Consider a series RLC circuit with $R = 2\Omega$, $L = 150mH$ and a 100V, 50Hz, AC voltage source. Calculate the capacitance such that the imaginary part of the series impedance is zero.*



Electric resonance

Electric resonance

In a series impedance, resonance occurs when, at a particular resonant frequency, the capacitive and inductive reactances of circuit elements cancel each other, i.e.,

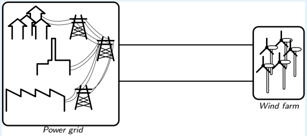
$$\mathrm{Im}\{\underline{Z}(R, L, C, \omega)\} = 0.$$

Consequences of series resonance

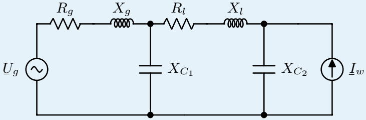
- In series resonant circuits, input voltage and currents are in phase
- Resonant circuits can generate very high voltages
- In particular, if $R = 0 \to I = \infty$, the impedance behaves as a short-circuit
- The frequency $\omega = \frac{1}{\sqrt{LC}}$, is known as the resonance frequency.

Electric resonance

Exercise 4



The circuit below represents the behaviour of a transmission line connecting a wind farm to a power grid.



Using $R_g = 0.04\Omega$, $X_g = 0.3\Omega$, $R_l = 0.8\Omega$, $X_l = 4\Omega$ *and* $X_C = 770\Omega$ *at* 50*Hz, find:*

- *the voltage across capacitor* C_2 *(parametrized by* $\underline{U}_g$ *and* $\underline{I}_w$*), and*
- *the resonance frequencies of the circuit.*

Electric resonance

The voltage across the capacitor C_2 is

$$\underline{U}_{C_2} = -jX_{C_2}\underline{Y}_A\underline{U}_g + \underline{Z}_B\underline{I}_w$$

where

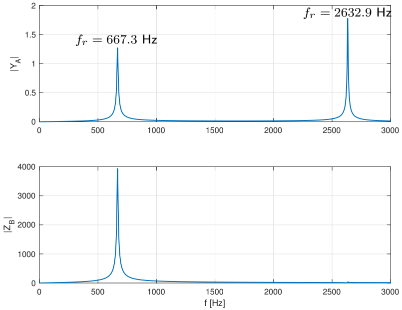
$$\underline{Y}_A = \frac{X_{C_1}}{\underline{Z}_g(X_{C_1}+X_{C_2}) + \underline{Z}_l X_{C_1} + j(\underline{Z}_l\underline{Z}_g - X_{C_1}X_{C_2})}$$

$$\underline{Z}_B = \frac{\underline{Z}_g\underline{Z}_l X_{C_2} + j(\underline{Z}_l+\underline{Z}_g)X_{C_1}X_{C_2}}{\underline{Z}_g(X_{C_1}+X_{C_2}) + \underline{Z}_l X_{C_1} + j(\underline{Z}_g\underline{Z}_l - X_{C_1}X_{C_2})}$$

Resonance occurs at ω_r when $\text{Im}\{\underline{Y}_A(\omega_r)\} = 0$ or $\text{Im}\{\underline{Z}_B(\omega_r)\} = 0$. Replacing $\underline{Z}_l = R_l + j\omega L_l$, $\underline{Z}_g = R_g + j\omega L_g$, $X_{C_1} = \frac{1}{\omega C_1}$ and $X_{C_2} = \frac{1}{\omega C_2}$, where L_l, L_g, C_1 and C_2 are obtained from the reactances at 50 Hz, we get

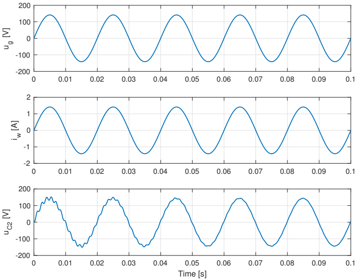
$$\omega_r = 4192.84 \text{ rad/s}$$

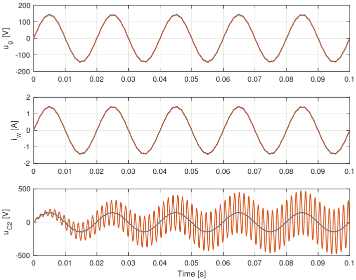
$$\omega_r = 16543.3 \text{ rad/s}$$



Electric resonance

Simulation results with $\underline{U}_g = 100$ V, $\underline{I}_w = 1$ A: (left) without higher frequencies, and (right) with additional noise/disturbance in the voltage and current, at the resonance frequency, $f_r = 667.3$ Hz, with 3% amplitude.





The resonance effect dramatically affect to the performance of the transmission line.

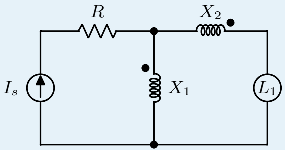
Outline

Exercises I

Exercise 5

[CM30] *Consider the circuit on the right. Find the measure of L_1:*

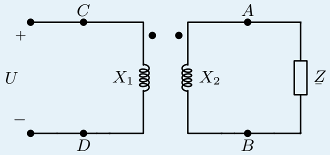
1. *if inductors X_1, X_2 are magnetically decoupled and L_1 is a voltmeter.*
2. *if inductors X_1, X_2 are magnetically coupled and L_1 is a voltmeter.*
3. *if inductors X_1, X_2 are magnetically coupled and L_1 is an ammeter.*



Exercises II

Exercise 6

[CM51] *Given the circuit below with a magnetic coupling the following parameters* $X_1 = 50\Omega$, $X_2 = 5\Omega$, *and* $\underline{Z} = 2 + j1\Omega$



Assuming perfect coupling, find:

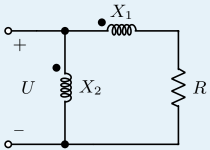
1. *The reactance of the capacitor that, connected between* A *and* B*, improves the power factor seen from* A, B *up to one.*
2. *The reactance of the capacitor that, connected between* C *and* D*, improves the power factor seen from* C, D *up to one.*

Exercises III

Exercise 7

[CM52] *The circuit on the right have the following parameters:* $U = 220V$, $R = 5\Omega$, $X_1 = 2\Omega$ *and* $X_2 = 3\Omega$. *Assuming that:*

1. X_1, X_2 *are not coupled, calculate the supplied current and the equivalent impedance seen from the voltage source.*
2. X_1, X_2 *are perfectly coupled, calculate the supplied current and the equivalent impedance seen from the voltage source.*
3. X_1, X_2 *are not coupled, calculate the value of the required capacitor that connected in parallel, ensures a unitary power factor.*
4. X_1, X_2 *are perfectly coupled, calculate the value of the required capacitor that connected in parallel, ensures a unitary power factor.*



Solutions I

Solution to Exercise 1

1. $\underline{I}_2 = \frac{R}{j\left((R+jX_2)X_1+RX_2\right)}\underline{U}_s$
2. $\underline{I}_2 = -\frac{jX_M}{X_1X_2-X_M^2}\underline{U}_s$
3. $\underline{I}_2 = \frac{R+jX_M}{jR(X_1+X_2-2X_M)}\underline{U}_s$
4. $\underline{I}_2 = 0$
5. $\underline{I}_2 = \frac{R+jX_M}{j\left((R+jX_2)(X_1-X_M)+(R+jX_M)(X_2-X_M)\right)}\underline{U}_s$

Solution to Exercise 2

$i(t) = 50\sqrt{2}\cos(t+45°) + 50\sqrt{2}\cos(3t-45°)$

Solutions II

Solution to Exercise 5

1. $L_1 = X_1 I_s$
2. $L_1 = (X_1 + X_m) I_s$
3. $L_1 = \frac{X_1 + X_m}{X_1 + X_2 + 2X_m} I_s$

Solution to Exercise 6

1. $X_C = 5\Omega$
2. $X_C = 25\Omega$

Solution to Exercise 7

1. $\underline{Z}_{\text{eq}} = 0.9 + j2.1\Omega$, $\underline{I} = 37.93 - j88.50\text{A}$
2. $\underline{Z}_{\text{eq}} = 0.06 + j2.99\Omega$, $\underline{I} = 1.48 - j73.33\text{A}$
3. $X_C = 2.486\Omega$
4. $X_C = 3\Omega$

Solutions III

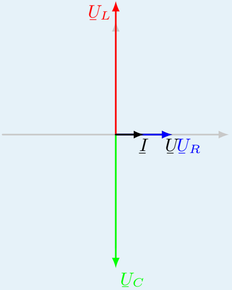
Solution to Exercise 3

$C = 67.55\mu\text{F}, \quad U_L = U_C = 2356.2\text{V}$

The voltage and current phasors are

$$\begin{aligned} \underline{I} &= 50\text{A} \\ \underline{U}_R &= 100\text{V} \\ \underline{U}_L &= 0 + j2356.19\text{V} \\ \underline{U}_C &= 0 - j2356.19\text{V} \end{aligned}$$

Phasor diagram (Schematic representation)



Solutions IV

Solution to Exercise 4

1. $\underline{U}_{C_2} = (1.006 - j0.0012)\underline{U}_g + (1.849 + j4.323)\underline{I}_w$
2. Resonance frequencies at:
   $\omega = 4192.84$ rad/s, $\omega = 16543.3$ rad/s

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