Electrical Systems

Lecture 8: Electrical power in three-phase circuits

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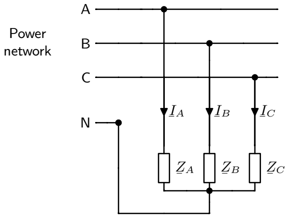
Outline

Instantaneous electrical power



Instantaneous three-phase electrical power

$$p(t) = p_A(t) + p_B(t) + p_C(t)$$

Given the three-phase load above, where

$$\begin{aligned} u_{AN}(t) &= U_{AN}\sqrt{2}\cos(\omega t + \phi_{Au}) \\ u_{BN}(t) &= U_{BN}\sqrt{2}\cos(\omega t + \phi_{Bu}) \\ u_{CN}(t) &= U_{CN}\sqrt{2}\cos(\omega t + \phi_{Cu}) \end{aligned} \qquad \begin{aligned} i_A(t) &= I_A\sqrt{2}\cos(\omega t + \phi_{Ai}) \\ i_B(t) &= I_B\sqrt{2}\cos(\omega t + \phi_{Bi}) \\ i_C(t) &= I_C\sqrt{2}\cos(\omega t + \phi_{Ci}) \end{aligned}$$

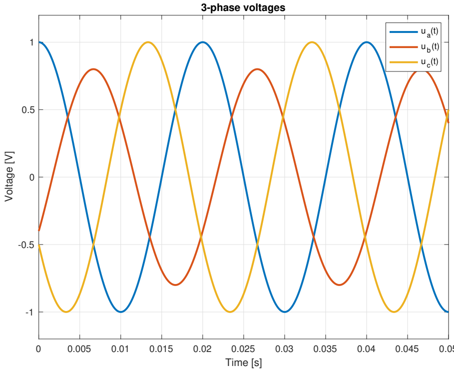
electrical power is

$$p(t) = u_{AN}(t)i_A(t) + u_{BN}(t)i_B(t) + u_{CN}(t)i_C(t)$$

Instantaneous electrical power

Numeric example, with $\underline{Z}_A = \underline{Z}_B = \underline{Z}_C = \frac{3}{2}\angle 30°\,\Omega$:

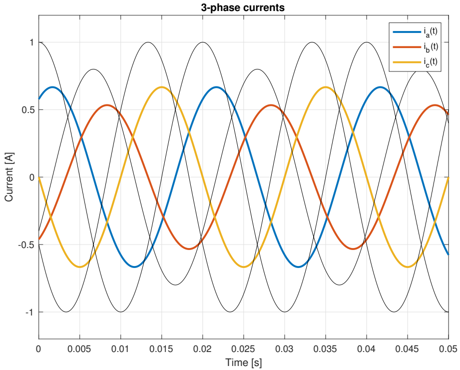
$$u_{AN}(t) = \cos(100\pi t)\,\text{V}$$
$$u_{BN}(t) = 0.8\cos\left(100\pi t - 2\pi/3\right)\text{V}$$
$$u_{CN}(t) = \cos\left(100\pi t + 2\pi/3\right)\text{V}$$



Instantaneous electrical power

Numeric example, with $\underline{Z}_A = \underline{Z}_B = \underline{Z}_C = \frac{3}{2}\angle 30^\circ \Omega$:

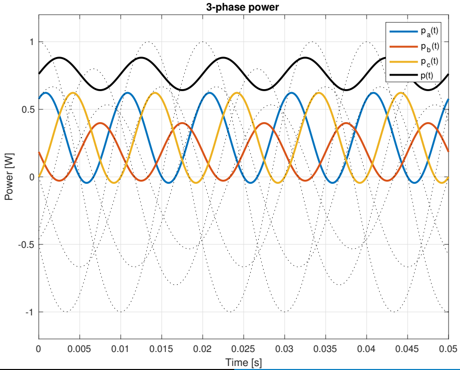
$$u_{AN}(t) = \cos(100\pi t)\text{V} \qquad i_A(t) = 2/3\cos(100\pi t - \pi/6)\text{A}$$
$$u_{BN}(t) = 0.8\cos(100\pi t - 2\pi/3)\text{ V} \qquad i_B(t) = 0.8 \cdot 2/3\cos(100\pi t - 2\pi/3 - \pi/6)\text{ A}$$
$$u_{CN}(t) = \cos(100\pi t + 2\pi/3)\text{ V} \qquad i_C(t) = 2/3\cos(100\pi t + 2\pi/3 - \pi/6)\text{ A}$$



Instantaneous electrical power

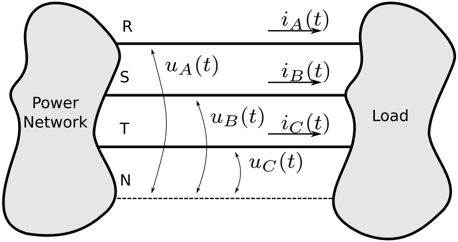
Numeric example, with $\underline{Z}_A = \underline{Z}_B = \underline{Z}_C = \frac{3}{2}\angle 30°\Omega$:

$$
\begin{aligned}
p(t) =& p_A(t) + p_B(t) + p_C(t) \\
=& 1/3(\cos(200\pi t) + 0.64\cos(200\pi t - 2\pi/3) + \cos(200\pi t + 2\pi/3)) + \sqrt{3}/3(1.32) \\
=& 0.762 - 0.12\cos(200\pi t - 2\pi/3)\text{W}
\end{aligned}
$$



Instantaneous electrical power

Consider now a balanced three-phase system



$$u_{AN}(t) = U_P\sqrt{2}\cos(\omega t) \qquad i_A(t) = I\sqrt{2}\cos(\omega t - \phi)$$

$$u_{BN}(t) = U_P\sqrt{2}\cos\left(\omega t - \frac{2\pi}{3}\right) \qquad i_B(t) = I\sqrt{2}\cos\left(\omega t - \phi - \frac{2\pi}{3}\right)$$

$$u_{CN}(t) = U_P\sqrt{2}\cos\left(\omega t + \frac{2\pi}{3}\right) \qquad i_C(t) = I\sqrt{2}\cos\left(\omega t - \phi + \frac{2\pi}{3}\right)$$

Instantaneous electrical power

Instantaneous electrical power in three-phase systems

$$p(t) = p_A(t) + p_B(t) + p_C(t)$$

$$\begin{aligned} p(t) &= 2U_{\mathrm{P}} I \big(\cos(\omega t)\cos(\omega t - \phi) + \cos(\omega t - 120^\circ)\cos(\omega t - \phi - 120^\circ) \\ &\quad + \cos(\omega t + 120^\circ)\cos(\omega t - \phi + 120^\circ) \big) \end{aligned}$$

Hint:

$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$ $\qquad$ $\cos(a) + \cos(a - 120^\circ) + +\cos(a + 120^\circ) = 0$

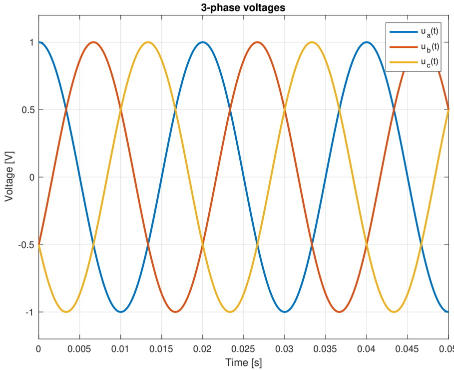
Instantaneous electrical power in balanced three-phase systems

$$p(t) = 3U_{\mathrm{P}} I \cos(\phi)$$

Instantaneous electrical power

Numeric example, with $\underline{Z}_A = \underline{Z}_B = \underline{Z}_C = \frac{3}{2}\angle 30^\circ \Omega$:

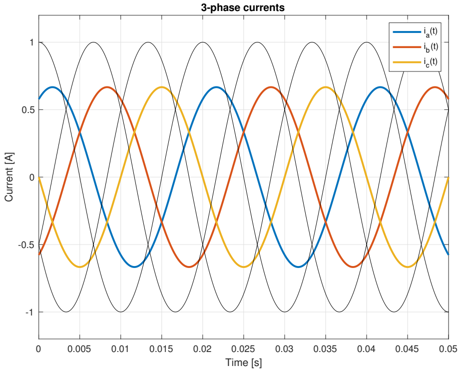
$$u_{AN}(t) = \cos(100\pi t) \text{V}$$
$$u_{BN}(t) = \cos(100\pi t - 2\pi/3) \text{V}$$
$$u_{CN}(t) = \cos(100\pi t + 2\pi/3) \text{V}$$



Instantaneous electrical power

Numeric example, with $\underline{Z}_A = \underline{Z}_B = \underline{Z}_C = \frac{3}{2}\angle 30°\,\Omega$:

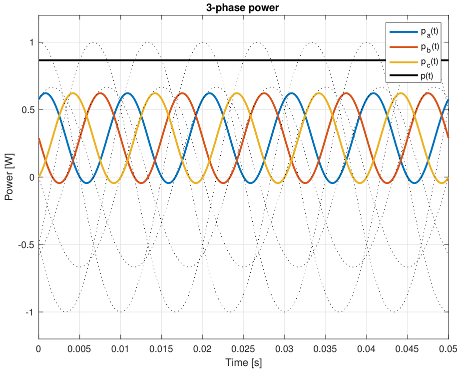
$$u_{AN}(t) = \cos(100\pi t)\,\text{V} \qquad i_A(t) = 2/3\cos(100\pi t - \pi/6)\,\text{A}$$
$$u_{BN}(t) = \cos\left(100\pi t - 2\pi/3\right)\,\text{V} \qquad i_B(t) = 2/3\cos\left(100\pi t - 2\pi/3 - \pi/6\right)\,\text{A}$$
$$u_{CN}(t) = \cos\left(100\pi t + 2\pi/3\right)\,\text{V} \qquad i_C(t) = 2/3\cos\left(100\pi t + 2\pi/3 - \pi/6\right)\,\text{A}$$



Instantaneous electrical power

Numeric example, with $\underline{Z}_A = \underline{Z}_B = \underline{Z}_C = \frac{3}{2}\angle 30°\Omega$:

$$\begin{aligned} p(t) =& 1/3\left(\cos(200\pi t) + \cos(\pi/6)\right) \\ &+ 1/3(\cos(200\pi t - 2\pi/3) + \cos(\pi/6)) \\ &+ 1/3(\cos(200\pi t + 2\pi/3) + \cos(\pi/6)) = 0.86\text{W} \end{aligned}$$



Instantaneous electrical power

Summarizing

- The instantaneous electrical power in three phase systems is the sum of the instantaneous powers in each phase.
- The instantaneous electrical power in **balanced three phase systems** is **constant**.

Outline

Electrical power in three-phase circuits

Total active power and per phase powers

It can be shown that the total active power is the sum of the phase active powers

$$P = P_A + P_B + P_C$$

where

$$P_i = \text{Re}(\underline{U}_i \underline{I}_i *) = U_i I_i \cos\phi_i, \quad i = A, B, C$$

The per phase reactive powers are defined as

$$Q_i = \text{Im}(\underline{U}_i \underline{I}_i *) = U_i I_i \sin\phi_i, \quad i = A, B, C$$

and the phase apparent powers become

$$S_i = U_i I_i = \sqrt{P_i^2 + Q_i^2}, \quad i = A, B, C$$

Electrical power in three-phase circuits

Arithmetic apparent power

The arithmetic apparent power is defined as

$$S_{Ar} = S_A + S_B + S_C$$

that implies

$$PF_{Ar} = \frac{P}{S_{Ar}}.$$

Vector apparent power

The vector apparent power is defined as

$$S_V = |\underline{S}_V|$$

where

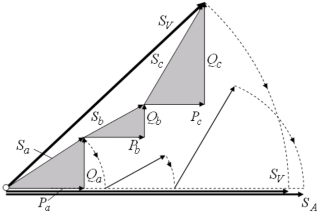
$$\underline{S}_V = P_A + P_B + P_C + j\left(Q_A + Q_B + Q_C\right) = P + jQ$$

that implies

$$PF_V = \frac{P}{S_V}.$$

Electrical power in three-phase circuits

A geometrical interpretation of arithemtic and vector apparent powers, S_{Ar} (labeled as S_A in the figure) and S_V, respectively. Figure extracted from IEEE Standard definitions 1459-2010.



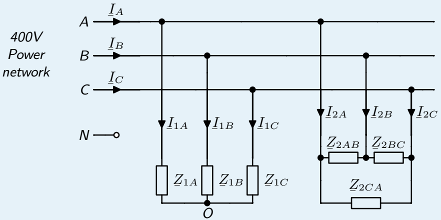
In balanced three-phase systems, both definitions give identical results. However, under unbalanced conditions one has $S_{Ar} \geq S_V$ which implies $PF_{Ar} \leq PF_V$.

Note:

IEEE Standard definitions 1459-2010 recommends to renounce to the arithmetic and vector apparent power definitions and replace them with the effective apparent power. However, for simplicity, **the vector apparent power is adopted in this course**.

Electrical power in three-phase circuits

Exercise 1



$$\underline{Z}_{1A} = 10\Omega,$$
$$\underline{Z}_{1B} = 10 - j10\Omega,$$
$$\underline{Z}_{1C} = 5 + j10\Omega,$$
$$\underline{Z}_{AB} = 20\Omega,$$
$$\underline{Z}_{BC} = j20\sqrt{3}\Omega,$$
$$\underline{Z}_{CA} = -j20\sqrt{3}\Omega.$$

Given the circuit above, find:

1. *the power grid currents* $(\underline{I}_A, \underline{I}_B, \underline{I}_C)$*, and*
2. *the grid powers* $(P, Q$ *and* $S)$*.*

Outline

Electrical power in balanced three-phase loads

The power for each i-phase (where $\alpha_A = 0$, $\alpha_B = -120^\circ$, $\alpha_C = 120^\circ$)

$$p_i(t) = 2U_\mathsf{P} I \big(\cos(\omega t + \alpha_i)\cos(\omega t - \phi + \alpha_i)\big)$$

can be split into a DC term plus and an oscillating term

$$p_i(t) = U_\mathsf{P} I \cos(\phi) + U_\mathsf{P} I \cos(2\omega t - \phi + 2\alpha_i),$$

or, alternatively,

$$p_i(t) = U_\mathsf{P} I \cos(\phi)(1 + \cos(2\omega t + 2\alpha_i)) + U_\mathsf{P} I \sin(\phi)\sin(2\omega t + 2\alpha_i)$$

Hint:

$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

Then

$$p(t) = 3U_\mathsf{P} I \cos(\phi) + U_\mathsf{P} I \sin(\phi)\underbrace{\big(\sin(2\omega t) + \sin(2\omega t + 120^\circ) + \sin(2\omega t - 120^\circ)\big)}_{=0}$$

Electrical power in balanced three-phase loads

Active power in balanced three-phase loads

$$P = 3U_\text{P} I \cos(\phi)$$

where U_P is the phase voltage and I is the line current, or using the line voltage $U = \frac{1}{\sqrt{3}} U_P$,

$$P = \sqrt{3} U I \cos(\phi).$$

Reactive power in balanced three-phase loads

$$Q = 3U_\text{P} I \sin(\phi) = \sqrt{3} U I \sin(\phi).$$

Apparent power in balanced three-phase loads

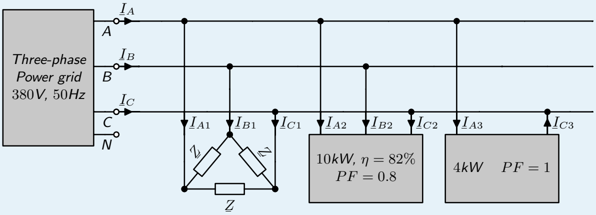
$$S = 3U_\text{P} I = \sqrt{3} U I.$$

Power factor in balanced three-phase loads

$$PF = \frac{P}{S}.$$

Electrical power in balanced three-phase loads

Exercise 2



From the electrical installation above, where $\underline{Z} = 20 + j20\Omega$, find:

1. *the power grid currents ($\underline{I}_A, \underline{I}_B, \underline{I}_C$), and*
2. *the grid powers (P, Q and S).*

Outline

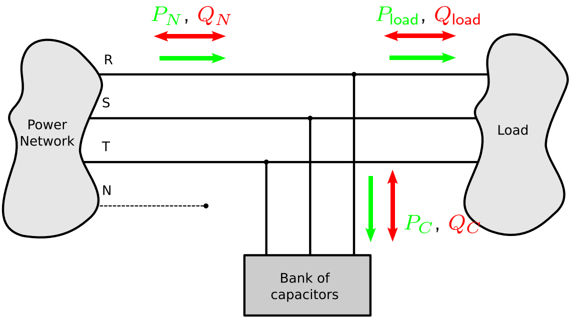
PF correction in balanced three-phase circuits

Exercise 3

Motivating example: *A three-phase 4.5kV transmission line connects a the power grid with a 5MW load placed 100m far from the power substation. If the resistivity of the line is* $4.5 \cdot 10^{-4}\Omega m^{-1}$*. Calculate for PF=0.8 and PF=1:*

- *the power losses, and*
- *the annual cost of the losses (Consider 5c€/kWh).*

PF correction in balanced three-phase circuits



P and Q of a 3-phase load

$$\begin{aligned} P_{\text{load}} &= 3U_{\text{P}} I_{\text{load}} \cos\phi_{\text{load}} \\ Q_{\text{load}} &= 3U_{\text{P}} I_{\text{load}} \sin\phi_{\text{load}} \end{aligned}$$

The total 3-phase power

$$\begin{aligned} \underline{S}_N &= P_N + jQ_N \\ &= P_{\text{load}} + P_C + j(Q_{\text{load}} + Q_C) \end{aligned}$$

PF correction in balanced three-phase circuits

P and Q of a Y-bank of capacitors

$$\begin{aligned} P_{C\mathsf{Y}} &= 0 \\ Q_{C\mathsf{Y}} &= -3\omega C_{\mathsf{Y}} U_{\mathsf{P}}^2 \end{aligned}$$

P and Q of a Δ-bank of capacitors

$$\begin{aligned} P_{C\Delta} &= 0 \\ Q_{C\Delta} &= -3\omega C_{\Delta} U_{\mathsf{L}}^2 \end{aligned}$$

Total reactive power

$$\begin{aligned} Q_N &= Q_{\text{load}} + Q_{C\mathsf{Y}} \\ &= Q_{\text{load}} - 3\omega C_{\mathsf{Y}} U_{\mathsf{P}}^2 \end{aligned}$$

Total reactive power

$$\begin{aligned} Q_N &= Q_{\text{load}} + Q_{C\Delta} \\ &= Q_{\text{load}} - 3\omega C_{\Delta} U_{\mathsf{L}}^2 \end{aligned}$$

Capacitance of the Y-bank

$$\begin{aligned} C_{\mathsf{Y}} &= \frac{P(\tan\phi_{\text{load}} - \tan\phi_N)}{3\omega U_{\mathsf{P}}^2} \\ &= \frac{P(\tan\phi_{\text{load}} - \tan\phi_N)}{\omega U_{\mathsf{L}}^2} \end{aligned}$$

Capacitance of the Δ-bank

$$\begin{aligned} C_{\Delta} &= \frac{P(\tan\phi_{\text{load}} - \tan\phi_N)}{3\omega U_{\mathsf{L}}^2} \\ &= \frac{P(\tan\phi_{\text{load}} - \tan\phi_N)}{9\omega U_{\mathsf{P}}^2} \end{aligned}$$

Outline

Exercises I

Exercise 4

CT1: *An electrical installation with 18 devices is connected to a symmetric and balanced 400V 50Hz power network. See table below.*

Num.	Conn.	Unitary P [kW]	Unitary Q [kvar]	Unitary S [kVA]	PF
3	D	20	–	–	0.6(i)
10	Y	4	2.5	–	–
5	Y	–	–	4	1

1. *Calculate the line currents.*
2. *Calculate the bank of capacitors required to set the power factor up to 0.96(i). Which is the new absorbed current?*

Exercises II

Exercise 5

CT2: *A company has the following (balanced) three-phase loads connected to a 380V power grid*

a) $P = 50kW$, $cos\varphi = 0.8(i)$, Y-connected load

b) $P = 24kW$, $Q = 12kvar$, Δ-connected load

c) $S = 20kVA$, $cos\varphi = 0.96(i)$, Δ-connected load

d) $P = 20kW$, $cos\varphi = 1$, Y-connected load

1. *Calculate the line currents.*
2. *Calculate equivalent impedances for each load.*
3. *With the obtained impedances, sketch the electrical circuit of the installation.*

Solutions I

Solution to Exercise 1

- $\underline{I}_{A1} = 5.13\angle-49.36°$A, $\underline{I}_{A2} = 30.55\angle 40.89°$A, $\underline{I}_{A} = 30.95\angle 31.36°$A
  $\underline{I}_{B1} = 27.84\angle-97.65°$A, $\underline{I}_{B2} = 30.55\angle-160.89°$A, $\underline{I}_{B} = 49.74\angle-130.91°$A
  $\underline{I}_{C1} = 31.49\angle 89.33°$A, $\underline{I}_{C2} = 11.55\angle-60°$A, $\underline{I}_{C} = 22.35\angle 74.05°$A
- $P_1 = 12.97$kW, $Q_1 = 2.16$kvar, $S_1 = 13.15$kVA
  $P_2 = 8$kW, $Q_2 = 0$kvar, $S_2 = 8$kVA
  $P = 20.97$kW, $Q = 2.16$kvar, $S = 21.08$kVA

Solutions II

Solution to Exercise 2

- $\underline{I}_{A1} = 23.27\angle -45^\circ$A, $\underline{I}_{B1} = 23.27\angle -165^\circ$A, $\underline{I}_{C1} = 23.27\angle 75^\circ$A
  $\underline{I}_{A2} = 23.16\angle -36.87^\circ$A, $\underline{I}_{B2} = 23.16\angle -156.87^\circ$A, $\underline{I}_{C2} = 26.47\angle 83.13^\circ$A
  $\underline{I}_{A3} = \underline{I}_{C3} = 10.52\angle -30^\circ$A, $\underline{I}_{B3} = 0$A
  $\underline{I}_{A} = 56.68\angle -38.92^\circ$A, $\underline{I}_{B} = 46.31\angle -160.94^\circ$A, $\underline{I}_{C} = 50.74\angle 90.36^\circ$A
- $P_1 = 10.83$kW, $Q_1 = 10.83$kvar, $S_1 = 15.3159$kVA,
  $P_2 = 12.1951$kW, $Q_2 = 9.1463$kvar, $S_2 = 15.2439$kVA,
  $P_3 = 4$kW, $Q_3 = 0$kvar, $S_3 = 4$kVA
  $P = 27.0251$kW, $Q = 19.9763$kvar, $S = 33.6067$kVA

Solution to Exercise 3

	Current [A]	Losses [kW]	Annual cost [€]
PF=0.8	801.9	86.8	38021.2
PF=1	641.5	55.6	24333.3

Solutions III

Solution to Exercise 4

1. $I = 230$A
2. $Q_c = 70$kvar, $I = 180.4$A

Solution to Exercise 5

1. $\underline{I}_A = 191.28\angle - 25.95°$A, $\underline{I}_B = a^2\underline{I}_A$A, $\underline{I}_C = a\underline{I}_A$A,
2. $\underline{Z}_a = 1.848 + j1.386\Omega$, $\underline{Z}_b = 14.440 + j7.220\Omega$, $\underline{Z}_c = 20.794 + j6.065\Omega$, $\underline{Z}_d = 7.220\Omega$

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