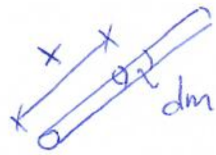
DYNAMICS OF SYSTEMS OF PARTICLES

Center of mass

$$\begin{cases} \text{of discrete body} & M\bar{r}_{CM} = \sum m_i \bar{r}_i \\ \text{of continuous body} & M\bar{r}_{CM} = \iiint \bar{r}\, dm \end{cases}$$

Concept: $x_{CM} = \dfrac{m_1 x_1 + m_2 x_2}{M_T}$

Example: $\lambda = \dfrac{M}{L} = \dfrac{dm}{dl} \Rightarrow dm = \lambda\, dx$



$$M x_{CM} = \int x\, dm = \int_0^L x \lambda\, dx = \lambda \left[\frac{x^2}{2} \right]_0^L$$

$$M x_{CM} = \lambda \frac{L^2}{2} = \frac{M}{L} \frac{L^2}{2} \Rightarrow \boxed{x_{CM} = \frac{L}{2}}$$

Potential Energy

The same energy as if all mass may be concentrated in the CM

$$U = M_T g h_{CM}$$

Motion of CM

$$M \frac{d\bar{r}_{CM}}{dt} = \sum m_i \frac{d\bar{r}_i}{dt} = \sum m_i \bar{v}_i$$

Conservation of linear momentum

→ one of the most important laws!"

For a particle: $\bar{p} = m\bar{v}$

$$\frac{d\bar{p}}{dt} = \frac{d(mv)}{dt} = m\bar{a}$$

$$\bar{F} = \frac{d\bar{p}}{dt}$$

For a system:

$$\left.\begin{aligned}\bar{P} &= \sum p_i = \sum m_i v_i \\ \bar{P} &= M\bar{v}_{CM}\end{aligned}\right\} \boxed{\frac{d\bar{P}}{dt} = M\bar{a}_{CM}}$$

$$\boxed{\bar{P} = \text{constant} \Leftrightarrow \sum F_i = F_{net} = \bar{0}}$$

Remark: $\bar{P}$ can be conserved, this does not imply energy is conserved.

$$m_0 \cdot v_0 + 0 = (m_0 + m_{water})\, v_f$$

$\bar{P}$ is constant.

$\{ E_p = E = mgh$ is constant? But then $m_f = m_0 + m_{water}$

$E_c = \frac{1}{2}mv^2$ changes!

Inelastic collision: { complete → remain stuck together; partial → general case.

Systems of variable mass: i.e. rockets

- simple one-stage rocket.

$v=0$  $p_0 = (m_{fuel} + m_{rocket}) \cdot 0 = 0$

ignition  $p_f = 0 = m_{rocket} \cdot v_r + m_{fuel} \cdot v_{fuel}$

$\overleftarrow{v_f}$ $\overrightarrow{v_r}$

In a real case, not all the fuel is exhausted in one-shot → mass changes continously:

$$p(t) = m(t)v(t) = mv$$

$$R \equiv \text{fuel exhaust rate} = \frac{dm}{dt}$$

$$p(t+\Delta t) = \underbrace{(m - R\Delta t)(v+\Delta v)}_{rocket} + \underbrace{R\Delta t(v - v_{exhaust})}_{gas}$$

$$p(t+\Delta t) = mv + m\Delta v - v_{exh} R \Delta t + \underbrace{\Delta t \cdot \Delta v}_{\simeq 0} \cdots$$

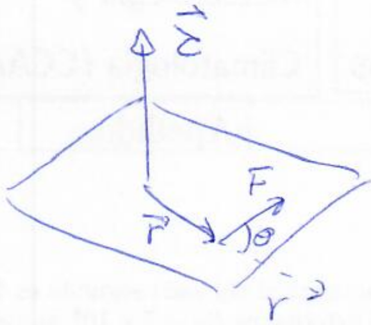
$$\frac{\Delta p}{\Delta t} = \frac{p(t+\Delta t) - p(t)}{\Delta t}$$

$$\frac{\Delta p}{\Delta t} = m\frac{\Delta v}{\Delta t} - v_{exh} \cdot R$$

Rotations and Torques

For one particle:

$$\vec{\tau} = \vec{r} \times \vec{F}$$



Torques imply rotations
↳

$$ds = r\, d\theta$$

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d^2\theta}{dt^2}$$

$$\bar{\tau} = \bar{r} \times \bar{F}$$

$$|\bar{\tau}| = |r|\,|F|\sin\theta$$

$$|\bar{\tau}| = |r|\,|F_{\text{tangential}}|$$

$$|\tau| = m a_t \cdot r$$

$$a_t = \alpha \cdot R$$

$$\boxed{|\tau| = m\,\alpha\, r^2}$$

For a system of particles

$$\Sigma \tau_i = \Sigma m_i r_i^2 \alpha$$

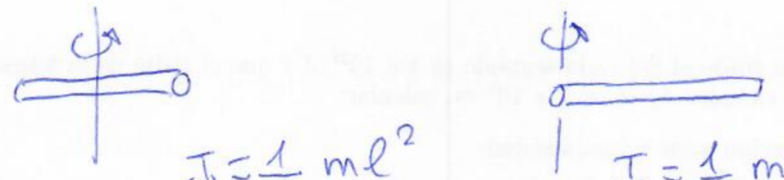
...dation

$$\boxed{\Sigma \tau_i = I \alpha} \leftrightarrow$$ similar ... but for rotations

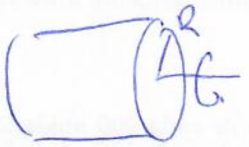
Torques ⇒ rotation!

$$I = \int r^2 \, dm$$

it depends on distance to rotation axis, hence on the axis



$I = \frac{1}{12} m l^2$ $\qquad$ $I = \frac{1}{3} m l^2$



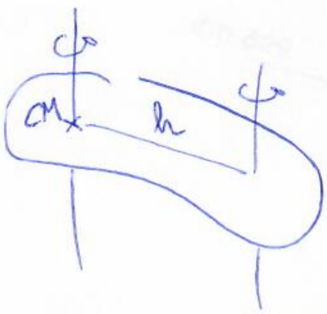
$I = \frac{1}{2} m R^2$

$I = \frac{2}{3} M R^2$

$I = m R^2$

$I = \frac{2}{5} M R^2$

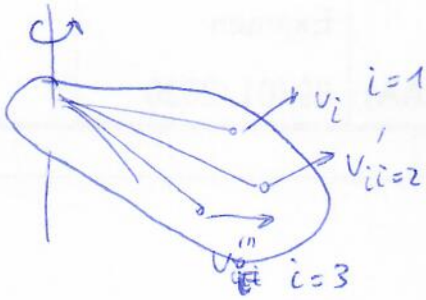
Parallel axis theorem



$$I = I_{cm} + M h^2$$

Example:

Angular moment and rotations



$$\vec{L} \equiv \vec{r} \times \vec{p}$$

$$L_i = r_i \times m_i v_i$$

$$L_i = m v_i r_i \vec{k}$$

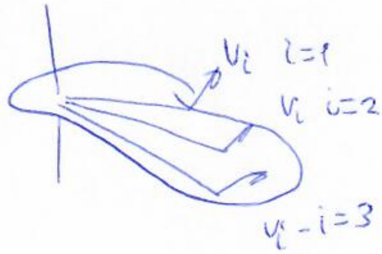
$$L_i = m r_i^2 \omega \vec{k}$$

$$L_T = \Sigma L_i = \Sigma m_i r_i^2 \omega \vec{k} = I \omega$$

$$\boxed{\vec{L}_T = I \vec{\omega}}$$

if $\vec{\tau} = 0 \Rightarrow \vec{L}$ is constant $\Rightarrow I\omega$ is constant

Kinetic energy and rotations



$$E_c = T_i = \frac{1}{2} m_i v_i^2$$

$$\Sigma E_{c_i} = \Sigma T_i = T_{\text{total rotation}}$$

$$T_{rot} = \frac{1}{2}(\Sigma m_i r_i^2)\omega^2$$

$$T_{rot} = \frac{1}{2} I \omega^2$$

Work and power

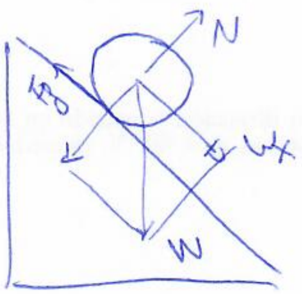
Rolling without sliding

$$s' = R\Theta$$

$$V_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$\left.\begin{aligned} V_{CM} &= R\omega \\ a_{CM} &= R\alpha \end{aligned}\right| \rightarrow \text{non-slip condition}$$

$$T_T = \underbrace{\frac{1}{2} I_{CM}\,\omega^2}_{\text{rotation}} + \underbrace{\frac{1}{2} M_T V_{CM}^2}_{\text{traslation}}$$



$$W_x - F_f = m a_{CM}$$

$$mg\sin\theta - \frac{\tau}{R} = m a_{CM}$$

$$mg\sin\theta - \underbrace{\left(I_{CM}\cdot\frac{a_{CM}}{R^2}\right)}_{(} = m a_{CM}$$

$$\tau = I\cdot\alpha = I\frac{a}{R}$$

$$a_{CM} = \frac{1}{1+\frac{I_{CM}}{mR^2}}\, g\sin\theta$$