6 Applications of Installments

Class 17

Applications: Bonds

Bonds

- These represent the long term debt of large corporations and government (fixed income securities)
- They provide a payment equal to their **nominal value** at expiration (the end of the life of the security) plus a regular and fixed payment, the coupon (expressed as a percentage of the nominal value of the bond)
- The **interest** is paid as a percentage of the nominal value of the bonds. This interest can be fixed or variable throughout the life of the bond issue. Variable interests are usually determined relative to some reference, e.g. libor, euribor, ...
- The **price** is usually described as percentage points of the norminal of the security
- Repayment of the principal is called **amortization**. Government bonds amortize at expiration. If at expiration the bond returns principal in excess of its nominal value then the bond is said to include a premium

Determining the price

- Example: a government debt issue with a nominal value of 1000€, annual 6% coupon, and expiry in 5 years
- The cash flows are received at the end of the year for the remaining life of the bond issue (today is 2012)

2013	2014	2015	2016	2017
60	60	60	60	1060

- How much would an investor be willing to pay for such a bond? This question serves as motivation to study price setting for bonds

Bond Valuation

- Determining the price (present value) of this bond depends on the market interest rate, that is, the return of assets that are traded in the market that have similar characteristics (risk, duration, ...)
- We look at the market for these returns
- Suppose previous issues of similar bonds are trading with a 7.7% annual return. This is what the investor is giving up in order to invest in this new bond
- Therefore, we discount these cash flows at the 7.7% return

$$VA = \frac{60}{1.077} + \frac{60}{1.077^2} + \frac{60}{1.077^3} + \frac{60}{1.077^4} + \frac{1060}{1.077^5} = 931.58€$$

Prices

- Bond prices are usually expressed as percentages (of the nominal amount)
- We can say that the price of our bond is 931.58€ or 93.518%
- Any bond is priced as

$$PV\,(\text{coupon payments}) + PV\,(\text{final payment})$$

- Thus, any bond can be valued as an annuity plus a single final payment

Example

- Consider the purchase of a 1000€ nominal bond, with 10 year maturity and yearly coupon of 3% (annual). Similar investments trade at a 3.2% return. Determine the price on this bond
- Price = PV(future payments)

$$PV = \frac{30}{1.032} + \frac{30}{1.032^2} + \ldots + \frac{1030}{1.032^{10}}.$$

- As the coupon is constant, we can calculate their PV using the annuity formula

$$PV = 30\left(\frac{1-(1+3.2\%)^{-10}}{3.2\%}\right) + \frac{1000}{1.032^{10}}.$$

- The resulting price is 970.79 (or 97.979%)

Discount rates

- So far, we have made the simplifying assumption that there is a single (constant) discount rate for all the future payments
- This is called a "flat term structure of interest rates"
- The basic formula to value an asset that makes several payments over time is

$$PV = \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_2)^2} + ... + \frac{C_n}{(1+r_n)^n}$$

  where r_t is the annual interest for single payment investments due in t years
- There is a different interest for each period of time
- The relationship between the interest rate and time (the corresponding period) is called the term structure of interest rates

Bond Valuation

- Suppose that the investor's best alternative is to deposit his money at the following interest rates

1 year	2 years	3 years	4 years	5 years
6.5%	7%	7.25%	7.5%	7.7%

- How much would the investor have to deposit today in order to generate the same cash flows as the first bond [nominal: 1000€, annual 6% coupon, 5 yr expiry] using the corresponding interests rates

1 year	2 years	3 years	4 years	5 years
$\frac{60}{1.065} = 56.34$	$\frac{60}{1.07^2} = 52.41$	$\frac{60}{1.0725^3} = 48.64$	$\frac{60}{1.075^4} = 44.93$	$\frac{1060}{1.077^5} = 731.52$

- The investor could lend 933.83 today and obtain these
- Then, if this is the best (only) investment alternative strategy then the price of the bond must be this one

What if the price is not 933.83: arbitrage

- Suppose the price is 900€. Could an investor make money at no cost to him?
- An investor could
  - buy the bond (pay 900€ and receive the interest and other payments)
  - borrow the above amounts: 56.34 for one year, 52.41 for two years, ...
- After one year, the investor can pay back the 56.34€ for one year plus the interest = 60€ using the payment from the bond
- After two years, the investor can pay back the 52.41€ for two years plus the interest = 60€ using the payment from the bond that year
- ... and so on until the fith year when he uses the 1060€ from the bond to cancel his final debt
- The net gains over the five years has been
  - cero for years 1, 2, ..., 5–the payment from the bond exactly cancelled the debt from the loans
  - a profit of 33.83€ today: he paid 900€ for the bond and obtained 933.83 from the loans
- This profit was obtained with no risk = arbitrage

Arbitrage

- If such opportunities to make profits via arbitrage exist, investors will try to exploit them. This will raise the price of the bond as investors increase the buying pressure on these bonds, and its price will increase. How far? Until the price has increases so that the opportunity for arbitrage disappears, i.e. when the price reaches 933.83
- In practice, these price adjustments occur (nearly) instantaneously so that these easy profit opportunities (arbitrage opportunities) are rarely found, and this is a result of the high degree of competition in financial markets
- The market for assets (financial markets) are in equilibrium when the value of assets is equal to the present value of their future payments, given the return from alternative investment opportunities (of similar maturity/expiry and risk)

Equilibrium prices

- Thus, the equilibrium (or fundamental) value of an asset is equal to the present value of its future payments

$$VA = \frac{60}{1.065} + \frac{60}{1.07^2} + \frac{60}{1.0725^3} + \frac{60}{1.075^4} + \frac{1060}{1.077^5} = 933.83€$$

- The price of the bond is said to be 93.38% (of its nominal value)
- It is very rare that the price of a bond is equal to its nominal value as the price of the bond varies every day due to
  - changes in discount rates as time passes
  - changes in interest rates for investing and borrowing in the future

Term structure and prices

- The present value of a typical bond is given by

$$PV = \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_2)^2} + \ldots + \frac{C_n}{(1+r_n)^n} + \frac{NV}{(1+r_n)^n}$$

  - The interest rates: $r_1, r_2, \ldots, r_n$ are called spot interest rates for different maturities
  - They define the term structure of interest rates (the relationship between interest rates and maturities)
  - They would be similar to the interest rates of alternative investments we have seen so far. Here, risk plays an important part–and hence whether the bonds are issued by a governmental entity or a company

Special cases

- Constant coupons and amortization at par (NV=nominal value)

$$\begin{aligned} VA &= P = \frac{C}{(1+r_1)} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C+NV}{(1+r_n)^n} \\ &= C\left(\sum_{t=1}^{n}\left(\frac{1}{(1+r)^t}\right)\right) + \frac{NV}{(1+r)^n} \end{aligned}$$

  Use annuities??? Only if the term structure is flat

- Corporate bond issues: there is greater risk so that the (discounting) interest rate should be higher because of the extra risk

$$r_1^C = r_1 + \phi_1,\ r_2^C = r_2 + \phi_2,\ \ldots\ r_n^C = r_n + \phi_n$$

$$VA = P = \frac{C}{(1+r_1^C)} + \frac{C}{(1+r_2^C)^2} + \ldots + \frac{C+NV}{(1+r_n^C)^n}$$

- Zero coupon bonds (STRIPs–Separated TRading of Interest and Principal)

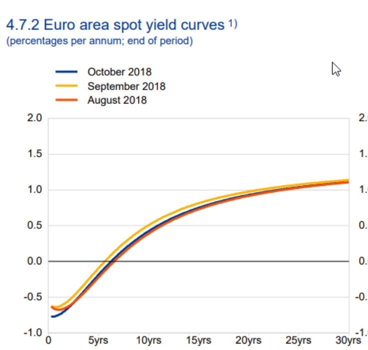
The term structure of interest rates

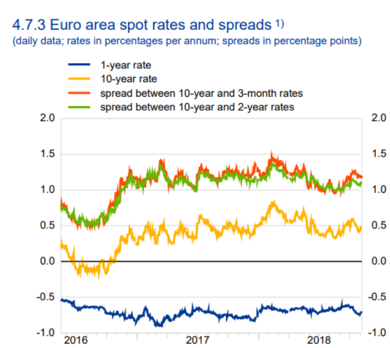
- The relationship between the interest rate and the term (the length of time of the investment) is called the term structure of interest rates
- The term structure is a graphic representation of this relationship: it could be a graph that is increasing, constant, decreasing ...

https://www.bde.es/webbde/es/estadis/infoest/tipos/tipos.html

The term structure of interest rates: curves

ECB: Statistics Bulletin (Financial Markets/Euro area yield curves)





The term structure of interest rates

- ECB: Statistics Bulletin (Financial Markets/Euro area yield curves)
- Th

Additional examples

- Today, 1-1-2004, you are offered a strip with a 10.000€ nominal value and an expiry date of 1-1-2009. The bond is amortized above par (that is with a premium, in this case of 20%) and the annual interest rate over 5 years is 4.5%. Calculate the price

$$P = \frac{1.2 \times 10000}{(1 + 4.5\%)^5} = 9629.41$$

Example

Today, 2-2-2005, you are a bond from El Tesoro (nominal 1.000€, 3.25% annual coupon, expiry date 2-2-2010). The bond is amortized with a premium of 10% and the annual interest rates over 1,2,3,4, and 5 years are 3%, 3.5%, 4%, 4.5%, 5.25% respectively. Calculate the price

$$P = \frac{32.5}{1+3\%} + \frac{32.5}{(1+3.5\%)^2} + \frac{32.5}{(1+4\%)^3} + \frac{32.5}{(1+4.5\%)^4} + \frac{32.5 + 1.100}{(1+5.25\%)^5} = 994.89€$$

Calculate the price of a bond on 1/9/2005 if the expiry date is 31/12/2009 and it pays an annual coupon of 7%. The term structure of interest rates is flat at 8.5%

993.47

Price-Return Relationship

- A higher interest rate implies that future payments will be "discounted more" and hence the price of the bond will be lower
- Thus, as interest rates increase, the price of bonds falls
- This highlights a basic bond investment rule: "As interest rates increase, bond prices will fall as the present value of future bond payments is obtained by discounting at a higher interest rate"

PRECIODE LOS BONOS A UN TIPO DE INTERÉS DETERMINADO				
Vencimiento Años	4%	5%	8%	10%
1	1,009.62 €	1,000.00 €	972.22 €	954.55 €
10	1,081.11 €	1,000.00 €	798.70 €	692.77 €
20	1,135.90 €	1,000.00 €	705.46 €	574.32 €
30	1,172.92 €	1,000.00 €	662.27 €	528.65 €

The return on bonds

- Consider a bond from an investor's point of view.
- If you buy a bond, where is the interest you obtain?
- Purchase (payment) possibly at a discount off the nominal value
- Regular payments obtained from coupons
- Example: cero coupon bond: nominal value 1.000€, expiry: 6 months, issue price 977€
  - Semi-annual return: 2.354%
  - Effective annual return: 4.76%
- What if there are intermediate coupon payments?

Yield–Internal Rate of Return

- When we discount future payments we could be discounting each payment at a different interest rate
- Now, we are asking for a single interest rate that when applied to the cash flows will return the value of the bond–this is the Internal Rate of Return (IRR) or yield
- In order to calculate the IRR one needs he price, payments and expiry of bond payments

Yield

- The equation for the IRR (y) is

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \ldots + \frac{C_n + N}{(1+y)^n}$$

- This (in general) is an n-th degree polynomial and hence a simple trial and error method: that is you select several values of y and compute the RHS to see if the result is equal to the price (on the RHS). The IRR solves this equation
- Note that from the above equation for the IRR one could also calculate the price of the bond

Example

- Suppose the term structure of interest rates is increasing, the one year interest rate is 5% and the two year annual interest rate is 6%. Compute the yield of a government bond with a 1000€ nominal value, a coupon of 4% per annum and a two year expiration (amortizing at par)

1. You have to calculate the price

$$P = \frac{40}{1.05} + \frac{1040}{1.06^2} = 963.69$$

2. Then you have to setup the equation for the yield

$$P = \frac{40}{1+y} + \frac{1040}{(1+y)^2}$$

Consider trying different values of y:

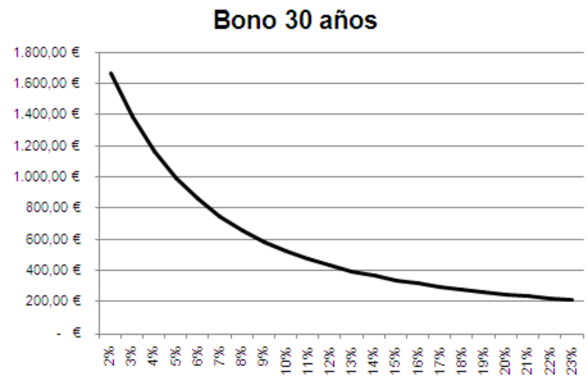
Example (cont)

$y = 5\%$ leads to a price of $P = 981.41$,
$y = 5.5\% \Rightarrow P = 972.32$,
$y = 5.9\% \Rightarrow P = 965.12$,
$y = 6.0\% \Rightarrow P = 963.33$, ...
The solution is

$$y = 5.98\%$$

- Note: there is no reason why the yield has to coincide with any of the interest rates used for discounting. In fact, it is an average of these.

Interest Rates and Prices



A 30 year bond with a 1.000€ nominal value, and a 5% coupon

IRR and returns

- From the above it follows that the IRR and the final return obtained from an investment can be different
- This issue does not arise for zero coupon bonds where the IRR coincides with the bond return
- In order to discount bonds, we will not use the interest rates obtained from the yields from bonds with coupons, nor from corporate debt. We use the spot interest rate obtained from zero coupon bonds from entities of maximum credit quality

Exercise

- A bond with a 1000€ nominal value has a four year maturity and a 10% coupon. It was issued at par (price = nominal value). Two years after its issuance, an investor wants to sell it (the market return is 15%). Determine
  - The theoretical sale price of the bond
  - Suppose the bond is sold at the price you have just calculated, determine the real return that the first investor has obtained, and the one the second investor will obtain if it holds the bond to maturity
- Determine the price price of the bond if the sale is done 2 years and 11 months after issuance

Solution

Nominal value	1.000,00 €
Issue date	1.000,00 €
Coupon	10%
	100,00 €
Yield	15%

Future payments		PV
1	100,00 €	86,96 €
2	1.100,00 €	831,76 €
Price		918,71 €

Seller

Payments		
0	- 1.000,00 €	
1	100,00 €	
2	1.018,71 €	
Rentabilidad		6,06%

Buyer

Payments		
0	- 918,71 €	
1	100,00 €	
2	1.100,00 €	
Rentabilidad		15,00%

After 2 years & 11 months

Future payments		PV
1/12	100,00 €	98,84 €
1+1/12	1.100,00 €	945,45 €
Price		1.044,29 €

Applications: Shares

Shares/Equity

- A share is a claim to a fraction of a company's capital. These are issued by the company and make the holder owner of the firm with the corresponding economic and voting rights:
  - The right to partake of the company's profits, if there is a dividend issue
  - The right to recover a fraction of the capital remaining after the company is liquidated
  - Preferential subscription rights for any new share issue
  - The right to attend and vote at the shareholder's meeting (management choice, statute changes, ...)

The price of shares

- A traditional way to determine the value of shares is to look at accounting data. On the balance sheet you find net assets (capital and reserves), and you divide them by the number of shares to obtain the (accounting) value of a share
- Thus, the accounting value of a share is

$$\text{Share value} = \frac{\text{net assets}}{\text{number of shares}}$$

- Even if the firm's accounts reflect the existing wealth of the firm, this method does not take into account the future expected profits, and the investor, more than wealth (past value), what he buys with a share is income (future profits)
- This model is very simple and static, as it does not take into account the profit generating potential of the firm, which in the end is what should determine the price of shares

Valuing common shares

- In previous lessons we learned how to value future cash flows/payments
- Shareholders receive money from the firm, in the shape of future dividends. Thus, the price (or present value) of a share is

$$PV\,(\text{share}) = PV\,(\text{expected future dividends})$$

- But, investors who buy shares also expect the share price to change and make capital gains ... how come the present value formula does not include capital gains?

Share prices

- In truth, shareholders obtain gains from two sources: dividends and capital gains
- How much would we be willing to pay for a share when we expect to obtain a dividend at the end of the year of 5€ and we expect to be able to sell the share for 110€, taking into account that the return on similar investments (in terms of risk) is 15%?

Share prices

- If I invest 100€ in some other (similar) investment, I obtain a return of 15%, that is 115€.
- If I buy this share today I will receive 110€ when I sell it plus 5€ dividends = a total of 115€—the same amount, so the price should be the same
- If we applied the discounting model:

$$P = \frac{5 + 110}{1 + 15\%} = 100€$$

- Thus, I will not pay more than 100€ for this share

In general

- How much would I be willing to pay for a share if I expect a dividend at for this year of Div_1 and I expect to sell the share for a price of P_1, when the return for similar investments is r?
- The answer is the amount that would generate a return of r, i.e. the present value of the dividend and the future (sale) price
$$PV = \frac{Div_1 + P_1}{1+r}$$
- I will not be willing to pay more than PV as I have alternative investment opportunities that generate the same return return (with a similar risk)
- The price today should be equal to the present value of the cash payments expected from holding the share
$$P_0 = \frac{Div_1 + P_1}{1+r}$$

What if the price is not equal to its PV?

- If $P_0 < PV$: then the share is undervalued ... everyone would want to buy this as its low price makes the share return higher than those for similar investments. This will generate buying pressure which increases the price ... how much? until the price equals its present value
- If $P_0 > PV$: then the share is overvalued ... holders of the share find that they can get better returns from other investments. This leads to selling pressure which lowers the price ... how much? until the price equals its present value

Similar investments are valued in a way that it offers the same expected return. This is a condition needed for financial markets to be in equilibrium

Future prices

- In order to determine the price today, we need the expected selling price next year
- How is the future price determined?
- how much would an investor be willing to pay for this share one year from today?

Determining the share price

- The investor will face next year the same problem as we do today, but looking forward one period at his future dividend and share price (holding the return of alternative investments constant, r):
$$P_1 = \frac{Div_2 + P_2}{1+r}$$
- If we substitute this expression back in the original one for P_0
$$\begin{aligned} P_0 &= \frac{Div_1 + P_1}{1+r} = \frac{Div_1}{1+r} + \frac{\frac{Div_2+P_2}{1+r}}{1+r} \\ &= \frac{Div_1}{1+r} + \frac{Div_2}{(1+r)^2} + \frac{P_2}{(1+r)^2} \end{aligned}$$
- This leaves us with the problem of estimating the price of the share two years from today

Looking into the future

- Repeating the process for years $3, 4, \ldots$

$$\begin{aligned} P_0 &= \frac{Div_1}{1+r} + \frac{Div_2}{(1+r)^2} + \ldots + \frac{Div_n + P_n}{(1+r)^n} \\ &= \sum_{t=1}^{n} \frac{Div_t}{(1+r)^t} + \frac{P_n}{(1+r)^n} \end{aligned}$$

- As n increases, $1/(1+r)^n$ goes to zero. Thus, dividends represent the greater part of the value of the share
- As the firm life is uncertain and possibly infinite (unless the firm goes into bankrupcy or is absorbed by another firm), the present value of the final price is approximately zero

The discounted dividends model

- The value of a share is equal to the sum of the present value of all the dividends it generates

$$P_0 = \sum_{t=1}^{\infty} \frac{Div_t}{(1+r)^t}$$

- This formula requires making a forecast of an infinite sequence of profits. Thus, it is not of practical use either to calculate the price of a given share, nor the return of a risky investment (knowing its current price)
- There are simplifying assumptions that allow the transformation of the above expression into something more manageable

Constant dividends

- Suppose a firm provides a constant stream of dividend payments (equal to Div). Then we are computing the present value of a perpetuity (infinite payments)

$$P_0 = Div a_{\infty|r} = \frac{Div}{r}$$

- From this formula we can determine what the return is for an invesment of a certain amount of risk and known price

$$r = \frac{Div}{P_0}$$

- This would give us the discount rate with which to value similar invesments–similar in terms of the risk of the shares with determine the return r

Dividends growing at a constant rate

- If dividends are growing at a constant rate we use the formula that applies to perpetuities growing at a constant (geometric) rate

$$P_0 = \frac{Div}{r-g}$$

- The formula can be used to obtain an estimate of r from Div, P and g

$$r = g + \frac{Div}{P_0}$$

- Thus, the compounded capitalization rate offered by the market is equal to the dividend return Div_1/P_0 plus the expected growth rate of dividends (g)

In General

- Shares are different from fixed income investments (and investments with no risk). This generates difficulties in order to determine their price/return
  - the present value formulas are hard to deal with when dividends are not constant or not growing at a constant rate
  - even if dividends grow at a constant rate, estimating this rate is not a simple task as it depends on future growth opportunities available to the firm
- Clearly, trying to estimate the market return for risky invesments using a discounted dividend model is no easy task. One would need a theory that quantifies in some precise way the relationship between risk and return (Portfolio theory)

Example IBX

- The dividend paid by one share of company IBX at the end of this year is expected to be 2.15€, and will continue growing at 11.2% each year. If the required return on an investment like IBX is 15.2% per year, what is the intrinsic value (price) of the share?

$$P = \frac{2.15€}{0.152 - 0.112} = 53.75$$

- What is today's expectation of next year's price?

$$P_1 = \frac{2.15\% \times 1.112}{.152 - .112} = 59.77$$

Example IBX

- Suppose an investor wants to buy IBX shares and sell them after one year (and after collecting the dividend for that year). Determine
  - The investor's total (expected) return
  - What was the return he obtained from the dividends, and what was the return from capital gains?
- Total (expected) future income: $2.15 + 59.77$. Return

$$\frac{2.15 + 59.77}{53.75} - 1 = 15.2\%$$

- Dividends: $2.15/53.75 = 4\%$, capital gain $59.77/53.75 = 1.112 \Longrightarrow 11.2\%$ return

Share dilution

- As we have seen, the value of a share depends on the future value of the dividends it will generate
- Suppose you have a company that generates 100,000€ in cash profits this year (at the end) and these cash profits will grow at 2% per year.
- Suppose also that the expected return for these type of company is 7%
- Determine the market value of the firm today
- Suppose that the firm issues 100,000 shares ... what will be the market price of one share?
- Suppose that the firm issues 200,000 shares ... what will be the market price of one share?
  - The effect on the price of shares produced by increasing the number of shares is called **dilution**

Share dilution in practice

- Do we see share dilution taking place?
  - yes and no
- We see two phenomena
  - seasoned equity offerings
  - dividends paid in shares

Dividends paid in shares: value neutral cash investment

- A company may choose to pay a dividend in the form of shares instead of cash
- Consider the company we used as an example: D_1: 100,000€ this year plus a growth of 2% with m =100,000 shares outstanding
- Price of 1 share is (using $d_1 = D_1/m$)

$$P_0 = \frac{d_1}{r-g} = \frac{1}{7\% - 2\%} = 20$$

  but also

$$P_0 = \frac{d_1 + P_1}{1+r} = \frac{1+P_1}{1+7\%}$$

  where

$$P_1 = \frac{d_2}{r-g} = \frac{1\,(1+2\%)}{7\% - 2\%} = 20.4, \quad P_0 = \frac{1+20.4}{5\%} = 20€$$

- At $t = 1$, the company management considers using the 100,000€ to increase production which will generate an extra cash flow next year C_2 of 5,000€ which will grow at 2% and is discounted at 7% (like the rest of the company)
  - this means that the company will generate cash at $t = 2 : \hat{D}_2 = D_2 + C_2$, at $t = 3$: $\hat{D}_3 = D_2\,(1+g) + C_2\,(1+g) = \hat{D}_2\,(1+g)$
  - therefore the value at $t = 1$ of the company is (using $D_2 = D_1\,(1+g)$)

$$V_1 = \frac{\hat{D}_2}{r-g} = \frac{100000\,(1+2\%) + 5000}{5\%} = 2140000$$

    notice that with this change, the value of the companyV_1 is the same as if they had paid the dividend $((20.4 + 1)\,100000)$

Dividends paid in shares: dilution

- Because management spends the cash to make the company bigger it has no cash, so it can do two things
  1. do not issue any dividends: then the new share price will be $2140000/100000 = 21.4$€ per share
  2. issue dividends in the form of shares: issue n new shares
- In the second case (issue new shares), the value of the company does not change, $V_1 = 2140000$, so that the price of each share after the dividend, $\hat{S}_1$, will be

$$\hat{S}_1 = \frac{V_1}{n+m}$$

- Notice that shareholders had old shares with price 21.4€ at $t = 1$, but now, for each old share they will own $1 + \frac{n}{m}$ new shares worth $\hat{S}_1$. Suppose that $n = 10000$ shares. Then the price of new shares will be:

$$\hat{S}_1 = \frac{V_1}{n+m} = \frac{2140000}{100000 + 10000} = 19.45€$$

but for each old share, now the investor has 1+10000/100000=1.1 shares, so that the total value of the shares is $1.1 * 19.45€ = 21.4€$

- We define the market value of the preferential (or preemptive) suscription rights, d, as the difference between the price per share before the announcement minus the price per share after the announcement

$$d = S_1 - \hat{S}_1 = 21.40 - 19.45 = 0.95$$

- Also notice that the old shareholders get the new shares "for free"–they receive new shares as a dividend payment so they do not have to pay for the new shares.

Seasoned Equity Offers

- Suppose that instead of using the 100.000€ from dividends, the company needs cash and wants to obtain it as equity capital (rather than as a loan/bond)
- We first keep the same relationships as before: we have the cash value of a share (old share) before the SEO, S_1, the cash value of a share (new share) after the SEO, $\hat{S}_1$, and the dilution effect of the SEO, d (so that $\hat{S}_1 = S_1 - d$).
- In addition, we need to consider the new share issue process. When the company issues new shares, existing shareholders have the right to purchase them first (priority rights) from the company–possibly at a special (lower than market) price.
- Related to this, the company announces two things
  1. the issue price of the new shares, E; this is the price, at which the new shares can be bought at from the company (as opposed to buying them at the (future) market price ($\hat{S}_1$).
  2. the ratio in which the rights to buy the new shares are assigned to the old shareholders.
- The ratio can be expressed in two ways: 1 new [n] shares for 10 old [m] shares, or a ratio of 1 to 10 ($n : m$)
- Taking into account that the new shares have to have the same market price as the price of the old shares after the issue ($\hat{S}_1$) the dilution effect will also be equal to the market price of the priority rights (d).
- There are two ways to have a new share
  1. Buy and old share (that was worth S_1) and sell (for cash) the priority rights:

  $$\hat{S}_1 = S_1 - d$$

  2. Buy m priority rights (from m old shares) which give you the right to buy n new shares at the company price of E each. This will net you n new shares:

  $$n\hat{S}_1 = m \times d + n \times E \iff \hat{S}_1 = \frac{1}{n}(m \times d + n \times E)$$
  $$\hat{S}_1 = \frac{m}{n} \times d + E$$

Seasoned Equity Offerings

If a company whose shares are trading at price S_1 issues n new shares for m old ones at a price E the value of the priority rights is obtained from

$$d = \frac{n}{m+n}(S_1 - E)$$

and the new price $\hat{S}_1 = S_1 - d$.

Seasoned Equity Offers: Example

- Suppose you have shares trading at 21.40€ and the company announces a SEO with an issue price (E) of 10€ and a ratio of 1 : 10 (*new* : *old*). What is the new share price?

$$\hat{S}_1 = 21.40 - d$$
$$\hat{S}_1 = \frac{1}{1}(10d + 1 \times 10)$$

$$1 \times (21.40 - d) = 10d + 10$$
$$21.40 - 10 = 11d$$
$$\Rightarrow d = 0.218 \quad \text{and } \hat{S}_1 = 21.18$$

- **Nominal value**: shares (like bonds) also have a nominal value which is unrelated to its market value.
- In the previous example, the nominal value of a share was not defined. We can suppose that the nominal value of the share was 10€.
- In general, in the SEO the issue price is usually announced as a percentage of the nominal value.
- Thus, in the previous example, we could have said that the nominal value of the shares was 10€ and that the new shares were issued *at par*, i.e. at a price of 100% (equal to the nominal value)
- Exercise: compare the result in this example, with the previous analysis (shares paid as dividends) where the shares were given to shareholders for free ($E = 0$)

Seasoned Equity Offers: Example II

- Suppose you have shares trading at 13.55€ and the company announces a SEO with an issue price (E) of 10€ and a ratio of 3 : 17 (*new* : *old*). What is the new share price?

$$\hat{S}_1 = 13.55 - d$$
$$\hat{S}_1 = \frac{1}{3}(17d + 3 \times 10)$$

$$3 \times 13.55 - 3d = 17d + 30$$
$$20d = 40.65 - 30$$
$$\Rightarrow d = 0.533 \quad \text{and } \hat{S}_1 = 13.02$$

Seasoned Equity Offers: Example III

- Suppose you have shares with a nominal value of 10€ which are trading at 14.00€ and the company announces a SEO with new shares issued at 120% and a ratio of 2 : 7 (*new* : *old*). What is the new share price?

$$\hat{S}_1 = S_1 - d = 14 - d$$

$$\hat{S}_1 = \frac{1}{2}(7d + 2 \times 120\% \times 10) = \frac{1}{2}(7d + 24)$$

$$14 - d = \frac{1}{2}(7d - 24)$$
$$d = \frac{1}{9}(2 \times 14 - 24) = \frac{4}{9} = 0.444$$
$$\hat{S}_1 = 13.556$$

Applications: Loans

Loans

- Simple loan
- The American repayment method
- The French repayment method
- Repayment with variable payments
- Repayment using constant loan reductions
- Changing interest rates
- Loan deferments
- Constant repayment amounts
- Early cancellation

Repaying a Loan

- In this class we will study different types of loans
- But, don't worry, you already know all the financial mathematics we will be using. In fact, we have already solved several problems involving loans
- What will be learn then? That there are several ways to repay a loan. When taking a loan, the contract generally specifies how the loan is to be repaid
- It is important to distinguish between the different ways of repaying a loan and to be able to manage increasingly complex financial transactions

Repaying a loan: structure

A loan is a financial transaction in which one of the parties (the lender) commits to giving a dated payment (C_0, t_0) to a counterparty (borrower) who commits to returning it over a given period (from t_0 to t_n) together with interest

Elements of a loan

Provision Dated payment, (C_0, t_0), provided by the lender at the start (origin) of the loan

Compensation Dated payments $(C_1, t_1), \ldots, (C_n, t_n)$ provided by the borrower. These payments are called "repayment amounts", and are normally paid at the end of each period (month, quarter, year, ...). Their purpose is to repay or cancel the loan

Financial Law One uses compound capitalization, although in very short term loans both parties may on occasion contract to use simple capitalization

Financial Equivalence Provision and compensation, as in every financial transaction, must be financially equivalent

Simple Loans

- Recall that you already know how to work with simple loans, that is, with loans with a single repayment amount, which includes the amount of the loan plus interest
- Example: You take out a loan of 3000€ for 5 years at a 6% efective APR.
- Solution: in 5 years you make a single payment of

$$3000\,(1+6\%)^5 = 4041.67€$$

- Nevertheless, most loans are never paid back in a single payment, but rather, are returned using several payments spread over a certain amount of time

Variables to consider when working with loans

- At time t you have
  - C_t: loan outstanding (pending)–amount of money the borrower owes the lender at the beginning of period t (before payment at date t)
  - a_t: total amount paid at date t. Includes interests and amortization (loan reduction)
  - I_t: interest payment–amount of money paid to the lender that corresponds to interest for period t
  - A_t: loan reduction–amount of money paid to the lender to reduce or cancel the amount owed
  - M_t: repaid loan–part of the loan (in €) that has been paid back to the lender

Variables and the evolution of a loan

Things to veryify

- Repayment amounts are used to pay interest and reduce the loan
- Interest payments are computed on the basis of amount of the loan outstanding
- The algebraic sum of all loan reductions is equal to the loan (the amount borrowed at the beginning)
- The debt outstanding for one period equals the debt outstanding in the period before before minus the loan reduction in that last period

Repayment Schedule

It is useful to depict the entire evolution of the loan variables from the beginning of the loan to the final cancellation. This is done on a table whose estructure is usually like the following

Period	Repayment Amounts	Interest Payment	Loan Reduction	Loan Outstanding	Repaid Loan
1	a_1	I_1	A_1	C_1	M_1
...	...	...	...	...	...
t	a_t	I_t	A_t	C_t	M_t
...	...	...	...	...	...
n	a_n	I_n	$A_n = C_n$	C_n	$M_n = C_0$

Repayment methods

- Loan contracts usually specify the method by which the loan will be repaid
- There are a number of ways in which a loan can be repaid
  - Single cancellation payment: the loan is repaid with a single cancellation payment. Different ways to pay interest lead to
    - Simple loan: all accumulated interest payment is paid one time, at the same time as the loan cancellation
    - American method: interest payments are made periodically while there is only one cancellation payment, at the end of the repayment period
  - Loan repayment by periodic installments
    - French method: constant repayment amounts
    - Variable periodic installment payments
  - Constant loan reduction method

Simple loan

- Simple financial transaction in which the provision and the compensation are a single dated payment: C_0 and C_n respectively
- By financial equivalence and compounding

$$C_0\left(1+r\right)^n = C_n$$

- The reserve (or loan pending) at date t is equal to

$$C_t = C_0\left(1+r\right)^t = C_n\left(1+r\right)^{-(n-t)}$$

- This is a special case where the repayment amounts are all zero, except the last one, and this last one includes all accrued (accumulated) interest, that is

$$\begin{aligned} a_1 &= \ldots = a_{n-1} = 0 \\ a_n &= A_n + I_n,\ A_n = C_0 \end{aligned}$$

American method

- This method has the peculiarity that the borrower makes regular interest payments on the total amount borrowed and cancels the loan all at one time, at the end
- From this it follows that all repayment amounts except the last one only include interest payments

$$\begin{aligned} a_t &= I_t = rC_0 \text{ for } t = 1, \ldots n-1 \\ a_n &= A_n + I_n,\ A_n = C_0 \end{aligned}$$

- As there are no loan cancellations prior to the end of the repayment period, all intermediate loan reductions are zero

$$A_1 = \ldots = A_{n-1} = 0$$

  which implies that the loan outstanding at the end of all these periods is the amount borrowed

$$C_1 = \ldots = C_{n-1} = C_0 \qquad C_n = 0$$

Example

Construct the repayment schedule for an 80.000€ loan with a 5 year repayment period that uses the American method, at a 6% nominal APR and annual interest payments

Period	Repayment Amounts	Interest Payment	Loan Reduction	Loan Outstanding	Repaid Loan
1	4800	4800	0	80000	0
2	4800	4800	0	80000	0
3	4800	4800	0	80000	0
4	4800	4800	80000	0	80000

Example

In order to finance the purchase of new equipment whose value is 2500€ today you arrange a loan with the supplier. This is a one year loan with a 9% nominal APR and interest payments every 2 months. You will pay only interest during for the first 5 payments, and will repay the loan at the end of the year. Determine the repayment amounts and construct the repayment schedule

$$r = \frac{9\%}{6} = 1.5\%, \quad a_t = 2500 \times 1.5\% = 37.50€$$

Example

Period	Repayment Amounts	Interest Payment	Loan Reduction	Loan Outstanding	Repaid Loan
1	37.50	37.50	0	2500	0
2	37.50	37.50	0	2500	0
3	37.50	37.50	0	2500	0
4	37.50	37.50	0	2500	0
5	37.50	37.50	0	2500	0
6	37.50	37.50	2500	0	2500

Bonds are loans

- When a government or a corporation need financing they issue securities (shares, bonds, ...)
- The buyer of these securities is lending money to these institutions in exchange for the right to collect regular payments plus the nominal value of the loan (except with shares)
- From the point of view of loan repayment
  - If the bond makes regular coupon payments, it corresponds to a loan using the American method
  - If it is a zero coupon bond it corresponds to a simple loan

Constant repayment amounts–French method

- This loan has constant repayment amounts so that

$$a_1 = a_2 = \ldots = a_n = a$$

- As provision and compensation have to be financially equivalent

$$C_0 = PV\left((a,1),(a,2),\ldots,(a,n)\right)$$

- If we assume a constant discount rate the computations for this loan are relatively easy
- Using financial equivalence and the fact that repayment amounts are constant

$$C_0 = a \cdot a_{n|r} \Rightarrow a = \frac{C}{a_{n|r}}$$

- Note that annual interest rates for monthly payments are usually nominal APRs

Constant repayment amounts

- The mathematical reserve or loan outstanding can be computed in several ways
  - Retrospective method
  $$C_t = C_0\left(1+r\right)^t - a \cdot s_{t|r}$$
  - Prospective method
  $$C_t = a \cdot a_{n-t|r}$$
  - Recurring method
  $$C_t = C_{t-1}\left(1+r\right) - a$$

Analysis

- If we compare the reserve for two consecutive periods

$$\begin{aligned} C_{t+1} &= C_t\left(1+r\right) - a \\ C_t &= C_{t-1}\left(1+r\right) - a \end{aligned}$$

  we notice the following about the difference

$$C_{t+1} - C_t = \left(C_t - C_{t-1}\right)\left(1+r\right)$$

  that is, the difference does not depend on the repayment amounts

- As $C_{t+1} = C_t - A_{t+1}$, this implies that two subsequent loan reduction payments are also independent of the repayment amount, in particular

$$A_t = A_{t-1}\left(1+r\right).$$

  Thus the loan repayment amounts are described by a geometric sequence

Example

Construct the repayment schedule for a 1 million euro loan that is repaid by four annual constant repayment amounts, where the loan rate is an 8% effective APR.

- First we determine the repayment amounts and then construct the schedule

$$a = \frac{C_0}{a_{n|r}} = \frac{1000000}{a_{4|8\%}} = 301921$$

Example

Period	Repayment Amounts	Interest Payment	Loan Reduction	Loan Outstanding	Repaid Loan
0	- €	- €	- €	1,000,000 €	
1	301,921 €	80,000 €	221,921 €	778,079 €	221,921 €
2	301,921 €	62,246 €	239,674 €	538,405 €	461,595 €
3	301,921 €	43,072 €	258,848 €	279,556 €	720,444 €
4	301,921 €	22,365 €	279,556 €	0 €	1,000,000 €

Example

You take out a loan for 30.000€ to be repaid over 30 years with monthly constant payments. Compute the monthly payment amount and the debt outstanding after 15 years

Example

- Calculating the payment amount

$$r = \frac{6,80\%}{12} = 0,567\% \quad a = \frac{C_0}{a_{30\times12|0,567\%}} = 1955,77€$$

- Debt outstanding

$$C_{15} = 1955,77 a_{15\times12|0,567\%} = 220.322,78€$$

Changing Interest Rates

- The (annual) forward curve is:

t=1	2	3	4
5%	4.5%	4%	4.5%

- Construct the repayment schedule for a a 35.000€ euro loan that is repaid by four annual constant repayment amounts
- Using financial equivalence

$$
\begin{aligned}
35.000 &= a\left[\frac{1}{1+5\%}+\frac{1}{1+5\%}\frac{1}{1+4.5\%}+\frac{1}{1+5\%}\frac{1}{1+4.5\%}\frac{1}{1+4\%}\right. \\
&\qquad \left.+\frac{1}{1+5\%}\frac{1}{1+4.5\%}\frac{1}{1+4\%}\frac{1}{1+4.5\%}\right] \\
a &= \frac{35.000}{3.5786}=9780,23€
\end{aligned}
$$

Example

Period	Repayment Amounts	Interest Payment	Loan Reduction	Loan Outstanding	Repaid Loan
0	- €	- €	- €	35,000 €	
1	9,780 €	1,750 €	8,030 €	26,970 €	8,030 €
2	9,780 €	1,214 €	8,567 €	18,403 €	16,597 €
3	9,780 €	736 €	9,044 €	9,359 €	25,641 €
4	9,780 €	421 €	9,359 €	- €	35,000 €

Loans with variable repayments

- The French method is the usual method for fixed interest rate loans, as well as for variable and personalized loans
- The usual interest rates used in actual loans tends to be variable, and a function of some reference index (like one year Euribor), for example Euribor + 0.25%. The usual references are the one year Euribor and the six month Euribor
- When the interest rate is indexed to Euribor, the way to compute the loan repayment amounts for each period is to apply the French method on each of the periods, taking the loan outstanding as the amount used to determine the payments at each moment in which the interest rate is revised, and taking the current interest rate as if it were to hold for the remaining live of the loan

Variable interest rate mortages

Example

You obtain a loan from this institution for 400.000€ which you will pay back over 35 years in monthly installments. The interest rate for the first semester will be calculated using the Euribor published in the BOE on March 3, 2009 (2,135%) plus the premium. The interest is revised every six months.

1. Compute the monthly payments for the first six months.
2. If the Euribor used as reference changes after six months to 1.5%, then the monthly payments will change. Compute the new montly payment.

Loan Deferments

- In certain loans, the lender gives the borrower conditions designed to facilitate repayment. Some such conditions involve the non-payment of interest obligations (payments), other times there is non-repayment of principal, or no payments at all over a specified period of time (grace period)
- Nevertheless, this does not mean that the borrower is exempt from the obligations derived from these non-payments. What happens is that the obligations are transferred to later dates

Loan Deferments

- Types of deferments
  - Partial deferment: during the initial periods (grace period) the borrower is exempt of principal repayment. This implies that repayments amounts are equal to interest payments. It also means that the loan outstanding does not change and is equal to the initial loan amount
  - Total deferment: during the grace period de borrower does not have to make any payments = zero repayment amounts. As nothing has been paid, not even interest, at the end of the grace period the amount owed has increased to include the unpaid interest

Example

Confirm the terms of the loan if you borrow the 1000€

Loan repayment with compound increasing payments

- These consist of loan terms that involve regular payments that are increasing as a geometric sequence
- The equation of financial equivalence is essentially equivalent to the one with constant payments, adjusted for the growth rate

$$C_0 = A\,(a,q)_{n|r}$$

$$C_0 = a\frac{\left(1-\left(\frac{1+g}{1+r}\right)^n\right)}{r-g}$$

Example

Michael wants to buy a car with a price of 18600€ and has several financing options. The option that best fits his lifestyle is one that involves a 600€ payment at the moment of purchase, and six semi-annual payments which increase by 2% every six months, at a 12% nominal APR. Compute the loan payments.

Constant Loan Repayments

- This a type of loan where the borrower makes principal loan repayments in every period, all of the same amount
- This implies that we are working with constant loan repayments–in each period the amount of the loan that is repaid is always the same
- Working with this condition as premise, we can derive the value of the other variables from the payment schedule

Constant Principal Repayment

- To construct the schedule we start with the principal repayment

$$C_0 = \sum_{t=1}^{n} A_t = nA \Rightarrow A_t = \frac{C_0}{n}.$$

- Then, the loan outstanding at the end of each period is $C_1 = C_0 - A$, $C_2 = C_1 - A = C_0 - 2A$

$$C_t = C_{t-1} - A = C_0 - tA.$$

- The interest obligation and loan amounts are

$$I_t = rC_{t-1}, \qquad a_t = I_t + A_t = rC_{t-1} + A$$

Example

Construct the loan schedule for a 3.000€ loan which is repaid in 5 payments (one per year) with constant principal repayment amounts and a 16% effective APR

Effective rates

- In previous classes we saw how to determine the effective rate for varying financial transaction and with varied clauses and conditions
- In any operation we can compute the effective rate for the lender,

Actual provision provided by the lender	= financially = equivalent	Actual compensation received by lender

Effective rates

- Similarly, the effective rate for the borrower is

Actual provision received by the borrower	= financially = equivalent	Actual compensation paid by borrower

- Note that these effective rates are computed using effective APR and compounding. If dated payments in our transaction are more frequent, say monthly, then the equation is setup with a monthly interest rate and this is then transformed into an effective APR

Example

- Mr Sanchez wants to buy a house priced at 300.000€. In order to finance the purchase he requests a loan from a financial institution which offers the following terms
  - Amount lent: 80% of the value of the house
  - Nominal APR: 3.1%
  - Loan period: 6 years
  - Repayment amount: constant amount each month
  - Guarantees: personal from Mr Sanchez plus two additional guarantors
  - Loan set up fee: 0.5% of the principal, payable at the start of the loan
- Determine
  - the repayment amounts
  - effective interest rate of lender

Early Cancellation

- You have obtained a 75.000€ loan. The terms of the loan are
  - Total deferment for the first three years
  - After the first three years, constant monthly payments for 12 additional years
  - For the grace period and for the first year of monthly payments, the interest is 4.1% nominal APR. Thereafter it will be EURIBOR + 0.75%
  - The setup costs for the loan are 1% of the principal
  - In case of early cancellation the penalty is 1% of the amount cancelled

Example

- Compute
  - Monthly payments for the fourth year of the loan
  - After the first four years of the loan, the interest rate is 3% nominal APR. Determine
    - The amount of the new monthly payments (if you keep the same termination date for the loan)
    - If you repay an extra 18.000€ on the principal with the last payment in year 4, determine the new monthly payments (if you keep the termination date the same)
    - If, when you repay the extra 18.000€ on the principal, you reduce the termination date of the loan and keep the same monthly payments, determine the number of years left on the loan