Matemáticas Financieras

Introduction

Class Notes

Outline

Contents

Overview

Instructor

- The instructor for this course
  - José Penalva
- Office Hours: Wednesday 11:00-12:00 & Friday 10:30-11:30 (**online**, for a person-in-person meeting you need to make an appointment at least 48 hours in advance to reserve an appropriate office).
- Materials and announcements – on Aula Global.
- Avoid email **except to make an appointment**:
  - do not expect emails to be responded to outside regular working hours (no nights and/or weekends)
  - do not expect anything but brief and kurt responses
  - do not expect responses from emails requesting information that is available on the university webpages (exams+holidays are on the university web, problems and class notes are on Aula Global).

Calculators and Computers

- Electronic aids: (must-have) **CALCULATORS** [normal scientific ones ... NOT too sophisticated] and a **LAPTOP**... Absolutely NO phones, electronic dictionaries, tablets, smart watches, ... during exams
- ALWAYS bring **BOTH** a calculator and a **fully charged laptop** to the problems class – we will use both to do problems and you need to work with them **individually**. If you do not have access to a laptop, borrow one from the **library**.

The Course

- The degree:

  GRADO EN **FINANZAS** Y CONTABILIDAD
  UNDERGRADUATE IN **FINANCE** AND ACCOUNTING

- This course provides the mathematical methods and techniques that are the foundations of the financial calculations for your degree and its application in jobs and your personal life
  - calculate payments from a savings/pension plan
  - calculate payments from a mortgage
  - calculate the interest rate in a loan
  - calculate the return on an investment
- This is more about finance than math ... the math is relatively simple, as you will see

Primary Objectives

- Introduce students to rigorous problem solving
- Provide the skills and knowlege to analyze basic financial problems and calculations in a riskless environment
- Set the basis for additional material in future courses on asset valuation (Economía Financiera and Gestión Financiera)
- Understand basic investment principles
- Understand basic financial instruments used by firms

Evaluation: Grades and Exams.

- The final class grade is determined as a combination of (% of final grade)
- Grade allocation (standard grades)
  - 50% final exam (with a minimum score of 4 out of 10)
  - 10% is obtained from online evaluations (quizzes)
  - 40% is obtained from 3 partial exams (tests during class time)
- Quizzes (10%)
  - the course is split into 3+1 blocks
  - in the first three blocks there are 3 quizzes each (total 9) and a 10th after the 3rd partial exam
  - you need to do them all — any missing quizzes count as 0
  - your grade is the arithmetic average of these 10
  - Your grade in the 10th quiz (if high) may help you if the rounding in your final grade matters
- Partial exam (40%)
  - 40% is obtained from 3 tests during class time WITH COMPUTERS–**bring a laptop**–if you need one you can borrow one from the library
  - They will include long questions to be done **on paper**

    * October 14th, class time (friday) **ROOM TBA**
    * November 11th, class time (friday) **ROOM TBA**
    * December 9th, class time (friday) **ROOM TBA**

- The three partial exams will not be weighted equally
  - We will take the best two grades
  - Each of these two grades will be worth 20% (40% total)
- You can miss one, the other two will count – no excuses needed, no excuses wanted
- The final exam (50%):
  - minimum score is 4 out of 10.
  - If you score less than 4 out of 10 your CLASS GRADE will be 100% the one in the final exam
  - It will include long questions to be done **on paper**
- Ordinary final exam: TBA
- Extra-ordinary exam: TBA (june)

Class Rescheduling

- The following classes will be rescheduled (rooms will be announced via Aula Global)
  - [NONE so far]

Organization of Learning

- The course is organized into three main blocks plus two additional periods:
  - Begining of term: week 1
  - Block 1: weeks 2-5
  - Block 2: weeks 6-9
  - Block 3: weeks 10-13
  - End of term: week 14
- Each main block consists of four weeks with one quiz each week for the first three weeks and a partial exam in the fourth week

A Standard Week

- Wednesday: new material is provided in the theory class (online)
- Exercises
  - Basic exercises provided in the form of auto-evaluation tests which provide solutions and are not timed or limited
  - Basic online exercises are NOT enough to pass the test ... just basic exercises to prepare you to do the **problem sets,** which are more representative of actual exam questions
  - Problem sets: similar to the exam questions, specially the written ones
- Friday: reduced groups meet in classroom. Objectives:
  - discuss issues from materials from the previous week (specially quiz questions)

  - discuss problems with the auto-evaluation results and the theory
  - work through more complicated and longer exam-type questions

- By Tuesday of the next week you have to complete the quiz: the quiz is done online, has a time limit, and is done in a secure exam-type environment and is similar to basic exercises plus some exam-like questions.
  - A quiz is NOT representative of a partial exam

Exercises

- There are basic exercises to practice key concepts on Aula Global. You should do these and repeat them until you know how to do them correctly.
- In addition, you will receive problem sets with applied exercises to give you a flavor of the kinds of questions to expect in exams. You should do as many as you can.

- **DETAILED SOLUTIONS** to **select** problems and questions are given out **only during class** time. If you want the detailed solution to a particular problem, bring YOUR solution to class or to office hours and it will be checked and corrected. The answer (final number only) will be made available for **ALL** problems. **Detailed solutions will not be posted on the web or Aula Global** (and never sent by email).
- We will make available all exercises prior to the practice class so that you can do them at home
- During class we will do some of these and spend time on solving them–you will be responsible for knowing how to do ALL exercises (see above for solutions) as well as new ones that use the same concepts
- Primarily, I will solve problems you have trouble with and/or propose a variation of an exercise from the take-home for you to do during class time

Start of Class Exercise

- At the start of class, every day, there will be one exercise to be done online
- The exercise will be timed and will be short
- Class partition matters: positive contributions will help your grade, negative contributions (being late, causing interruptions, ...) will harm your grade
- These factors will be used to round off continuous evaluation grades and will affect the instructor's availability to help you outside of class and office hours

1 Introduction

Mathematics in Financial Mathematics

- The **mathematics** you will be using in this class are relatively simple and you should already be familiar with them
- Primary exercises:
  1. Work with powers: Compute the answer to: $2100(1+0.05)^8 = ...$
  2. Solve simple equations: Solve for x in: $x(1+0.03)^7 = 12870$
- Calculations in a "complex" question. Solve:

$$1+r=\sqrt{\frac{51130}{46590\,(1+0.5\%)}}$$

- The complex part of the course is the **financial** part, which you will need to translate a problem into mathematics and then solve: Suppose you bought shares of a company on 1/1/2010 and paid 46590€ for them. Six months later you decided to sell them and obtained 51130€. Determine the real effective APR for you from the transaction taking into account that you have to pay a broker fee of 0.5% of the total cash value of transactions at the moment of purchase.

Introduction to Financial Mathematics

1. The elements of a financial transaction
2. Time value of money and financial equivalence
3. Financial Laws

The elements of a financial transaction

Fortune Smiles upon You

You have just received an email informing you that you have won a 100€ price. You can choose whether you want to receive the 100€ today, or if you prefer to receive 100€ two years from now.

When will you prefer to receive the money $\Downarrow$ If you prefer the money today

This is because you, even if you are not conscious of it, already know some Financial Mathematics. You already grasp a fundamental concept of Financial Mathematics.

One Euro today is worth more than one Euro in the future

Financial Transaction

A Financial Transaction is an intertemporal exchange of dated payments

- You had 100€ and you saved them.
- In effect, you have loaned your money to the bank.
- In exchange, the bank will pay you some compensation, for example a 10% annual interest rate
- If you leave your money in the savings account, next year you will have a balance on the account of

$$\text{Balance}_1 = 100 + 10\% \times 100 = 110€$$

Interest

- When you lend money, the amount you lend is the **principal.**
- The lender sacrifices consumption today and gives the borrower the opportunity to consume today (the money).
- It is logical for the borrower to obtain some gain from this transfer of consumption possibilities and to put a price for giving up his own consumption.
- This price is called **interest**. Interest is the price a lender charges others for the use of his money.

Interest

- Interest can be expressed as a percentage, 3.5%, or in basis points, 350 bps
- If the interest rate goes from 3.5% to 4%, the increase can be expressed as
  - The rate of interest has increased 0.5%
  - The rate of interest has increased 50 bps

Elements of a financial transaction

- In this simple example you will find all five elements that define a financial transaction

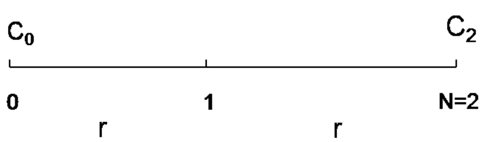
Provision	You have deposited 100€
Financial law	The bank pays you 10% annual interest
Compounding Period	The interest is paid once a year
Time	You leave the money there for 2 years
Compensation	Two years later you receive 121€

- Financial mathematics allows us to obtain one element if we know the other four

Determine the compensation

Let

- C_0 denote the provision
- r be the interest rate for one (compound) period
- compound financial law
- N the number of periods the transaction lasts

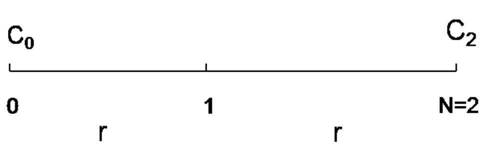


Knowing that today you deposit 100€ (known provision) at 10% interest (compound financial law) for 2 years (time), Financial Mathematics allows you to determine that the bank will pay me back 121€

Determine the interest rate

Let

- C_0 denote the provision
- r be the interest rate for one (compound) period
- compound financial law
- N the number of periods the transaction lasts

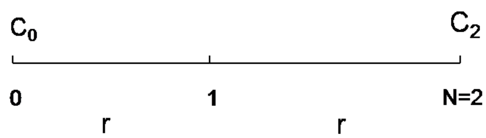


Knowing that today you deposit 100€ (known provision) for 2 years (time), and that the bank has paid you back 121€ using the compound financial law, I can obtain the interest rate using Financial Mathematics

Determine the time

Let

- C_0 denote the provision
- r be the interest rate for one (compound) period
- compound financial law
- N the number of periods the transaction lasts

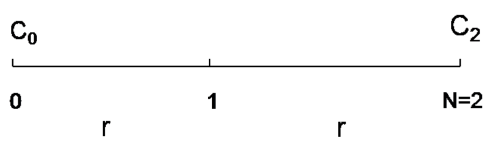


Knowing that today you deposit 100€ (known provision), that the bank gives you a 10% interest rate, and that the bank has paid you back 121€, Financial Mathematics allow me to obtain the amount of time the money was at the bank

Determine the provision

Let

- C_0 denote the provision
- r be the interest rate for one (compound) period
- compound financial law
- N the number of periods the transaction lasts



Knowing that today you obtain 121€ (known compensation) at 10% interest (known financial law) for 2 years (time), Financial Mathematics allows you to determine that you deposited 100€

The players in a financial transaction

- The players (parties) in the above financial transaction are:
  - You, the **lender**, who financed the operation (loan a financial amount)
  - The bank, the **borrower** who pays back the initial financing via a single payment to the lender, a payment that has been obtained according to a given financial law and over a given amount of time.

Degrees of Complexity in a Financial Transaction

- Exchange single dated payments: C
  - 1 period of length τ, interest rate per τ, capitalization period τ
  - n periods of length τ, interest rate per τ, capitalization period τ
- Multiple dated payments: $C_1, \ldots, C_n$

- each C_j; n_j periods of length τ, interest rate per τ, capitalization period τ

- Exchange single dated payments: C
  - n periods of length τ_0, interest rate per τ_1, capitalization period τ_1
  - n periods of length τ_0, interest rate per τ_1, capitalization period τ_2
- Multiple dated payments: $C_1, \ldots, C_n$
  - each C_j; n_j periods of length τ_0, interest rate per τ_1, capitalization period τ_2

Time value of money, financial equivalence

Financial capital–dated payment

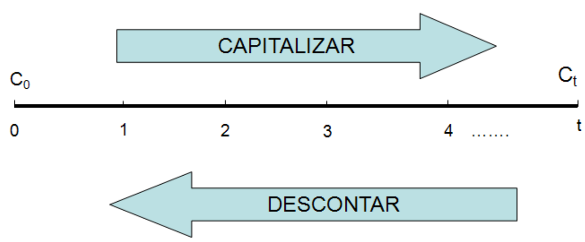
- If we return to your 100€ price, the economic value of this amount does not depend solely on the 100€, but also the time you receive this amount.
- In our example, you were offered 100€ today (100,0); you were also offered the same 100 euros two years from now (100,2). It is obvious that these are different financial amounts.
- Why? Because of the **time value of money**.

Time value of money

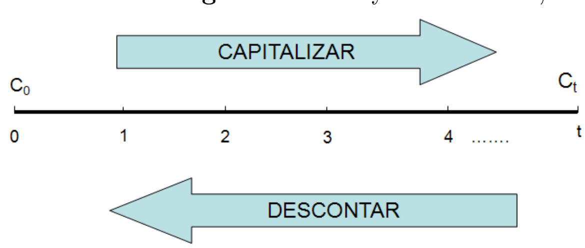
- Economic agents, when selecting between equal amounts of money, prefer to receive the money before rather than after.
- Time influences decisions in that the present is preferred to the future (and also more is preferred to less)
  - (100€,1/1/08) vs (100€,1/1/07)
  - (150€,1/1/08) vs (100€,1/1/08)
  - (100€,1/1/08) vs (105€,1/1/09) ¿?
- We need a way of determining given an amount at date t what is the **equivalent** amount at $t+1$, or viceversa.

Two types of financial valuations

- In order to determine the final value (C_t) we have to move money into the future. This is called **capitalization**: "100€ today, one year from now has a capitalized value of 110€, at a 10% annual interest."



- In order to determine the present value (C_0) we have to move money backwards in time. This we refer to as **discounting**. "110 € one year from now, has today a value of 100€, at a 10% annual interest."



.... and using a financial law which, for now, does not matter.

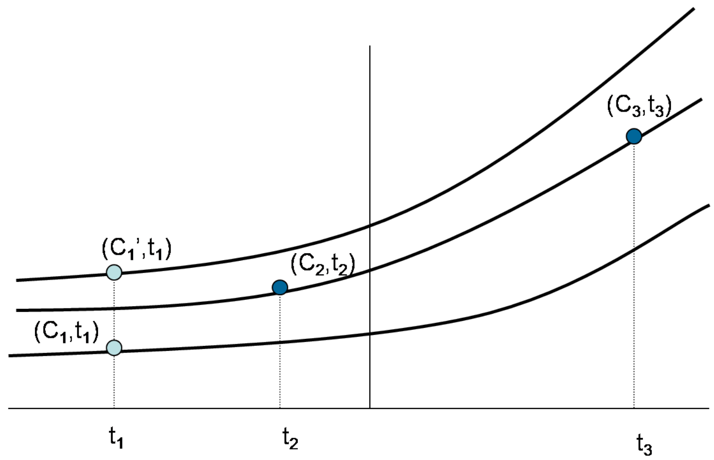


Figure 1: Graphically it is relatively easy to compare financial amounts, but how about numerically?

One period capitalization and discounting

- If the interest for one period is r then

  Capitalization X€ at the beginning the period are equivalent to $X(1+r)$€ at the end of the period

  Discounting Y€ at the end of the period are equivalent to $Y/(1+r)$€ at the beginning of the period

- Example:
  - if the two-year interest rate is 12% then 12000€ today is .....€ in 2 years
  - if the 6months interest rate is 4% then 800€ on 1/1/2019 is .....€ on 1/7/2018

Time value of money

- We have seen that if we were to deposit 100 euros at an annual interest rate of 10% we would obtain 110€ after one year.
- This means that (100,0) and (110,1) are financially equivalent. For all those that so wish (and can obtain a 10% interest rate) having 100€ today is equivalent to having 110€ one year from today.

Financial equivalence line

Future & Present Value

Future value (FV) and present value (PV)

- The future value of an investment is the amount of money you can obtain at the end for the investment plus interest.
  - 100€ invested for a year at a 5% annual interest rate becomes 105€.
  - An investor is indifferent between receiving 100 today or 105 one year from now, but not between 100 today and something a little less than 105 one year from now.
- The present value of 105€ one year from now, given a 5% interest rate is 100€

BOTH FINANCIAL PAYMENTS ARE FINANCIALLY EQUIVALENT AT A 5% INTEREST RATE AND A FINANCIAL LAW WHICH WE WILL NOW CONSIDER.

Financial Laws

Financial equivalence and financial laws

- A financial law is an agreement between two parties in a financial transaction on the method that is used to move money over time (also called accumulation or discounting function).

In a financial transaction the provision and the compensation are financially equivalent: "what you give is what you get"

- Regardless of how simple the financial transaction, financial equivalence between a provision C_0 today, and another payment t periods from now (which defines the compensation, C_t) requires knowing the financial law that is to be used.
- Let us take advantage of the calculations we made in today's financial transaction.
- There are only two ways to move money over time: forwards (towards the future) and backwards (towards the past).

Future value (FV) and present value (PV)

- The future value of an investment is the amount of money you can obtain at the end for the investment plus interest.
  - 100€ invested for a year at a 5% annual interest rate becomes 105€.
  - An investor is indifferent between receiving 100 today or 105 one year from now, but not between 100 today and something a little less than 105 one year from now.
- The present value of 105€ one year from now, given a 5% interest rate is 100€

BOTH FINANCIAL PAYMENTS ARE FINANCIALLY EQUIVALENT AT A 5% INTEREST RATE AND A FINANCIAL LAW WHICH WE WILL NOW CONSIDER.

Multiple Periods: Financial laws

- There are financial laws for capitalization and for discounting.
- Capitalization laws
  - Simple capitalization
  - Compound capitalization
- Discounting laws
  - [Simple commercial discounting]
  - Simple rational discounting
  - Compound discounting

Multiple Periods: Financial laws

- The main financial laws go in pairs: discounting being the reverse of capitalization
- Simple (rational) Laws
  - Simple capitalization
  - Simple rational discounting
- Compounding laws
  - Compound capitalization
  - Compound discounting
- Simple commercial discounting – only used for discounting

2 Classical Financial Laws

Class 2

Simple vs Compound Financial Laws

Capitalization

- Financial operations can be arranged under different capitalization regimes (rules for generating interest)
- Capitalization is to add the interest earned to an investment
- We distinguish two financial capitalization laws:

  Simple Capitalization: used for short-term transactions (less than a year), such as savings accounts, short term bonds, fixed return deposits with less than a one year horizon

  Compound Capitalization: used for transactions with a one year or longer horizon (fixed rate deposits, computation of annual payments (annuities)).

- Nevertheless, this is not the primary difference between simple and compound capitalization. The main difference lies in what happens to the interest payments generated from the transaction.

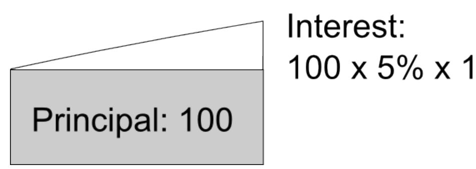
Capitalization

- Financial transactions may use different procedures for the capitalization of interest (interest generation)
- Capitalization is defined as the process of adding the interest generated by an initial capital/investment to that same capital/investment
- We distinguish two financial capitalization systems:
  - Simple Capitalization
    * Transactions that last less than a year: current accounts, letters of crédit, lines of credit, short-term government debt, saving deposits for less than a year.
  - Compound Capitalization
    * Transactions that last for a year or more: savings accounts, annuities, debt repayment, setting up a fund.

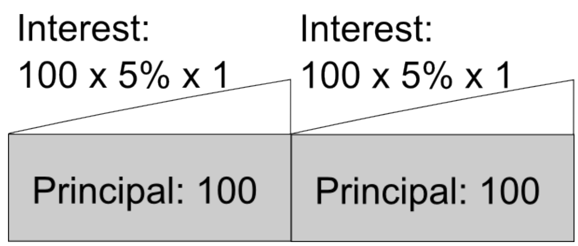
Simple Capitalization

- In transactions that operate under simple capitalization, the interest payment is computed as a constant proportion of the initial investment
- If r denotes the interest rate, n the duration of the transaction (in units of the compounding period), and C the amount invested, then

$$FV = C\left[1 + rn\right] = \underbrace{C}_{\text{principal}} + \underbrace{C \times r \times n}_{\text{interest}}$$



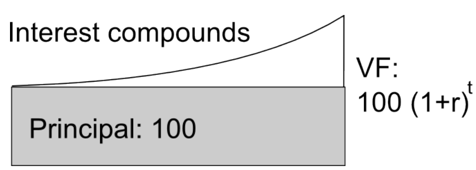
Simple Capitalization



Annual Compounding

- In transactions with compounding: at the end of the year you compute the interest payments, and these are added to the amount invested.
- An amount C invested at an annual nominal interest rate r over n years grows as:

$$FV = C\left(1 + r\right)^n$$



Financial Factor

Financial factors

- Simple capitalization

$$100\left(1 + 0.05 \times \mathbf{2}\right) = 110$$

Capitalization factor is

$$\left(1 + 0.05 \times \mathbf{2}\right) = 1.10$$

- Compounded capitalization

$$100\,(1+0.05)^{\mathbf{2}} = 110.25$$

Capitalization factor is

$$(1+0.05)^{\mathbf{2}} = 1.1025$$

Capitalization vs Discounting

Capitalization and Discounting

- Take as fixed the financial law (compounding), a compounding period (1 year), an interest rate (5%), time (2 years)
- For a given provision (100€ today) you can determine the financially equivalent compensation, x, using:

$$100\,(1+5\%)^2 = x$$

where x is the FV of 100€ capitalized over 2 periods

- Similarly, if I told you that after two years you had 1102,50€ in the account and I asked you what the initial deposit was, you could answer by solving for y in

$$y\,(1+5\%)^2 = 1102,5$$

Capitalization and Discounting

- or you could solve the equivalent equation

$$y = \frac{1102,5}{(1+5\%)^2} = 1000.$$

- You have just determined the Present Value of 1102,5€ in two years, and you have used the financial law for compound discounting
- Four laws work in pairs in this way
  - Compound capitalization and compound discounting
  - Simple capitalization and simple rational discounting
- Simple commercial discounting has not equivalent capitalization law

Financial factors

- Simple discounting

$$100 = \frac{110}{1+0.05\times\mathbf{2}}$$

Discount factor is

$$\frac{1}{1+0.05\times\mathbf{2}} = \frac{1}{1.10}$$

- Compound discounting

$$100 = \frac{110.25}{(1+0.05)^{\mathbf{2}}}$$

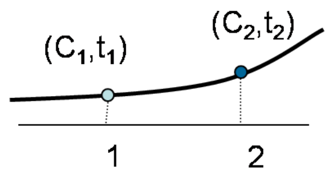
Discount factor is

$$\frac{1}{(1+0.05)^{\mathbf{2}}} = \frac{1}{1.1025}$$

Future and Present Value (cont)

Future Value (FV) and Present Value (PV)

We have two ways to compare payments (C_1, t_1) and (C_2, t_2)



1. Capitalize and compare in the future (use capitalization factor)

$$C_1 (1+r)^{t_2 - t_1} \qquad \text{or} \qquad C_1 (1 + (t_2 - t_1) r)$$

2. Discount and compare in the past (use discount factor)

$$C_2 \frac{1}{(1+r)^{t_2 - t_1}} \qquad \text{or} \qquad C_2 \frac{1}{(1 + (t_2 - t_1) r)}$$

Future Value (FV) and Present Value (PV)

- The future value of an investment is the amount that you can obtain at the end after you add all interest payments
  - 100€ invested for a year at an annual interest of 5% generates 105€.

$$100 (1 + 5\%) = 105$$

  - An investor is indifferent between receiving 100 today or 105 one year from today, but not between 100 now and a little less than 105 one year from now. Both dated payments are financially equivalent.
- The present value of 105€ one year from now, today at an annual interest rate of 5% is 100€

$$PV = \frac{105}{1 + 5\%} = 100$$

Examples

- Compute the FV of 1000€, five years from today, at 5% annual interest rate with simple capitalization

$$FV = ...$$

- Compute the PV of (5000€, 6 years) with simple capitalization and a 3% annual interest rate

$$PV = ...$$

- Compute the FV of 1000€, five years from today, at 5% annual interest rate with compound capitalization
- Compute the PV of (5000€, 6 years) with compound capitalization and a 3% annual interest rate

Working with Multiple Payments

Financial Sum

- In order to calculate the joint value of several dated payments, one has to move them to the same point in time and then add them.
- We will capitalize or discount, depending on where the investment is located and at which point in time we want the final sum to be located.

Financial sum

- Suppose that you are offered the following two alternatives
  - Option 1, receive: $\{(1000, 2); (8000, 4); (15000, 6)\}$
  - Option 2, receive: $\{(7000, 1); (8000, 4); (7000, 5)\}$
- As you receive the amounts you deposit them with a financial institution that offers you a 4% annual interest with compounding.
  - Which option will you choose?
  - In order to decide we need to determine the value of the financial sum of each of the two options at one point in time

Financial Sum

- Alternatives
  - Option 1, receive: $\{(1000, 2); (8000, 4); (15000, 6)\}$
  - Option 2, receive: $\{(7000, 1); (8000, 4); (7000, 5)\}$
- As you receive the amounts you deposit them with a financial institution that offers you a 4% annual interest with compounding.

$$
\begin{aligned}
FV_{Opt\,1} &= 1.000\,(1+4\%)^4 + 8.000\,(1+4\%)^2 + 15.000 \\
&= 24.822,66 \\
FV_{Opt\,2} &= 7.000\,(1+4\%)^5 + 8.000\,(1+4\%)^2 + 7.000\,(1.04) \\
&= 24.449,37
\end{aligned}
$$

The preferred option is the first one

Present Value

- To compute the present value in the same problem
  - Option 1, receive: $\{(1000, 2); (8000, 4); (15000, 6)\}$
  - Option 2, receive: $\{(7000, 1); (8000, 4); (7000, 5)\}$
- You can compute the present value of each of the financial payments and sum them.

$$
\begin{aligned}
PV_{Opt\,1} &= 1.000\,(1+4\%)^{-2} + 8.000\,(1+4\%)^{-4} + 15.000\,(1+4\%)^{-6} \\
&= \\
PV_{Opt\,2} &= 7.000\,(1+4\%)^{-1} + 8.000\,(1+4\%)^{-4} + 7.000\,(1.04)^{-5} \\
&=
\end{aligned}
$$

Fractional Capitalization

Fractions of a Period

- So far we have calculated the PV and FV using factors that correspond with the law being used, and measuring time by the number of periods we were moving the money
- What happens when you need to capitalize for 4 years and 9 months? How does one compute interest for periods that are less than one year?
- To work with periods of time that are less than a year in an transaction with yearly compounding (one period = one year):
  - Moving one month = moving 1/12th of a year
  - One week = 1/52nd of a year
  - One day = 1/365ths of a year

Simple capitalization

- The interest obtained over a fraction of a year, $\frac{1}{m}$, is:

$$Cr/m$$

  so that the final value after $1/m$ of a year is

$$VF = C\left(1 + r\frac{1}{m}\right)$$

- If you deposit 1000€ at a 5% annual interest rate (with simple capitalization) for 6 months you obtain:

$$VF = 1000\left(1 + 5\%\frac{6}{12}\right) = 1025$$

Example

- You have just invested 10.000€ in a savings account that offers an annual simple interest rate of 9%. ¿What will be the balance on the account after 8 months?
- It comes down to working out the final value after 8 months
  - You take the annual interest rate and capitalize over 8/12 of a year

$$VF = 10.000\left(1 + 9\% \times \frac{8}{12}\right) = 10.600$$

  - Not that can also be interpreted as: transform the interest into a monthly interest and capitalize over 8 months

$$VF = 10.000\left(1 + \frac{9\%}{12} \times 8\right) = 10.600$$

    But this requires a change in the "capitalization" period of the interest rate ... something we will see later.

Actual year or commercial year

- Suppose you have a liquidity crunch and you need 1000€ for 60 days. You go to the bank and ask them for the money. The interest rate is 12% annual.
  - We know that one has to use the same unit of measurement for the interest and time
  - An actual year has 365 days (except leap years that have 366), so that the length of the loan is 60/365 of a year
  $$FV = 10000\left(1 + 12\%\frac{60}{365}\right) = 1.019,72$$
  - Nevertheless, it is common practice to work with a commercial year. A commercial year has 360 days, which are made up of 12 months, each with 30 days. In this case, the duration of the loan is 60/360 of a year
  $$FV = 10000\left(1 + 12\%\frac{60}{360}\right) = 1.020$$

Day Count Conventions

- To determine how many days in a month and in a year to use in our calculations we have to determine what is the **day count convention** for the transaction/interest rate we are working with
- Examples of day count conventions
  - 30:360 (commercial year)
  - 30:365
  - Actual:actual
  - Actual: 360
  - Actual:365
- The first element refers to the number of days in a month and the second to the number of days in a year–how to count days for the numerator and how to count days for denominator when calculating the fraction of a year $\frac{k}{m}$
- The day count convention is only applied on the fractional part of the year–first count years, and then the fraction of the last (or first) year.
- Calculate actual
  1. Calculate number of years: find end date (day-month) that is less than one year from start date (endYr), count number of years from there to the end (endYr to end date)
  2. Calculate fraction of the year: count number of days from date start to endYr and divide by number of days from dayYr to dayYr minus one year
- Calculating 30 day differences with 30:360
  1. Adjust end of month as follows:
     - European 30/360 (30E/360): if day = 31, use day = 30
  2. number of years + (30*(moEnd - moStart) + (daEnd - daStart))/360 — if moEnd < moStart then use 12+moEnd-moStart)

Compound Capitalization

- With a yearly interest rate we use years as the unit of time
- Capitalize 100€ at a 5% annual interest rate with compounding
  - for 2 years
  $$FV = 100\,(1+5\%)^2 = 110,25$$
  - for 2 and a half years
  $$FV = 100\,(1+5\%)^{2.5} = 112,97$$
  - for 6 months
  $$FV = 100\,(1+5\%)^{1/2} = 102,46$$
  - and even for 100 days (with Actual:365)
  $$FV = 100\,(1+5\%)^{100/365} = 101,34$$

Exercises

1. Compute the final value by December 31st, 2015 of 100€ which you will deposit on December 31st, 2013. The interest rate is 5% annual with annual compounding.
2. The equivalent value today of 100€ one month from today using simple capitalization with a 12% annual interest rate (30:360)
3. Total capital you will have in one month time if you invest your money in two financial institutions: one uses 12% simple annual capitalization, while the other uses 12.5% annual compound capitalization. You deposit 1000€ in each institution
4. Mr Smith can save 10,000€ each year, at the end of the year. Compute the total amount of money he will have five years from now (January 1) if he invests his money at 12% annual compound capitalization
5. I have 40,000€ invested in a money-market account which expires in 4 years time. I receive a 7% annual compound interest rate. I have been offered to invest those 40,000€ in a business that will turn it into 140,000€ in 4 years. What do you recommend I should do?
6. A friend has offered me to invest 15,000€ and get 26,435€ back five years from now. Is this an interesting investment opportunity if I want to receive a 10% annual return? How about if I want a 15% return?
7. How many years will it take for my investments to multiply by 3 if I invest at an 8% annual interest rate with annual compounding?
8. Using 5% annual compound capitalization do the following
   (a) calculate the value today of receiving 100€ three months from today
   (b) The present value of 1000€ a year and a half from today
   (c) The future value of 1000€ at the end of one semester

3 Interest Rates

Class 3

Nominal and Effective Interest Rates

Capitalization Period and Interest Rate

- For now we have kept the capitalization period and the time unit of the interest rate the same
- We have used the time unit of the interest rate/capitalization period to determine how to count the number of periods to capitalize/discount whether
  - whole number
  - fractions
- ... but what if the time unit of the interest rate is not the same as that of the capitalization period (of the transaction) ... what if you have an annual interest rate with monthly capitalization?
  - Which one should we use to count the number of periods to capitalize?
  - How do we adjust the interest rate and the capitalization period to be the same?

The Time Unit of a Transaction

- If the time unit of the interest rate is different from that of the capitalization period of the transaction, **financial equivalence** is preserved only if you use the time unit of the **capitalization period**
- This implies that financial equivalence requires you to **change the time unit of the interest rate** to match that of the capitalization period

Nominal and Effective interest rates

- There are **TWO** ways to change the time unit of the interest rate and it depends on whether the interest rate is **nominal** or **effective**
- If you have a nominal annual interest rate and the capitalization period is monthly, you have to use the (equivalent) nominal monthly interest rate to preserve financial equivalence
- If you have an effective annual interest rate and the capitalization is every 6 months, you have to use the (equivalent) effective semestral interest rate to preserve financial equivalence

Equivalent Nominal Interest Rates

- To adjust nominal interest rates you make the change by **multiplying the interest rate** by the ratio of the two time periods
- If you have a **nominal** annual interest rate of r and you want to calculate the **nominal** interest rate over six months then
  - you have the following ratio: 1 semester is $\frac{1}{2}$ years
  - the semester interest rate is $\frac{1}{2}$ times the annual interest rate
- Exercise: convert a 6% annual nominal interest rate into an equivalent semester interest rate
- If you have a nominal interest rate for a trimester (3 months) and you want a nominal interest rate for a quatermester (4 months) then
  - the ratio is 1 quatermester (4 months) is $\frac{4}{3}$ trimesters (3 months)
  - the interest rate per quatermester is $\frac{4}{3}$ times the interest rate per trimester.

- Fractional interest rates for an annual interest rate of r are

$$\text{monthly: } r^{(12)} = \frac{r}{12}, \text{ trimester: } r^{(4)} = \frac{r}{4}, \text{ semester: } r^{(2)} = \frac{r}{2}.$$

Important Note
It is very important to preserve a uniform unit of measurement for the interest rate and time–for financial equivalence, determined by the capitalization period.

If time is measured in years, you capitalize/discount with an annual interest rate
If time is measured in months, you capitalize/discount with a monthly interest rate
....

Equivalent Effective Interest Rates

- To adjust effective interest rates you make the adjust the capitalization factor by **raising it to the power** given by the ratio of the two time periods
- If you have an **effective** annual interest rate of r and you want to calculate the **effective** interest rate over six months then
  - you have the following ratio: 1 semester is $\frac{1}{2}$ years
  - the semester capitalization factor, $1 + r_{sem}$, is a power $\frac{1}{2}$ of the annual capitalization factor, $1 + r_{ann}$,

$$1 + r_{sem} = (1 + r_{ann})^{1/2},$$

Effective Interest Rates

- In general, the **effective interest rate** $r^{(m)}$ **equivalent** to an annual effective interest rate of r

$$r^{(m)} = (1 + r)^{1/m} - 1,$$

  where m is the number of times in which a year is divided
  - To obtain an interest rate per semester, $m = 2$
  - To obtain a weekly interest rate, $m = 52$
  - To obtain a monthly interest rate, $m = 12$

Annual Nominal and Effective Rates

- The most common interest rates used to describe the intererst of a transaction are annual interest rates
- Many times these annual interest rates are **nominal interest rates** (nominal APR-annual percentage rates)
- Some times they are effective. In these cases we talk of the **effective annual interest rate** the APR
- In anglosaxon environments one distinguishes APR and effective APR

  APR The total interest obtained from investing at compound interest rate for a year

  Effective APR (TAE) The annual compound interest rate that equates the provision actually given with the actual compensation received (we will see later)

- In simple cases both the APR and effective APR are the same.

Interest Rates and Financial Laws

Interest Rates and Financial Laws

- As you will have noticed there is a strong relationship between
  - nominal interest rates and the simple capitalization/discounting rule
  - effective interest rates and the compound capitalization/discounting rule
- However, you do find combinations that mix them up
- For example: an amount 1000€ is invested in a deposit account that offers an annual nominal interest rate of 6% for 5 years with monthly capitalization
- Here we have
  - an annual nominal interest rate
  - the compound capitalization financial law
  - a capitalization period of 1 month
- To find the future value of C in 5 years we need to convert everything to the time unit of the financial transaction, given by the capitalization period: one month
  1. convert the interest rate: **nominal** annual to **nominal** monthly: 6% per year $\Rightarrow \frac{1}{12}6 = \frac{1}{2}\%$ per month
  2. convert the capitalization time, 5 years, to months: 5 years $\Rightarrow 5 \times 12 = 60$ months
  3. apply standard compound capitalization

$$FV = 1000 \times \left(1 + \frac{1}{2}\%\right)^{60} = 1348,85€$$

Nominal to Effective Rates

- A usual exercise, related to a key calculation needed when comparing investments or loans is to convert nominal interest rates to effective rates
- The key concept to remember when doing this is that the conversion from nominal to effective (or viceversa) **depends on the transaction**, and more specifically, on the transaction's capitalization period
- As we saw, the capitalization period determines the interest rate in the transaction
  - monthly capitalization requires monthly interest rates
  - yearly capitalization requires annual interest rates
- Thus, to convert a 6% nominal APR into an effective APR
  - for a transaction with monthly capitalization

$$1 + r = \left(1 + 6\%\frac{1}{12}\right)^{12}$$

  - for a transaction with capitalization every four months

$$1 + r = \left(1 + 6\%\frac{4}{12}\right)^{3}$$

- And of course we can change a **1% monthly nominal interest rate into a semester effective rate**
  - for a transaction with annual capitalization

$$1 + r = (1 + 1\% \times 12)^{\frac{1}{2}}$$

  for a transaction with capitalization every 3 months

$$1 + r = (1 + 1\% \times 3)^{2}$$

The Path of the Righteous

- The general rule for converting nominal into effective is:
  - convert given nominal to nominal for the capitalization period
  - **nominal at capitalization period = effective at capitalization period**
  - convert effective at capitalization period to desired effective
- The general rule for converting effective into nominal is:
  - convert given effective to effecitve for the capitalization period
  - **effective at capitalization period = nominal at capitalization period**
  - convert nominal at capitalization period to desired nominal

Compounding in semesters

- To see why the above process work, consider a problem where interest can be generated each semester (semester compounding period)
- An investment of 100 for a year, at a 5% annual **nominal** interest rate with capitalization in semesters will generate payments of $\frac{1}{2}5\%$ (**2,5€**) after 6 months,
- As interest is compounded, at the end of the year you would obtain

$$100\,(1+2,5\%)\,(1+2,5\%) = 100\,(1+2,5\%)^2 = 105,062€$$

- The relevant elements for this transaction are:
  - nominal annual interest rate: 5% [**nominal APR**]
  - effective interest rate per semester: $r^{(2)} = \frac{5\%}{2} = 2,5\%$
  - effective interest rate (annual): $(1+2,5\%)^2 - 1 = 5,062\%$ [**APR**]

Quarterly compounding

- Interest is paid every quarter (every 3 months)
- An investment of 100 for one year at an nominal APR of 5% with quarterly compounding generates interest payments
  - after 3 months: interest on the principal is $r^{(4)} = 5\%/4 = 1,25\%$
  - after 6 months
  - after 9 months
  - after 12 months
- As interest is left in the account and compounds every quarter, at the end the balance on the account will be

$$\begin{aligned}&100\,(1+1,25\%)\,(1+1,25\%)\,(1+1,25\%)\,(1+1,25\%)\\ &= 100\,(1+1,25\%)^4 = 105,094€\end{aligned}$$

- The effective annual interest rate is $5,094\%$

The Devil's Formulas

- Let m be the capitalization period described by the number of periods in one year ($m = 2$ for a semester, $m = 4$ for quarters, $m = 12$ for months, ...) and r is the nominal APR. Then the equivalent annual effective rate, r^* is equal to

$$r^* = \left(1 + \frac{r}{m}\right)^m - 1$$

- And, similarly,

$$r = \left((1 + r^*)^{1/m} - 1\right) m$$

In general

- The above procedure can be generalized for any capitalization period. Let C be an amount of money, m be the number of periods in one year ($m = 2$ for a semester, $m = 4$ for quarters, $m = 12$ for months, ...) and r is the nominal APR

$$VF = C\left(1 + \frac{r}{m}\right)^m$$

- What happens with daily compounding?
- As $m \to \infty$, we get the continuous compounding formula

$$VF = Ce^r$$

Continuous Compounding

- Consider the sequence of equivalent effective rates for a annual nominal interest rate of i: $r_{(1)}$, $r_{(2)}$, $r_{(4)}$, $r_{(6)}$, $r_{(12)}$, $r_{(52)}$, $r_{(360)}$

$$1 + r_{(m)} = \left(1 + \frac{i}{m}\right)^m.$$

- The limit as $m \to \infty$ of the right hand side is the exponential function

$$\lim_{m\to\infty}\left(1 + \frac{i}{m}\right)^m = \exp(i).$$

- That is, the PV of 100€ in 6 months with a continuous compound interest of 3% per year is

$$PV(100, 6\text{ months}) = 100 \times \exp\left(-\frac{1}{2} \times 3\%\right) = \frac{100}{\exp\left(\frac{1}{2} \times 3\%\right)}.$$

A Useful "Trick"

- We said earlier that there is a strong relationship between
  - nominal interest rates and simple capitalization/discounting
  - effective interest rates and compound capitalization/discounting
- Recall the earlier example: you have just invested 10.000€ in a savings account that offers an annual simple interest rate of 9%. ¿What will be the balance on the account after 8 months?
- We could solve this by

- You take the annual interest rate and capitalize over 8/12 of a year
- Or transform the interest into a monthly interest and capitalize over 8 months

$$FV = 10.000 \left(1 + \frac{9\%}{12} \times 8\right) = 10.600$$

- Similarly, suppose you invest 2.000€ for 8 months in an account with monthly capitalization and a 12% effective APR
  - You can take the annual interest rate and convert it into monthly

$$1 + r = (1 + 12\%)^{1/12}$$
$$FV = 20.000\,(1 + r)^8$$

  - Or, you can convert 8 months into $\frac{8}{12}$ of a year and calculate the FV

The TWO Exceptions

Important Note
You can avoid capitalization period and compute financial equivalence using the time period of the interest rate ...

... for compound financial laws with effective interest rates
... for simple (rational) financial laws with nominal interest rates

BUT you have to ensure that the time unit for counting the passage of time is the SAME as that of the interest rate

TIN & TAE

TIN and TAE

- As we saw earlier there are cases where you have an annual nominal rate and you want to calculate an annual effective rate, and the transaction may have a small capitalization period
- This is typical of savings accounts
- A financial institution will announce the interest rate in annual terms
  - the nominal rate is called the TIN [in Spanish] or nominal APR [in English]
  - the effective rate is called the TAE [in Spanish] or effective APR [in English]
- To get from one to the other, say if the capitalization rate in monthly, we need to go through the capitalization rate

$$1 + TAE = \left(1 + TIN \times \frac{1}{12}\right)^{\frac{1}{12}}$$

Example

› más información › contratar

*T.A.E. calculada para cualquier importe superior a un céntimo de euro. Abono mensual de intereses. Tipo de interés nominal anual 2,23 %. La cuenta NARANJA no admite domiciliación de recibos. ING DIRECT NV Sucursal en España.

RBE nº 2500/04

Example

If you deposit 1000€ for a year in the account this would be your balance at the end of each month

Month	Balance
1	1,001.86 €
2	1,003.72 €
3	1,005.58 €
4	1,007.45 €
5	1,009.32 €
6	1,011.20 €
7	1,013.08 €
8	1,014.96 €
9	1,016.85 €
10	1,018.74 €
11	1,020.63 €
12	1,022.53 €

Can you replicate the calculations?

- The contracted amount is the nominal APR, 2,23%
- The effective monthly interest rate is
  - $2,23\%/12 = 0.00185833333333$
  - which is rounded to 4 significant digits: 0.1858%
- Then, the balance is calculated for each month

$$1.000\,(1 + 0.1858\%) = 1,001.8580$$

- which is again rounded off to the nearest cent: 1.001, 86€
- ...
- The TAE (effective APR) is calculated on the basis of the monthly interest rate

$$(1 + 0.1858\%)^{12} - 1 = 0.02252526$$

which again is rounded to the nearest BP: 2, 25%

Effective APR (T.A.E.)

- As you can see in the example, financial contracts identify interests in two forms
  - nominal APR (TIN): annual nominal interest rate
  - effective APR (TAE): annual effective interest rate
- In order to facilitate transparency of financial transactions, most governments regulate the way interest rates are publicized.

(Real) Effective APR
For now, the effective APR is the annual interest rate using compounding.

- Later on we will look at how this interest rate is calculated to account for commissions and other financing costs

Example

- You need to ask for a loan and you visit three different financial institutions and they offer you the following conditions

  Bank A 7% annual interest and compounding

  Bank B 6,5% nominal APR with monthly compounding

  Bank C Quarterly interest payments with a quarterly interest of 1,7%

- The loan is for 7.000€ and it will be repaid two years from now. Compare the three alternatives and calculate the amount (FV) that will be repaid

Example

- Bank A: effective APR is 7%

$$7.000\,(1+7\%)^2 = 8.014,30€$$

- Bank B: nominal APR of 6,5% converted into monthly interest to do the compounding (for 24 months)

$$r = \frac{6,5\%}{12} \Rightarrow 7.000\left(1+\frac{6,5\%}{12}\right)^{24} = 7.969€$$

- Bank C: quarterly interest is $1,7\%$ compounded over 8 quarters

$$7.000\,(1+1,7\%)^8 = 8.010€$$

- Or, we could just compare effective APR

$$7\% \quad \left(1+\frac{6,5\%}{12}\right)^{12} - 1 = 6,7\%, \quad \left(1+\frac{7\%}{4}\right)^{4} - 1 = 7,19\%$$

Exercise to think about & research
Which of the following investment vehicles do you prefer?

- A savings account with a 12% nominal APR and annual compounding?
- A savings account with a 11.7% nominal APR and semestral compounding?
- A savings account with an 11.5% nominal APR and continuous compounding?

The Many Interests in a Transaction

Usually, you will find more than one interest rate associate with a transaction

- The interest rate that is given to you
- The effective APR of the "pure"/"raw" transaction
- The effective APR of the whole transaction
  - For the borrower
  - For the lender

For now, some of these are the same effective APR, but this will change as we introduce additional elements into the financial transactions, such as fees, commissions, taxes, ...

Class 5

Forward and Spot Rates

Changing Interest Rates

- What happens if we start a transaction at one interest rate, say 3%, and after a year, the interest rate changes, say to 2%?
- This just implies that you have to change the rate at which you capitalize/discount ... just be careful that you make sure you match interest rates with the time periods which correspond to them

Exercise

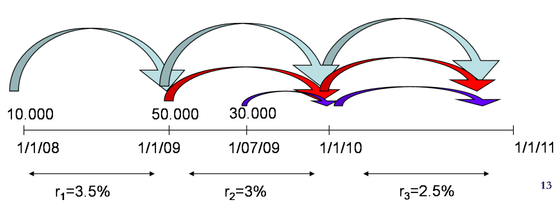
- An investor deposits the following amounts in a savings account:

$$\{(10.000, 1/1/2018), (50.000, 1/1/2019), (30.000, 1/7/2019)\}.$$

- Suppose that the interest rate on the savings account is
  - 2018: 3.5% annual
  - 2019: 3% annual
  - 2020: 2.5% annual

  Calculate the amount the investor will receive on 1/1/2021

ANSWER:



Solution: Time Steps

The key is to realize what is going on during each year:

- 2018: initial balance: 10.000, final balance: $10.000\,(1+3.5\%) = 10.350$
- 2019: initial balance: $10.350 + 50.000 = 60.350$
  - balance 1/7: $60.350(1+3\%)^{1/2} = 61{,}248.56$, $+30.000$
  - final balance: $91{,}248.56\,(1+3\%)^{1/2} = 92{,}607.17$
- 2020: initial balance: $92{,}607.17$, final balance $92{,}607.17\,(1+2.5\%) = 94{,}922.35$

Solution: Amounts

Alternatively, we can focus on each of the payments and the interest each one generates

- 10.000 earns 3.5% for one year, 3% for another, and 2.5% in the final year

$$10.000\,(1+3.5\%)\,(1+3\%)\,(1+2.5\%) = 10{,}927.01$$

- 50.000 earns 3% for one year, and 2.5% in the final year

$$50.000\,(1+3\%)\,(1+2.5\%) = 52{,}787.50$$

- 30.000 earns 3% for 6 months and 2.5% in the final year

$$30.000\,(1+3\%)^{1/2}\,(1+2.5\%) = 31{,}207.84$$

- In total you obtain

$$VF = 10{,}927.01 + 52{,}787.50 + 31{,}207.84 = 94{,}922.35$$

Time

- As we have effective rates and compound capitalization one has to keep the unit of time constant for the interest and the number of periods but we can change the time unit
- We used years, but we could do the same exercise with semesters.
- Semester interest rates
  - 2018: $(1+3.5\%)^{1/2} = 1 + 1.7349\%$
  - 2019: $(1+3\%)^{1/2} = 1 + 1.4889\%$
  - 2020: $(1+2.5\%)^{1/2} = 1 + 1.2423\%$
- And the solution

$$\begin{aligned} FV &= 10.000\,(1+1.7349\%)^2\,(1+1.4889\%)^2\,(1+1.2423\%)^2 \\ &\quad +50.000\,(1+1.4889\%)^2\,(1+1.2423\%)^2 \\ &\quad +30.000\,(1+1.4889\%)\,(1+1.2423\%)^2 \\ &= 94{,}922.35 \end{aligned}$$

Spot vs Forward Rates

- Consider a version of the previous example
- The interest during
  - 2018: was a 5% annual rate
  - 2019: was a 2% annual rate
  - 2020: was a 4% annual rate
- These interest rates (from the beginning of year 201x to the end of year 201x) when we look at them from the year 2018, they are called **forward rates**: you deposit C euros today, and
  - after 1 year you have $C\,(1+5\%)$
  - after 2 years you have $C\,(1+5\%)\,(1+2\%)$
  - after 3 years you have $C\,(1+5\%)\,(1+2\%)\,(1+4\%)$
- Another way to look at this is to say that at the beginning of 2018
  - a 1 year investment has a 5% annual (effective) return
  - a 2 year investment has a 3,49% annual (effective) return
  - a 3 year investment has a 3,66% annual (effective) return
- These last interest rates are called **spot rates**
- Both interest rates are equivalent ... in fact they are related by the following relationship
  - let f_t be the forward rate for the year t
  - let r_t be the spot rate for an investment from the start of the first year to the end of year t
  - then

$$(1+r_t)^t = (1+f_1)\,(1+f_2)\ldots(1+f_t)$$

- EXERCISE: verify that the interest rates in the example are indeed equivalent

Example

Suppose the interest rates are $r_1 = 0.3\%$, $r_2 = 0.9\%$, $r_3 = 1.2\%$, $r_4 = 1.5\%$ then you can compute the equivalent forward rates using

$$(1+r_1) = (1+f_1) \Rightarrow f_1 = 0.3\%$$

$$\begin{aligned}(1+r_2)^2 &= (1+f_1)\,(1+f_2)\\ 1+f_2 &= \frac{1.009^2}{1.003} = 1.0150 \Rightarrow f_2 = 1.50\%\end{aligned}$$

$$\begin{aligned}(1+r_3)^3 &= (1+f_1)\,(1+f_2)\,(1+f_3) = (1+r_2)^2\,(1+f_3)\\ 1+f_3 &= \frac{1.012^3}{1.009^2} = 1.0180 \Rightarrow f_2 = 1.80\%\\ 1+f_4 &= \frac{1.015^4}{1.012^3} = 1.0241 \Rightarrow f_2 = 2.41\%\end{aligned}$$

Effective APR as a comparison tool

The Effective APR of a raw transaction

- A raw transaction is just a list of the dated payments that are given (provision) and the list of dated payments that are received (compensation)
- Take any simple transaction, i.e. exchanging of one dated payment for another, (X, t) for (Y, τ)
- Example: exchange (1000, year 1) for (1200, year 5)
- For this simple raw transaction one can determine the IMPLIED EFFECTIVE APR — that is the effective APR that would make one dated payment financially equivalent to the other dated payment with (annual) compounding
- This implied effective APR, r, solves

$$
\begin{aligned}
X(1+r)^{(\tau-t)} &= Y \\
1000(1+r)^4 &= 1200 \\
\Rightarrow r &= 4.664\%
\end{aligned}
$$

Effective APRs

- A friend of yours tells you about an investment in a new financial product that has been created in Holland
- The product is a special savings account that offers 3% nominal APR with monthly capitalization
- However, because it is only available in Holland, you have to pay an intermediary a fee to make an investment
- The fee is flat, that is it is constant an independent of how much you want to invest 10€
- As we will see, there are a number of different APRs that are relevant
- The first is the effective APR of the savings account (as published in the Dutch press): as any Dutch person can invest without fees, their effective APR (and the legal one in the advert) is

$$\left(1 + \frac{3\%}{12}\right)^{12} - 1 = 3.042\%$$

The return on an investment

- However, if you have 1000€ which you want to invest for a year, then you:
  - pay 10€ to the intermediary
  - the intermediary deposits 990€ in the savings account
  - one year later you receive 990€ plus interest (30.11€)
- What is the real (annual) interest of your investment?
  - you pay 1000€, 10€ as fees and 990€ deposited into the account
  - you receive 990€+30.11€ in one year
- The raw transaction for you is $(1000, 0)$ exchanged for $(1020.11, 1 \text{ year})$, i.e. a 2.011% effective APR

Same product, different effective APR

- The "official" effective APR: 3.042%
- The effective APR of a Spanish investor of 1000€ for 1 year
- Suppose you invest 1500€ for two years:

$$2000\,(1+r)^2 = (2000-10)\left(1+\frac{3\%}{12}\right)^{24}$$
$$(1+r)^2 = \frac{2112.90}{2000} \Rightarrow r = 2.784\%$$

- Each investor has a different return for his/her investment, a different "true" effective APR, which is calculated with the "raw" transaction. In the last example
  - pay $(2000, 0)$ and receive $(2,112.90, 2 \text{ years})$

Effective APR as a comparison tool

- When comparing different investments (or loans) you can use the effective APR as a shorthand to compare them
  - investment: highest effective APR is best
  - loans/financing: lowest effective APR is best
- BUT, you have to use the "true" effective APR, the one that is computed with the raw dated payments associated with each possibility

4 Financial Operations & TAE

Class 7

Bank Discounting

Commercial discounting

- We have seen how to discount dated payments using rational discounting.
- Unfortunately, when discounting dated payments in a business environment one finds a very extended practice: the use of commercial discounting [cash discounting]
- We will now see what this law is and how it is applied in one of the most common financial operations amongst firms, bank discounting of commercial paper (discount receivables)

Bank discounting

- The usual method of payment for a sale is commercial paper (promissory note/IOU (pagaré), bill of exchange (letra), etc): this is a document issued by the seller and with which the buyer commits to making a cash payment equal to the price of the good after a specified amount of time.
- The seller will need to continue with his production activities and may need liquidity. In this case he can go to a financial institution and discount the paper. In exchange for this service he will incur a financial cost – the bank buys the "receivable" at a discount
- Mathematically the operation is simple (short term) and the financial law used is **commercial (or cash) discounting**.

Bank discount

- The financial institution advances the money, the amount stated on paper (V_N) and substracts the amount that corresponds to the time left until expiration.
  - V_N: the nominal value on paper (the amount that will be received in the future)
  - n: the number of days until you receive V_N
  - $d\%$: the discount rate (usually, annual)
  - E: the cash amount you receive today
- Thus, a commercial paper with a nominal value (V_N) that is due in n days, with a discount rate of $d\%$ per annum, will receive a cash amount equal to:

$$E = V_N \left(1 - \frac{n}{360} d\%\right)$$

Example

- Suppose that ORALVA S.A. sells produce worth 100.000€ to EVER S.L. EVER decides to pay in 90 days (net 90), but during this period ORALVA needs cash. They can go to the bank and discount the right to receive this payment, the bill. If the bank charges an 8% interest on the commercial discounting transaction, how much will ORALVA receive today?
  - Nominal: $V_N = 100.000$€
  - Interest charged: $V_N d(n/360) = 100.000 \times 0.08 \times (90/360) = 2.000$€
  - Cash received: $E = 100.000 - 2.000 = 98.000$€

$$E = 100.000 \left(1 - \frac{90}{360} 8\%\right)$$

Example

- What would have occurred under rational discounting?

$$\frac{100.000}{(1 + \frac{90}{360} 8\%)} = 98.039,22\text{€}$$

  ORALVA would have received more money
- Note that this financial transaction is equivalent to a **loan**:
  - ORALVA receives money today
  - ORALVA pays (back) in the future–by handing over the money that it will receive from her client later on

Implied interest rate

- Suppose we compute the annual interest rate for the transaction (loan). What we find is that it is not 8%:
  - The transaction corresponds to the exchange of one dated payment for another receive $(98.000, 0)$ in exchange for $(100.000, 90 \text{ days})$
- If the interest would have been 8% then the final value of 98.000€ is

$$98.000 \left(1 + \frac{90}{360} 8\%\right) = 99.960\text{€}$$

- But, the right amount is 100.000 in 90 days, so the interest rate must be higher ... **WHY?**
- Because in commercial discounting, the interest payments are calculated on the final (nominal) value, and not on the initial (present) amount

Bank discount

- The effective cost for the client, and the return for the bank are calculated so as to include, not only interest payments, but also commissions and other costs.
- For example: there usually is an additional cashing commission charged as a percentage of the nominal amount on paper (e.g. 0.8%–or 80bps, on V_N) in order to compensate the bank for the paperwork involved in collecting the amount from the buyer. Furthermore, this commission has a floor/minimum value for each commercial paper (for example, 4 €)
- Once these commercial characteristics have been taken into account and added/substracted from the compensation and/or the provision one obtains the real provision and the real compensation
- Using the real compensation and provision one can compute the return obtained by the bank and the cost paid by the client

The "real" transaction

- The real transaction as perceived by the bank
- The real transaction as perceived by the client

The financial cost of discounting

- The cost or return of a transaction is obtained by financially equating the provision and compensation received, applying the capitalization law.
- As bank discounting is a particular transaction of financial institutions, it is subject to the Circular 8/90 of the Banco de España. Then, the financial cost can be calculated using the same one used by these institutions, that is, the interest as computed using compound capitalization in years, which equates provision and compensation.

Example

- Let us return to the previous example, where EVER S.L. writes a bill for 100.000€ net 90 to ORALVA S.A. The latter takes the paper to a bank that offers an 8% discount and hence receives

$$E = 100.000\left(1 - \frac{90}{360}8\%\right) = 98.000.$$

- In addition, suppose the bank charges a 1% commission as a transaction cost, so that ORALVA receives

$$E = 100.000\left(1 - \frac{90}{360}8\%\right) - 100.000 \times 0.01 = 97.000$$

- The cost of the transaction, r_{TAE}, for the client is obtained from the following calculation

$$97.000\,(1 + r_{TAE})^{90/365} = 100.000 \Rightarrow r_{TAE} = 13.15\%$$

  which in this case coincides with the return obtained by the bank on the transaction

Discounting batches

- The usual process for discounting involves a batch of commercial paper, not a single one
- For example: Canasa deposits a batch of three bills of 50.000€ each, and they are due in 30, 60 and 90 days, respectively. The discount rate is 12% annual. Compute the amount of cash received.

Camasa and its batch of paper

In order to solve this problem you have several options:

1. You can compute the cash amount received from each bill and add them

$$\begin{aligned} E &= 50.000\left(1-\frac{30}{360}12\%\right)=49.500 \\ E &= 50.000\left(1-\frac{60}{360}12\%\right)=49.000 \\ E &= 50.000\left(1-\frac{90}{360}12\%\right)=48.500 \end{aligned}$$

   Total: $49.500+49.000+48.500=147.000$€

2. You can compute the average due date ... use that to construct a single, equivalent, commercial paper, and then discount that one paper

   (a) the average due date is computed as follows

$$VM=\frac{50.000\times 30+50.000\times 60+50.000\times 90}{50.000+50.000+50.000}=\frac{9.000.000}{150.000}=60$$

   (b) the equivalent paper has a nominal value of $50.000+50.000+50.000=150.000$€ and a due date equal to $VM=60$ days

   (c) the discounted value of the equivalent paper is

$$150.000\left(1-\frac{60}{360}12\%\right)=147.000€$$

The average due date

- In the previous example, the average due date was the arithmetic average of the three due dates ... this is because the nominal values of the three bills was the same
- In general, when the nominal values of the bills are not the same, the average due date is the weighted average of the due dates of the bills, where the weights are the relative values of each of the bills
- Thus, when discounting to bills: $(50.000, 30)$ and $(100.000, 120)$, the average due date is

$$VM=\frac{50.000}{50.000+100.000}30+\frac{100.000}{50.000+100.000}120=90$$

The VM calculated as the weighted average of the bills always works

Forfeit

- Sometimes, the bank offers what is called a **forfeit**, instead of the usual interest plus commission rates applied in the usual commercial discount
- This forfeit usually is applied in terms of a minimum number of days
- That is, if the due date is less than the minimum, the bank discounts the bill using the minimum number of days

Forfeit: example

Canasa wants to discount a bill: 80.000€ due in 10 days. The bank applies a discount rate of 12%, with a minimum forfeit of 15 days. Compute the cash received and the financing cost (simple capitalization)

- Using the actual due date they should receive

$$E = 80.000\left(1 - 12\%\frac{10}{360}\right) = 79,773.34€$$

which corresponds to a financing cost of

$$79,773,34\left(1 + \frac{10}{360}r\right) = 80.000 \Rightarrow r = 12,04\%$$

- Instead, applying the forfeit they receive

$$E = 80.000\left(1 - 12\%\frac{15}{360}\right) = 79,600€$$

which corresponds to a financing cost of

$$79,600\left(1 + \frac{10}{365}r\right) = 80.000 \Rightarrow r = 18,34\%$$

Supplier Financing

The cost of supplier financing

- It is quite common between firms to finance each other, that is the seller gives the buyer a certain amount of time to pay him back.
- This is not to say that using this method of financing one's purchases (via suppliers) is free. There is a financing cost, though this cost is not explicit. The implicit cost is there, and it usually is very high.
- First of all, one has to recognize that there is a debt, and hence a financial transaction, there. When you do not pay for your purchases today, in cash, you borrow the money, and if it is the supplier who is letting you postpone the payment, it is the supplier who is, in effect, lending you the money to make the purchase.

Example

- Canasa has just received at its loading dock a shipment from a supplier of raw materials worth 10.000€. The bill for the purchase is net 60 (due in 60 days). The supplier also offers you the opportunity to pay upfront, today, and if you do you will receive a 2.5% discount on the purchase price. What is the financial cost incurred by Canasa if they accept financing from this supplier?
- Canasa has two options:
  - pay the supplier today. The discount is 250 € (2.5% of 10.000) so that the total amount paid is 9.750€
  - pay 10.000 in 60 days time

Financing costs

- By postponing the payment, Canasa is accepting a 60 day loan of 9.750€ which they will return in 60 days and pay 10.000€.
- What is the financing cost from this supplier?

- $10.000 = 9.750 +$ interest of 250€
- the loan is for 60 days so that

$$9.750\left(1+r\frac{60}{365}\right)=10.000 \Rightarrow r=15,60\%$$

- This is a loan at a 15.60% nominal APR. For sure, Canasa can find a cheaper source of finance and pay 9750€ today!
- It is highly recommend to identify the financing cost so that the financial decisions we take are the best for us and our company.

Compound discounting

- In order to compute the present value of a dated payment, (C, t_1), to date t_0 using simple commercial discounting we solve

$$PV = C_1\left(1-d\left(t_1-t_0\right)\right)$$

- Taking into account that we can also do the discounting using compounding you get

$$PV=\frac{C_1}{(1+r)^{t_1-t_0}}$$

- From this, and using $t_1 - t_0 = 1$, you can obtain the compound interest, r, that corresponds to the commercial discounting interest, d, for a one year transaction

$$(1+r)^{-1}=1-d \Rightarrow r=\frac{d}{1-d}$$

Example

- A firm wants to discount paper for 5.000€ which is due in 4 months. Three financial institutions make the following offers each
  - 9% annual with simple commercial discounting
  - 9.5% annual with simple rational discounting
  - 9,77% annual with compound discounting
- What is the best option for the holder of the commercial paper?

Example: answer

- Let us work out how much cash the holder would obtain with each of the three options
  - 9% annual with simple commercial discounting

$$E=5.000\left(1-9\%\frac{4}{12}\right)=4,850$$

  - 9.5% annual with simple rational discounting

$$E=\frac{5.000}{\left(1+9,5\%\frac{4}{12}\right)}=4,846.53$$

  - 9,77% annual with compound discounting

$$E=\frac{5.000}{(1+9.77\%)^{4/12}}=4.847,32$$

Example: answer

- Thus, the best option is the first one, which leaves him with the most amount of cash
- Nevertheless, in case the discount rate would have been the same under all three financial laws, it would not have been necessary to do any calculations: the greatest amount of cash is provided by simple rational discounting, which generates the greatest present value.