Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 20

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Outline

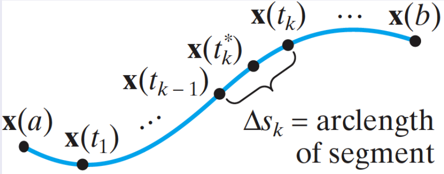
1. Scalar and Vector Line Integrals
   - Scalar line integral
   - Vector line integral
   - Differential form of the line integral
   - Effect of reparametrization
   - Closed and simples curves
2. Green's Theorem
   - Definition
   - Examples

Outline

Scalar Line Integral as a limit of a Riemann sum

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^3$ be a path of class C^1
- Let $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a continuous function
- Suppose that domain X contains the image of $\mathbf{x}$, so that the composite $f(\mathbf{x}(t))$ is defined
- As with every other integral, the scalar line integral is a limit of appropriate Riemann sums
- Consider a partition of $[a, b]$

$$a = t_0 < t_1 < \cdots < t_k < \cdots < t_n = b$$



Scalar Line Integral as a limit of a Riemann sum

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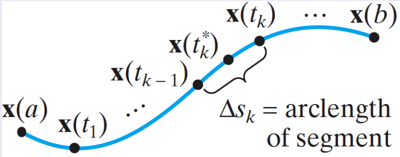


- Let us think of
  - The image of the path $\mathbf{x}$ as representing an idealized wire in space
  - $f(\mathbf{x}(t))$ as the electrical charge density of the wire
- Then, the Riemann sum approximates the total charge of the wire

$$\text{Total charge} = \lim_{\text{all } \mathbf{\Delta t_k} \to 0} \sum_{k=1}^{n} f(\mathbf{x}(t_k^*))\Delta s_k$$

Definition 1.1: Scalar Line Integral

- The scalar line integral of f along the C^1 path $\mathbf{x}$ is

$$\int_a^b f(\mathbf{x}(t))\|\mathbf{x}'(t)\|dt$$

- We denote this integral

$$\int_{\mathbf{x}} f\ ds$$

Remarks

- The line integral represents a sum of values of f along $\mathbf{x}$, times "infinitesimal" pieces of **arclength** of $\mathbf{x}$

Remarks

- Definition 1.1 can be made for arbitrary n, that is, for functions f defined on domains in $\mathbb{R}^n$ for arbitrary n

Remarks

- We can still define the scalar line integral if
  - $\mathbf{x}$ is not of class C^1, but only "piecewise" C^1
  - $f(\mathbf{x}(t))$ is only piecewise continuous

Example 1

- Let $f(x, y, z) = xy + z$ and $\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3$ be the helix

$$\mathbf{x}(t) = (\cos t, \sin t, t)$$

- We compute

$$\int_{\mathbf{x}} f\ ds = \int_0^{2\pi} f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| dt$$

- First, from the double-angle formula

$$\begin{aligned} f(\mathbf{x}(t)) &= \cos t \sin t + t = \frac{1}{2} \sin 2t + t \\ \mathbf{x}'(t) &= (-\sin t, \cos t, 1) \\ \|\mathbf{x}'(t)\| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \end{aligned}$$

Example 1

$$f(x,y,z) = xy + z \quad \text{and} \quad \mathbf{x}(t) = (\cos t, \sin t, t)$$

$$f(\mathbf{x}(t)) = \cos t \sin t + t = \frac{1}{2}\sin 2t + t$$

$$\mathbf{x}'(t) = (-\sin t, \cos t, 1), \quad \|\mathbf{x}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

- Thus

$$\int_{\mathbf{x}} f \, ds = \int_0^{2\pi} f(\mathbf{x}(t))\|\mathbf{x}'(t)\| dt = \int_0^{2\pi} \left(\frac{1}{2}\sin 2t + t\right)\sqrt{2}\, dt$$

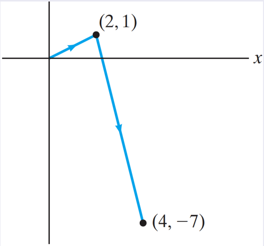
$$= \sqrt{2}\int_0^{2\pi} \left(\frac{1}{2}\sin 2t + t\right) dt = \sqrt{2}\left(-\frac{1}{4}\cos 2t + \frac{1}{2}t^2\right)\Big|_0^{2\pi}$$

$$= \sqrt{2}\left(\left(-\frac{1}{4} + 2\pi^2\right) - \left(-\frac{1}{4} + 0\right)\right) = 2\sqrt{2}\pi^2$$

Example 2

- Let $f(x,y) = y - x$ and let $\mathbf{x} : [0,3] \to \mathbb{R}^2$ be the planar path

$$\mathbf{x}(t) = \begin{cases} (2t, t) & \text{if } 0 \le t \le 1 \\ (t+1, 5-4t) & \text{if } 1 < t \le 3 \end{cases}$$



- Hence, $\mathbf{x}$ is piecewise C^1 path
- The two path segments defined for t in $[0,1]$ and for t in $[1,3]$ are each of class C^1

Example 2

- Let $f(x, y) = y - x$ and let $\mathbf{x} : [0, 3] \to \mathbb{R}^2$ be the planar path

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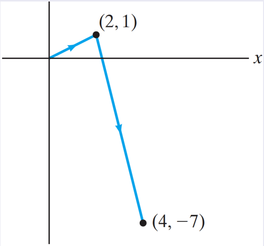

- Thus

$$\int_{\mathbf{x}} f\, ds = \int_{\mathbf{x_1}} f\, ds + \int_{\mathbf{x_2}} f\, ds$$

Example 2

- Let $f(x,y) = y - x$ and let $\mathbf{x} : [0,3] \to \mathbb{R}^2$ be the planar path

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- Thus

$$\int_{\mathbf{x}} f \; ds = \int_{\mathbf{x_1}} f \; ds + \int_{\mathbf{x_2}} f \; ds$$

  where

  - $\mathbf{x_1}(t) = (2t, t)$ for $0 \leq t \leq 1$
  - $\mathbf{x_2}(t) = (t+1, 5-4t)$ for $1 < t \leq 3$

- It is easy to see that

$$\|\mathbf{x_1}'(t)\| = \sqrt{5} \quad \text{and} \quad \|\mathbf{x_2}'(t)\| = \sqrt{17}$$

Example 2

- Let $f(x,y) = y - x$ and let $\mathbf{x} : [0,3] \to \mathbb{R}^2$ be the planar path

$$\mathbf{x}(t) = \begin{cases} (2t, t) & \text{if } 0 \leq t \leq 1 \\ (t+1, 5-4t) & \text{if } 1 < t \leq 3 \end{cases}$$

$$\|\mathbf{x_1}'(t)\| = \sqrt{5} \quad \text{and} \quad \|\mathbf{x_2}'(t)\| = \sqrt{17}$$

- Thus

$$\int_{\mathbf{x_1}} f\ ds = \int_0^1 f(\mathbf{x_1}(t))\|\mathbf{x_1}'(t)\|dt = \int_0^1 (t-2t)\cdot\sqrt{5}\ dt = -\frac{\sqrt{5}}{2}t^2\Big|_0^1 = -\frac{\sqrt{5}}{2}$$

$$\int_{\mathbf{x_2}} f\ ds = \int_1^3 f(\mathbf{x_2}(t))\|\mathbf{x_2}'(t)\|dt = \int_1^3 ((5-4t)-(t+1))\cdot\sqrt{17}dt$$

$$= \sqrt{17}\left(4t - \frac{5}{2}t^2\right)\Big|_1^3 = -12\sqrt{17}$$

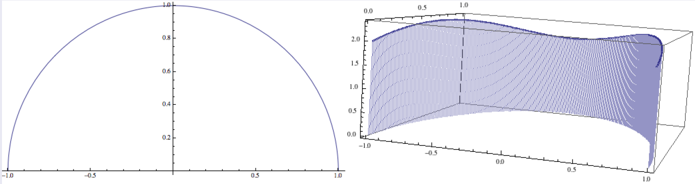
Geometric Interpretation of Scalar Line Integrals

- Let $f(x,y) = 2 + x^2 y$ and let $\mathbf{x} : [0,\pi] \to \mathbb{R}^2$ be the planar path

$$\mathbf{x}(t) = (\cos t, \sin t), \quad 0 \leq t \leq \pi$$

- Then

$$f(\mathbf{x}(t)) = f(x(t), y(t)) = 2 + \cos^2 t \sin t$$



- The line integral of f along $\mathbf{x}$ is the area of the “fence“ whose
  - Path is governed by $\mathbf{x}$
  - Height is governed by f

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Definition 1.2

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a path of class C^1
- Let $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a vector field
- Suppose that X contains the image of $\mathbf{x}$ and assume that $\mathbf{F}$ varies continuously along $\mathbf{x}$
- The vector line integral of $\mathbf{F}$ along $\mathbf{x} : [a, b] \to \mathbb{R}^n$, is

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt$$

Remarks

- As with scalar line integrals, we may define the vector line integrals when $\mathbf{x}$ is a piecewise C^1 path
- We juste need to break up the integral in a suitable manner

Example 3

- Let $\mathbf{F}$ be the radial vector field on $\mathbb{R}^3$ given by

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- Let $\mathbf{x} : [0,1] \to \mathbb{R}^3$ be the path

$$\mathbf{x}(t) = (t, 3t^2, 2t^3)$$

- Then

$$\begin{aligned}
\mathbf{x}'(t) &= (1, 6t, 6t^2) \\
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt \\
&= \int_0^1 \left(t\mathbf{i} + 3t^2\mathbf{j} + 2t^3\mathbf{k}\right) \cdot \left(\mathbf{i} + 6t\mathbf{j} + 6t^2\mathbf{k}\right) dt \\
&= \int_0^1 \left(t + 18t^3 + 12t^5\right) dt = \left(\frac{1}{2}t^2 + \frac{9}{2}t^4 + 2t^6\right)\Bigg|_0^1 = 7
\end{aligned}$$

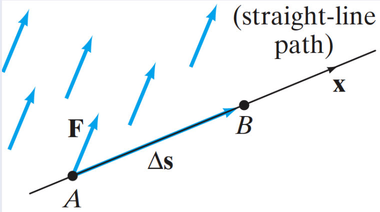
Physical Interpretation of Vector Line Integrals

- Consider **F** to be a force field in space
- Then, the vector line integral could represent the work done by **F** on a particle as the particle moves along the path **x**

$$\text{Total Work} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

Simplified example

- Suppose **F** is a constant vector field and **x** is a straight-line



Physical Interpretation of Vector Line Integrals

- Consider $\mathbf{F}$ to be a force field in space
- Then, the vector line integral could represent the work done by $\mathbf{F}$ on a particle as the particle moves along the path $\mathbf{x}$

$$\text{Total Work} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

Simplified example

- Suppose $\mathbf{F}$ is a constant vector field and $\mathbf{x}$ is a straight-line
- Then, the work done by $\mathbf{F}$ in moving a particle from one point A along $\mathbf{x}$ to another point B is given by

$$\text{Work} = \mathbf{F} \cdot \Delta\mathbf{s} = \mathbf{F} \cdot (B - A)$$

Differential Geometry Interpretation

- Suppose $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a C^1 path with $\mathbf{x}'(t) \neq \mathbf{0}$ for $a \leq t \leq b$
- Recall that we define the unit tangent vector $\mathbf{T}$ to $\mathbf{x}$ by normalizing the velocity

$$\mathbf{T} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$$

- Then

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt \\ &= \int_a^b \left(\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T}(t)\right) \|\mathbf{x}'(t)\| dt = \int_{\mathbf{x}} \left(\mathbf{F} \cdot \mathbf{T}\right) ds \end{aligned}$$

Differential Geometry Interpretation

- Suppose $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a C^1 path with $\mathbf{x}'(t) \neq \mathbf{0}$ for $a \leq t \leq b$
- Then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \quad = \quad = \int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{T}) \, ds$$

- Since the dot product $\mathbf{F} \cdot \mathbf{T}$ is a scalar quantity, we have written the original vector line integral as a scalar line integral
- It represents the (scalar) line integral of the tangential component of $\mathbf{F}$ along the path

Differential Geometry Interpretation

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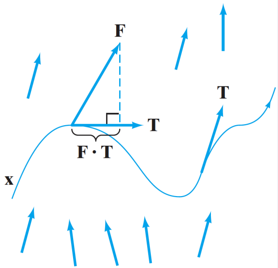
- Then
$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{T})\, ds$$



Example 4

- The circle $x^2 + y^2 = 9$ may be parametrized by

$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases} ,0 \leq t \leq 2\pi$$

- Hence, a unit tangent vector is

$$\mathbf{T} = \frac{-3\sin t\mathbf{i} + 3\cos t\mathbf{j}}{\sqrt{9\sin^2 t + 9\cos^2 t}} = -\sin t\mathbf{i} + \cos t\mathbf{j} = \frac{-y\mathbf{i} + x\mathbf{j}}{3}$$

- Now consider the radial vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ on $\mathbb{R}^2$
- At every point along the circle we have

$$\mathbf{F} \cdot \mathbf{T} = (x\mathbf{i} + y\mathbf{j}) \cdot \left(\frac{-y\mathbf{i} + x\mathbf{j}}{3}\right) = 0$$

Example 4

$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}, 0 \le t \le 2\pi, \mathbf{T} = \frac{-y\mathbf{i} + x\mathbf{j}}{3} \text{ and } \mathbf{F} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{T} = 0$$

- Thus, **F** is always perpendicular to the curve, and

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{T})\, ds = \int_{\mathbf{x}} 0\, ds = 0$$

Considering **F** as a force,
no work is done

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Differential Form of the Line Integral

- Suppose that $\mathbf{x}(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$, is a C^1 path
- Consider a continuous vector field $\mathbf{F}$ written as

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

- Then, from Definition 1.2 of the vector line integral, we have

$$\begin{aligned}
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b (M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}) \\
&\cdot \quad (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k})\, dt \\
&= \int_a^b (M(x, y, z)x'(t) + N(x, y, z)y'(t) + P(x, y, z)z'(t))\, dt
\end{aligned}$$

Recall that $dx = x'(t)dt, dy = y'(t)dt, dz = z'(t)dt$

$$= \int_{\mathbf{x}} M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$$

Differential Form of the Line Integral

- Suppose that $\mathbf{x}(t) = (x(t), y(t), z(t)), a \leq t \leq b$, is a C^1 path
- Consider a continuous vector field $\mathbf{F}$ written as

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- Then, from Definition 1.2 of the vector line integral, we have

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$$

- A notational alternative is

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} M\ dx + N\ dy + P\ dz$$

The **differential form**
of the line integral

Differential Form of the Line Integral

- Suppose that $\mathbf{x}(t) = (x(t), y(t), z(t)), a \leq t \leq b$, is a C^1 path
- Consider a continuous vector field $\mathbf{F}$ written as

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

- Then, from Definition 1.2 of the vector line integral, we have

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$$

- A alternative notation is

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} M\ dx + N\ dy + P\ dz$$

- $M\ dx + N\ dy + P\ dz$ is itself called a differential form
- $M\ dx + N\ dy + P\ dz$ should be evaluated using the parametric equations for x, y, and z

Example 5

- Let $\mathbf{x}$ be the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $0 \leq t \leq 1$
- We compute
$$\int_{\mathbf{x}} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$
- Along the path, we have
$x = t \Rightarrow dx = dt, y = t^2 \Rightarrow dy = 2tdt, z = t^3 \Rightarrow dz = 3t^2dt$
- Therefore
$$\int_{\mathbf{x}} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$
$$= \int_0^1 (t^2+t^3)dt + (t+t^3)2tdt + (t+t^2)3t^2dt$$
$$= \int_0^1 \left(5t^4+4t^3+3t^2\right)dt = \left.\left(t^5+t^4+t^3\right)\right|_0^1 = 3$$

Outline

The Effect of Reparametrization

- The unit tangent vector to a path depends on the geometry of the underlying curve

It doesn't depend on
the particular parametrization

- We might expect the line integral likewise to depend only on the image curve
- For example, consider the following two paths in the plane

$$\mathbf{x} : [0, 2\pi] \to \mathbb{R}^2, \quad \mathbf{x}(t) = (\cos t, \sin t)$$
$$\mathbf{y} : [0, \pi] \to \mathbb{R}^2, \quad \mathbf{y}(t) = (\cos 2t, \sin 2t)$$

- Both **x** and **y** trace out a circle once in a counterclockwise sense
- If we let $u(t) = 2t$, then we see that $\mathbf{y}(t) = \mathbf{x}(u(t))$

Definition 1.3

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 path
- Consider another C^1 path $\mathbf{y} : [c, d] \to \mathbb{R}^n$
- We say that $\mathbf{y}$ is a reparametrization of $\mathbf{x}$ if there is a one-one and onto function $u : [c, d] \to [a, b]$ of class C^1
  - With inverse $u^{-1} : [a, b] \to [c, d]$ that is also of class C^1
  - Such that $\mathbf{y}(t) = \mathbf{x}(u(t))$, that is, $\mathbf{y} = \mathbf{x} \circ u$

Remark

- Thus, any reparametrization of a path must have the same underlying image curve as the original path

Example 6

- Consider the path

$$\mathbf{x}(t) = (1 + 2t, 2 - t, 3 + 5t), \quad 0 \leq t \leq 1$$

- It traces the line segment from the point $(1, 2, 3)$ to the point $(3, 1, 8)$

1. So does the path

$$\mathbf{y}(t) = (1 + 2t^2, 2 - t^2, 3 + 5t^2), \quad 0 \leq t \leq 1$$

- We have that $\mathbf{y}$ is a reparametrization of $\mathbf{x}$ via the change of variable

$$u(t) = t^2$$

Example 6

- Consider the path

$$\mathbf{x}(t) = (1 + 2t, 2 - t, 3 + 5t), \quad 0 \leq t \leq 1$$

- It traces the line segment from the point $(1, 2, 3)$ to the point $(3, 1, 8)$

2. We consider now the path $\mathbf{z} : [-1, 1] \to \mathbb{R}^3$

$$\mathbf{z}(t) = (1 + 2t^2, 2 - t^2, 3 + 5t^2), \quad -1 \leq t \leq 1$$

- It is not a reparametrization of $\mathbf{x}$
- We also have $\mathbf{z}(t) = \mathbf{x}(u(t))$, where $u(t) = t^2$
- But in this case u maps $[-1, 1]$ onto $[0, 1]$ in a way that is not one-one

Example 6

- Consider the path

$$\mathbf{x}(t) = (1 + 2t, 2 - t, 3 + 5t), \quad 0 \leq t \leq 1$$

- It traces the line segment from the point $(1, 2, 3)$ to the point $(3, 1, 8)$

3. We finally consider the path $\mathbf{w} : [0, 1] \to \mathbb{R}^3$

$$\mathbf{w}(t) = (3 - 2t, 1 + t, 8 - 5t), \quad 0 \leq t \leq 1$$

- It is a reparametrization of $\mathbf{x}$
- We have $\mathbf{w}(t) = \mathbf{x}(1 - t)$
- So the function $u : [0, 1] \to [0, 1]$ given by $u(t) = 1 - t$ provides the change of variable for the reparametrization.

Geometrically, $\mathbf{w}$ traces the line segment between $(1, 2, 3)$ and $(3, 1, 8)$ in the opposite direction to $\mathbf{x}$

Reparametrization and Orientation

- Let $\mathbf{y} : [c, d] \to \mathbb{R}^n$ be a reparametrization of $\mathbf{x} : [a, b] \to \mathbb{R}^n$ via the change of variable $u : [c, d] \to [a, b]$
- Then, since u is one-one, onto, and continuous, we must have either
  1. $u(c) = a$ and $u(d) = b$, or
  2. $u(c) = b$ and $u(d) = a$
- In case 1, we say that $\mathbf{y}$ (or u) is orientation-preserving

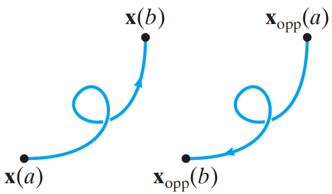
  $\mathbf{y}$ traces out the same image curve
  in the same direction that $\mathbf{x}$ does
- In case 2, we say that $\mathbf{y}$ (or u) is orientation-reversing

  $\mathbf{y}$ traces out the same image curve
  in the opposite direction that $\mathbf{x}$ does

Example 7

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be any C^1 path
- Then, we may define the opposite path $\mathbf{x}_{opp} : [a, b] \to \mathbb{R}^n$ by

$$\mathbf{x}_{opp}(t) = \mathbf{x}(a + b - t)$$



- That is, $\mathbf{x}_{opp}(t) = \mathbf{x}(u(t))$, where $u : [a, b] \to [a, b]$ is given by

$$u(t) = a + b - t$$

- Clearly, then, $\mathbf{x}_{opp}(t)$ is an orientation-reversing reparametrization of $\mathbf{x}$

Reparametrization and Velocity

In addition to reversing orientation,
a reparametrization of a path can change the speed

- This follows readily from the chain rule

  Speed of $\mathbf{y} = \|\mathbf{y}'(t)\| = |u'(t)|\|\mathbf{x}'(t)\| = |u'(t)| \cdot$ (Speed of $\mathbf{x}$)

- Since u is one-one, it follows that either
  - $u'(t) \geq 0$ for all $t \in [a, b]$ or
  - $u'(t) \leq 0$ for all $t \in [a, b]$
- The first case occurs when $\mathbf{y}$ is orientation-preserving
- The second case occurs when $\mathbf{y}$ is orientation-reversing

Theorem 1.4

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 path
- Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function whose domain X contains the image of $\mathbf{x}$
- If $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is any reparametrization of $\mathbf{x}$, then

$$\int_{\mathbf{y}} f \; ds = \int_{\mathbf{x}} f \; ds$$

Remark

- Theorems 1.4 tell us that scalar line integrals are independent of the way we might choose to reparametrize a path

Theorem 1.5

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 path
- Let $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field whose domain X contains the image of $\mathbf{x}$
- If $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is any reparametrization of $\mathbf{x}$, then

  1. If $\mathbf{y}$ is orientation-preserving, then

  $$\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

  2. If $\mathbf{y}$ is orientation-reversing, then

  $$\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

Theorem 1.5

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 path
- Let $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field whose domain X contains the image of $\mathbf{x}$
- If $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is any reparametrization of $\mathbf{x}$, then

$$\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \quad \text{or} \quad \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

Remark

- Theorems 1.5 tell us that vector line integrals are independent of reparametrization up to a sign
- This sign depends only on whether the reparametrization preserves or reverses orientation

Example 8

- Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, and consider the following three paths between $(0,0)$ and $(1,1)$

$$\begin{aligned} \mathbf{x}(t) &= (t,t), & 0 \leq t \leq 1 \\ \mathbf{y}(t) &= (2t,2t), & 0 \leq t \leq \frac{1}{2} \\ \mathbf{z}(t) &= (1-t,1-t), & 0 \leq t \leq 1 \end{aligned}$$

- The three paths are all reparametrizations of one another
- $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ all trace the line segment between $(0,0)$ and $(1,1)$
  - $\mathbf{x}$ and $\mathbf{y}$ from $(0,0)$ to $(1,1)$, and
  - $\mathbf{z}$ from $(1,1)$ to $(0,0)$
- We can compare the values of the line integrals of $\mathbf{F}$ along these paths
- The results of these calculations must agree with what Theorem 1.5 predicts

Example 8

- Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, and consider the following three paths between $(0,0)$ and $(1,1)$

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$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt = \int_0^1 (t\mathbf{i} + t\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) dt \\ &= \int_0^1 2t\, dt = \left. t^2 \right|_0^1 = 1 \end{aligned}$$

Example 8

- Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, and consider the following three paths between $(0,0)$ and $(1,1)$

$$\begin{aligned} \mathbf{x}(t) &= (t,t), & 0 \le t \le 1 \\ \mathbf{y}(t) &= (2t,2t), & 0 \le t \le \frac{1}{2} \\ \mathbf{z}(t) &= (1-t,1-t), & 0 \le t \le 1 \end{aligned}$$

$$\begin{aligned} \int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{\frac{1}{2}} \mathbf{F}(\mathbf{y}(t))\cdot \mathbf{y}'(t)dt = \int_0^{\frac{1}{2}} (2t\mathbf{i}+2t\mathbf{j})\cdot(2\mathbf{i}+2\mathbf{j})dt \\ &= \int_0^{\frac{1}{2}} 8t\, dt = 4t^2\Big|_0^{\frac{1}{2}} = 1 \end{aligned}$$

Example 8

- Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, and consider the following three paths between $(0,0)$ and $(1,1)$

$$\begin{aligned} \mathbf{x}(t) &= (t,t), & 0 \le t \le 1 \\ \mathbf{y}(t) &= (2t,2t), & 0 \le t \le \frac{1}{2} \\ \mathbf{z}(t) &= (1-t,1-t), & 0 \le t \le 1 \end{aligned}$$

$$\begin{aligned} \int_{\mathbf{z}} \mathbf{F}\cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{z}(t))\cdot \mathbf{z}'(t)dt \\ &= \int_0^1 ((1-t)\mathbf{i} + (1-t)\mathbf{j})\cdot(-\mathbf{i}-\mathbf{j})dt \\ &= \int_0^1 2(t-1)\,dt = (t-1)^2\Big|_0^1 = -1 \end{aligned}$$

Outline

Closed and Simple Curves

- Theorems 1.4 and 1.5 enable us to define line integrals over curves rather than over parametrized paths
- To be more explicit, we say that a piecewise C^1 path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is closed if $\mathbf{x}(a) = \mathbf{x}(b)$
- We say that the path $\mathbf{x}$ is simple if it has no self-intersections

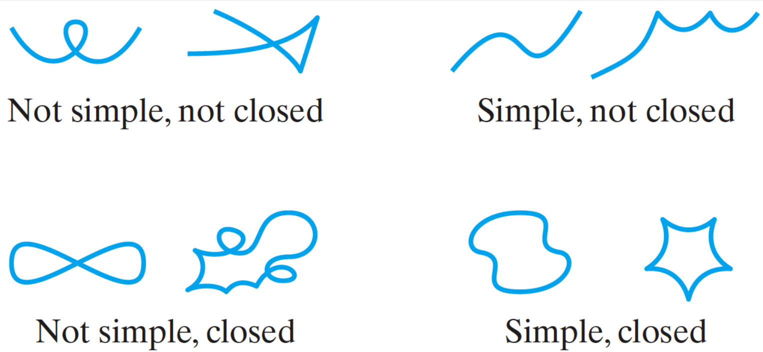
That is, if $\mathbf{x}$ is one-one on $[a, b]$,
except possibly that $\mathbf{x}(a)$ may equal $\mathbf{x}(b)$

- Then, by a curve C, we now mean the image of a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$

This path is one-one except possibly
at finitely many points of $[a, b]$

- The (nearly) one-one path $\mathbf{x}$ will be called a parametrization of C

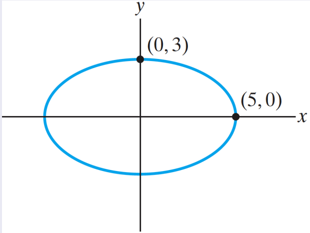
Closed and Simple Curves



Example 9

- Consider the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$



- It is a simple, closed curve that may be parametrized by either

$$\mathbf{x}(t) = (5\cos t, 3\sin t), \quad \mathbf{x} : [0, 2\pi] \to \mathbb{R}^2$$

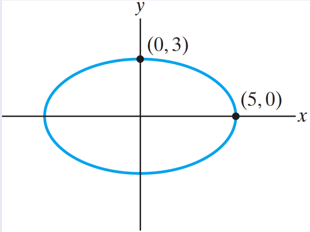
or

$$\mathbf{y}(t) = (5\cos 2(\pi - t), 3\sin 2(\pi - t)), \quad \mathbf{y} : [0, \pi] \to \mathbb{R}^2$$

Example 9

- Consider the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$



- Consider now the path

$$\mathbf{z}(t) = (5\cos t, 3\sin t), \quad \mathbf{z} : [0, 6\pi] \to \mathbb{R}^2$$

- It is not a parametrization, since it traces the ellipse three times as t increases from 0 to 6π. $\mathbf{z}$ is not one-one.

Example 10

- Let C be the upper semicircle of radius 2, centered at $(0,0)$ and oriented counterclockwise from $(2,0)$ to $(-2,0)$
- We calculate

$$\int_C (x^2 - y^2 + 1) ds$$

- We can choose any parametrization for C, for instance,

$$\begin{aligned} \mathbf{x}(t) &= (2\cos t, 2\sin t), \quad 0 \le t \le \pi \\ &\text{or} \\ \mathbf{y}(t) &= (-2\cos 2t, -2\sin 2t), \quad -\frac{\pi}{2} \le t \le 0 \end{aligned}$$

- Note that $\mathbf{y}(t) = \mathbf{x}(2t + \pi)$

Example 10

- Let C be the upper semicircle of radius 2, centered at $(0,0)$ and oriented counterclockwise from $(2,0)$ to $(-2,0)$
- We calculate

$$\int_C (x^2 - y^2 + 1) ds$$

$$\mathbf{x}(t) = (2\cos t, 2\sin t), \quad 0 \leq t \leq \pi$$

- Then

$$\int_C (x^2 - y^2 + 1) ds = \int_{\mathbf{x}} (x^2 - y^2 + 1) ds$$

$$= \int_0^{\pi} \left(4\cos^2 t - 4\sin^2 t + 1\right) \sqrt{4\sin^2 t + 4\cos^2 t}\, dt$$

By the double-angle formula $\cos(2t) = \cos^2 t - \sin^2 t$

$$= \int_0^{\pi} (4\cos 2t + 1)\, 2\, dt = 2\left(\sin 2t + t\right)\Big|_0^{\pi} = 2\pi$$

Example 10

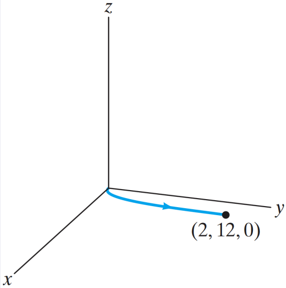
- Let C be the upper semicircle of radius 2, centered at $(0,0)$ and oriented counterclockwise from $(2,0)$ to $(-2,0)$
- We calculate
$$\int_C (x^2 - y^2 + 1)ds$$
$$\mathbf{y}(t) = (-2\cos 2t, -2\sin 2t), \quad -\frac{\pi}{2} \leq t \leq 0$$
- Then
$$\int_C (x^2 - y^2 + 1)ds = \int_{\mathbf{y}} (x^2 - y^2 + 1)ds$$
$$= \int_{-\pi/2}^{0} \left(4\cos^2 2t - 4\sin^2 2t + 1\right) \sqrt{16\sin^2 2t + 16\cos^2 2t}\, dt$$
By the double-angle formula
$$= \int_{-\pi/2}^{0} (4\cos 4t + 1)\, 4\, dt = 4\left(\sin 4t + t\right)\Big|_{-\pi/2}^{0} = 2\pi$$

Example 11

- Consider the force

$$\mathbf{F} = x\mathbf{i} - y\mathbf{j} + (x + y + z)\mathbf{k}$$

- We calculate the work done by the force F on a particle that moves
    - Along the parabola $y = 3x^2$, z $= 0$
    - From the origin to the point $(2, 12, 0)$



Example 11

- Consider the force

$$\mathbf{F} = x\mathbf{i} - y\mathbf{j} + (x+y+z)\mathbf{k}$$

$$\text{Along } y = 3x^2,\ z = 0, \text{ from } (0,0,0) \text{ to } (2,12,0)$$

- We parametrize the parabola by

$$x = t, y = 3t^2, z = 0 \text{ for } 0 \leq t \leq 2$$

- Then, by Definition 1.2

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{x}(t))\cdot \mathbf{x}'(t)dt$$

$$= \int_0^2 \left(t, -3t^2, t+3t^2\right)\cdot(1, 6t, 0)\,dt = \int_0^2 \left(t - 18t^3\right)\,dt$$

$$= \left(\frac{1}{2}t^2 - \frac{9}{2}t^4\right)\Big|_0^2 = 2 - 72 = -70$$

Example 11

- Consider the force

$$\mathbf{F} = x\mathbf{i} - y\mathbf{j} + (x + y + z)\mathbf{k}$$

Along $y = 3x^2$, z $= 0$, from $(0, 0, 0)$ to $(2, 12, 0)$

- We parametrize the parabola by

$$x = t, y = 3t^2, z = 0 \text{ for } 0 \leq t \leq 2$$

- Then, by Definition 1.2

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt = -70$$

- The meaning of the negative sign is that by moving along the curve in the indicated direction, work is done against the force

Example 11

- Consider the force

$$\mathbf{F} = x\mathbf{i} - y\mathbf{j} + (x + y + z)\mathbf{k}$$

Along $y = 3x^2$, z $= 0$, from $(0, 0, 0)$ to $(2, 12, 0)$

- We parametrize the parabola by

$$x = t, y = 3t^2, z = 0 \text{ for } 0 \leq t \leq 2$$

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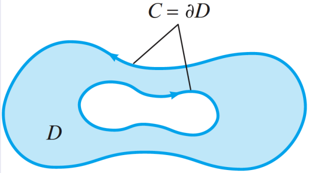
- If we orient the curve the opposite way, then the work done in moving from $(2, 12, 0)$ to $(0, 0, 0)$ would be 70

Outline

Theorem 2.1: Green's Theorem

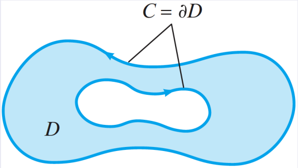
- Let D be a closed, bounded region in $\mathbb{R}^2$
- Assume its boundary $C = \partial D$ consists of finitely many simple, closed, piecewise C^1 curves
- Orient the curves of C so that D is on the left as one traverses C



- If $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)j$ is a vector field of class C^1 throughout D, then

$$\oint_C M dx + N dy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Theorem 2.1: Green's Theorem



- If $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)j$ is a vector field of class C^1 throughout D, then

$$\oint_C M dx + N dy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

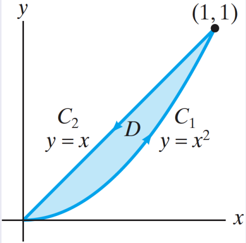
- The symbol $\oint_C$ indicates that the line integral is taken over one or more closed curves

Green's Theorem relates the vector line integral around a closed curve C in $\mathbb{R}^2$ to an appropriate double integral over the plane region D bounded by C

Outline

Example 1

- Let $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ and let D be the first quadrant region bounded by the line $y = x$ and the parabola $y = x^2$

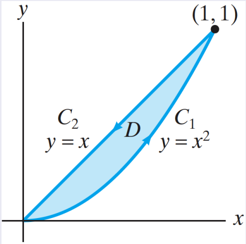


- ∂D is oriented counterclockwise, the orientation stipulated by the statement of Green's Theorem
- We can calculate

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial D} xy\, dx + y^2\, dy$$

Example 1

- Let $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ and let D be the first quadrant region bounded by the line $y = x$ and the parabola $y = x^2$



- We need to parametrize the two C^1 pieces of ∂D separately

$$C_1 : \begin{cases} x = t \\ y = t^2 \end{cases}, 0 \leq t \leq 1 \quad \text{and} \quad C_2 : \begin{cases} x = 1 - t \\ y = 1 - t \end{cases}, 0 \leq t \leq 1$$

- Note the orientations of C_1 and C_2

Example 1

$\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$, D be the first quadrant bounded by $y = x$ and $y = x^2$

$C_1 : \begin{cases} x = t \\ y = t^2 \end{cases}, 0 \leq t \leq 1$ and $C_2 : \begin{cases} x = 1 - t \\ y = 1 - t \end{cases}, 0 \leq t \leq 1$

- Then

$$\oint_{\partial D} xy\ dx + y^2\ dy = \oint_{C_1} xy\ dx + y^2\ dy + \oint_{C_2} xy\ dx + y^2\ dy$$

$$= \int_0^1 \left(t \cdot t^2 + t^4 \cdot 2t\right) dt + \int_0^1 \left((1-t)^2 + (1-t)^2\right)(-dt)$$

$$= \int_0^1 \left(t^3 + 2t^5\right) dt + \int_0^1 2(1-t)^2(-dt)$$

$$= \left(\frac{1}{4}t^4 + \frac{2}{6}t^6\right)\Bigg|_0^1 + \left(\frac{2}{3}(1-t)^3\right)\Bigg|_0^1 = \frac{1}{4} + \frac{2}{6} - \frac{2}{3} = -\frac{1}{12}$$

Example 1

$\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$, D be the first quadrant bounded by $y = x$ and $y = x^2$

$$C_1 : \begin{cases} x = t \\ y = t^2 \end{cases}, 0 \leq t \leq 1 \text{ and } C_2 : \begin{cases} x = 1 - t \\ y = 1 - t \end{cases}, 0 \leq t \leq 1$$

- On the other hand

$$\int\int_D \left(\frac{\partial}{\partial x}\left(y^2\right) - \frac{\partial}{\partial y}(xy) \right) dx dy = \int_0^1 \int_{x^2}^{x} -x \, dy dx$$

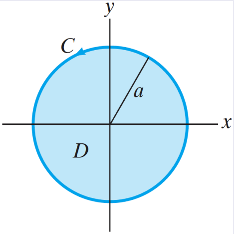
$$= \int_0^1 -x\left(x - x^2\right) \, dx = \int_0^1 \left(x^3 - x^2\right) \, dx = \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 \right)\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

- The line integral and the double integral agree

Example 2

- Let C be the circle of radius a, oriented counterclockwise
- Then, C is the boundary of the disk D of radius a



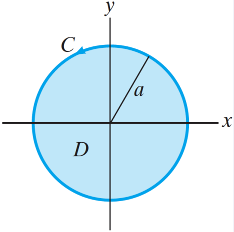
- We calculate the line integral

$$\oint_C -y\,dx + x\,dy$$

- Although we can parametrize C and thus evaluate the line integral, it is easier to employ Green's Theorem instead

Example 2

- Let C be the circle of radius a, oriented counterclockwise
- Then, C is the boundary of the disk D of radius a



- We calculate line integral

$$\oint_C -y\, dx + x\, dy = \int\int_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right) dxdy$$

$$= \int\int_D 2\, dxdy = 2(\text{Area of } D) = 2\pi a^2$$

Generalization of Example 2

- Suppose D is any region to which Green's Theorem can be applied
- Then, orienting ∂D appropriately, we have

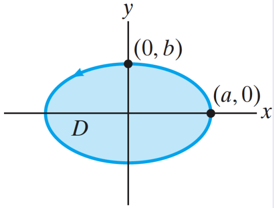
$$\frac{1}{2}\oint_{\partial D} -y\ dx + x\ dy = \frac{1}{2}\int\int_D 2\ dxdy = \text{Area of } D$$

- Thus, we can calculate the area of a region (two-dimensional) by using line integrals (one-dimensional)

Example 3

- We compute the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



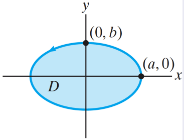
- The ellipse itself may be parametrized counterclockwise by

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, 0 \leq t \leq 2\pi$$

Example 3

- We compute the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, 0 \le t \le 2\pi$$



- Then

$$\text{Area of ellipse} = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t\,dt) + a\cos t(b\cos t\,dt)$$

$$= \frac{1}{2}\int_0^{2\pi} \left(ab\sin^2 t + ab\cos^2 t\right)dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab$$