Problems

Problem 7.1 Let f and g be differentiable functions in $\mathbb{R}$. Write down the derivative of the following functions in their respective domains:

(i) $h(x) = \sqrt{f(x)^2 + g(x)^2}$;

(ii) $h(x) = \arctan\left(\dfrac{f(x)}{g(x)}\right)$;

(iii) $h(x) = f\big(g(x)\big)e^{f(x)}$;

(iv) $h(x) = \log\big(g(x)\sin f(x)\big)$;

(v) $h(x) = f(x)^{g(x)}$;

(vi) $h(x) = \dfrac{1}{\log\big(f(x) + g(x)^2\big)}$.

Problem 7.2

(a) Make up a continuous function in $\mathbb{R}$ which vanishes for $|x| \geqslant 2$ and equals 1 for $|x| \leqslant 1$.

(b) Do it again, but this time make sure that the function is differentiable in $\mathbb{R}$.

Problem 7.3 Check that the following functions satisfy the specified differential equations, where c, c_1, and c_2 are constants:

(i) $f(x) = \dfrac{c}{x}$ satisfies $xf' + f = 0$;

(ii) $f(x) = x\tan x$ satisfies $xf' - f - f^2 = x^2$;

(iii) $f(x) = c_1 \sin 3x + c_2 \cos 3x$ satisfies $f'' + 9f = 0$;

(iv) $f(x) = c_1 e^{3x} + c_2 e^{-3x}$ satisfies $f'' - 9f = 0$;

(v) $f(x) = c_1 e^{2x} + c_2 e^{5x}$ satisfies $f'' - 7f' + 10f = 0$;

(vi) $f(x) = \log(c_1 e^x + e^{-x}) + c_2$ satisfies $f'' + (f')^2 = 1$.

Problem 7.4 Prove the identities (valid only in the specified regions)

(i) $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}$, for $x > 0$;

(ii) $\arctan\dfrac{1+x}{1-x} - \arctan x = \dfrac{\pi}{4}$, for $x < 1$;

(iii) $2\arctan x + \arcsin\dfrac{2x}{1+x^2} = \pi$, for $x \geqslant 1$.

HINT: Differentiate the equation and check one point of the specified region.

Problem 7.5 At which points does the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent?

Problem 7.6 Given the function

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

calculate the angle between the tangents on the left and on the right of its graph at $x = 0$.

Problem 7.7 Find the sets where the function $f(x) = \sqrt{x+2}\arccos(x+2)$ is continuous and differentiable.

Problem 7.8 Calculate the smallest α for which $f(x) = |\alpha x^2 - x + 3|$ is differentiable in $\mathbb{R}$

Problem 7.10 Given the function

$$f(x)=\begin{cases} a+bx^2, & |x|\leqslant c,\\ \dfrac{1}{|x|}, & |x|>c, \end{cases} \qquad c>0,$$

find a and b so that it is continuous and differentiable in $\mathbb{R}$.

Problem 7.11 Given the function

$$f(x)=\begin{cases} \dfrac{3-x^2}{2}, & x<1,\\ \dfrac{1}{x}, & x\geqslant 1, \end{cases}$$

(a) determine the sets where it is continuous and where it is differentiable;

(b) check that the mean value theorem can be applied to this function in $[0,2]$ by determining the point(s) $c\in(0,2)$ where the theorem holds.

Problem 7.12 Function $f(x)=1-x^{2/3}$ vanishes in $x=\pm1$; however $f'(x)\neq 0$ in $(-1,1)$. Find which hypothesis of Rolle's theorem is not satisfied.

Problem 7.13 Prove, using Rolle's theorem, the following statements about a function f that is continuous in $[a,b]$ and differentiable in (a,b):

(i) If f vanishes $k(\geqslant 2)$ times in $[a,b]$ then f' vanishes at least $k-1$ times in $[a,b]$.

(ii) If f is n-times differentiable in (a,b) and vanishes in $n+1$ different points of $[a,b]$, then $f^{(n)}$ vanishes at least once in $[a,b]$.

Problem 7.14 Using the mean value theorem, find an approximation to $26^{2/3}$ and $\log(3/2)$.

Problem 7.15 Calculate the limits

(i) $\displaystyle\lim_{x\to0}\frac{e^x-\sin x-1}{x^2}$;

(ii) $\displaystyle\lim_{x\to0}\frac{\log|\sin 7x|}{\log|\sin x|}$;

(iii) $\displaystyle\lim_{x\to1^+}\log x\log(x-1)$;

(iv) $\displaystyle\lim_{x\to\infty}x^{1/x}$;

(v) $\displaystyle\lim_{x\to0}\frac{(1+x)^{1+x}-1-x-x^2}{x^3}$;

(vi) $\displaystyle\lim_{x\to\infty}x\left(\tan\frac{2}{x}-\tan\frac{1}{x}\right)$.

Problem 7.16 Suppose $h(x)$ is a twice-differentiable function and let

$$f(x)=\begin{cases} \dfrac{h(x)}{x^2}, & x\neq 0,\\ 1, & x=0. \end{cases}$$

Calculate $h(0)$, $h'(0)$, and $h''(0)$ so that f is continuous.

Problem 7.17 Calculate the limits

(i) $\displaystyle\lim_{x\to\infty}x\left[\left(1+\frac{1}{x}\right)^x-e\right]$;

(iii) $\displaystyle\lim_{x\to\infty}\left(\frac{2^{1/x}+18^{1/x}}{2}\right)^x$;

(a) calculate $f(0)$;

(b) calculate $f'(0)$;

(c) calculate $\lim_{x\to 0} \dfrac{(f\circ f)(2x)}{f^{-1}(3x)}$.

Problem 7.19 The equation $e^{-f}f' = 2+\tan x$ together with the condition $f(0) = 1$ define a one-to-one, differentiable function in the interval $[-\pi/4, \pi/4]$. If $g(x) = f^{-1}(x+1)$, calculate the limit

$$\lim_{x\to 0} \frac{e^x - e^{-\sin x}}{g(x)}.$$

Problem 7.20 Let $f(x) = |x^3(x-4)| - 1$.

(a) Find where f is continuous and where it is differentiable.

(b) Determine its extrema.

(c) Prove that $f(x) = 0$ has a unique solution in $[0,1]$.

Problem 7.21 Solve these optimisation problems:

(a) A factory that produces tomato sauce wants to can it in cylindrical cans of a fixed volume V. Determine their radius r and height h so that their fabrication consumes the least possible material.

(b) A recipient with square bottom and no cap must be covered by a thin layer of lead. If the volume of the recipient must be 32 litres, which dimensions should it have so that it requires the least possible amount of lead?

(c) Find two numbers $x, y > 0$ such that $x + y = 20$ and x^2y^3 is maximum.

(d) Find the rectangle inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$ with its sides parallel to the axes of the ellipse, such that its area is maximum.

(e) With a tangent to the parabola $y = 6 - x^2$ and the positive axes one can make a triangle. Determine which of those triangles has the smallest area and compute it.

(f) We need to construct a box with no cap with the shape of a parallelepiped whose base is an equilateral triangle, and whose volume is 128 cm^3. If the material for the base costs 0.20 euros/cm^2 and that for the lateral surfaces costs 0.10 euros/cm^2, what are the dimensions of the cheapest such box?

(g) A right triangle ABC has vertex A at the origin, vertex B on the circumference $(x-1)^2 + y^2 = 1$ —side AB is the hypothenuse of the triangle— and side AC on the horizontal axis. Calculate the location of C that maximises the area of the triangle.

(h) Let $\text{P} = (x_0, y_0)$ be a point of the first quadrant $(x_0, y_0 > 0)$. A straight line through P cuts the axes at $\text{A} = (x_0+\alpha, 0)$ and $\text{B} = (0, y_0+\beta)$. Calculate $\alpha > 0$ and $\beta > 0$ so as to minimise

  (i) the length of segment AB;

  (ii) the sum of the lengths of OA and OB;

  (iii) the area of the triangle OAB.

  HINT: Triangle similarity implies $\beta = x_0y_0/\alpha$.

Problem 7.22 Prove the following inequalities:

(b) Prove that the previous inequality is equivalent to $e^x > x^e$ for all $x > 0$, $x \neq e$.

Problem 7.24 Determine the number of solutions of the following equations in the specified domains:

(i) $x^7 + 4x = 3$ in $\mathbb{R}$;

(ii) $x^5 = 5x - 6$ in $\mathbb{R}$;

(iii) $x^4 - 4x^3 = 1$ in $\mathbb{R}$;

(iv) $\sin x = 2x - 1$ in $\mathbb{R}$;

(v) $x^x = 2$ in $[1, \infty)$;

(vi) $x^2 = \log \frac{1}{x}$ in $(1, \infty)$.