Problems

Problem 8.1 Write the Taylor polynomial $P_{5,0}(x)$ for these functions:

(i) $e^x \sin x$;
(ii) $e^{-x^2} \cos 2x$;
(iii) $\sin x \cos 2x$;
(iv) $e^x \log(1-x)$;
(v) $\sin^2 x$;
(vi) $\dfrac{1}{1-x^3}$.

Problem 8.2 Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x-4$.

Problem 8.3 Write the Taylor polynomial $P_{n,a}(x)$ for these functions around the specified a:

(i) $f(x) = 1/x$ around $a = -1$;
(ii) $f(x) = xe^{-2x}$ around $a = 0$;
(iii) $f(x) = (1+e^x)^2$ around $a = 0$;
(iv) $f(x) = \sin x$ around $a = \pi$.

Problem 8.4 Write the Taylor polynomial $P_{n,0}(x)$ of the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Problem 8.6 Calculate the following limits using Taylor's theorem:

(i) $\lim\limits_{x\to 0} \dfrac{e^x - \sin x - 1}{x^2}$;
(ii) $\lim\limits_{x\to 0} \dfrac{\sin x - x + x^3/6}{x^5}$;
(iii) $\lim\limits_{x\to 0} \dfrac{\cos x - \sqrt{1-x}}{\sin x}$;
(iv) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$;
(v) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x(1-\cos 3x)}$;
(vi) $\lim\limits_{x\to 0} \dfrac{\cos x + e^x - x - 2}{x^3}$;
(vii) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$;
(viii) $\lim\limits_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{x} - \cot x\right)$;
(ix) $\lim\limits_{x\to\infty} x^{3/2}\left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$;
(x) $\lim\limits_{x\to\infty} \left[x - x^2 \log\left(1 + \dfrac{1}{x}\right)\right]$.

Problem 8.8 Determine the first nonzero order in the Taylor expansion of the following functions:

(i) $f(x) = \tan(\sin x) - \sin(\tan x)$;
(ii) $f(x) = \dfrac{1}{R^2} - \dfrac{1}{(R+x)^2}$;
(iii) $f(x) = \sqrt[3]{\dfrac{1+x}{1-x}} - \sqrt[3]{\dfrac{1-x}{1+x}}$.

Problem 8.15 Given the function $f(x) = \cos x + e^x$,

(i) find its Taylor polynomial $P_{3,0}(x)$;
(ii) estimate an upper bound for the error incurred if $-1/4 \leqslant x \leqslant 1/4$.

Problem 8.16 What is the smallest degree Taylor polynomial necessary to approximate the function $f(x) = e^x$ in $[-1, 1]$ with at least three exact decimal places?