Problems

Problem 9.1 Obtain the following immediate (or nearly so) primitives:

(i) $\int \frac{dx}{\cos^2 x}$; (ii) $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$; (iii) $\int \frac{x}{(x^2+1)^{5/2}}\,dx$; (iv) $\int \frac{1+\sin x}{1+\cos x}\,dx$; (v) $\int \frac{dx}{1-\sin x}\,dx$; (vi) $\int \frac{x}{\sqrt{1+x^2}}\,dx$; (vii) $\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$; (viii) $\int \frac{\cos^3 x}{\sin^4 x}\,dx$; (ix) $\int x^3\sqrt{1-x^2}\,dx$.

HINTS: (iv) multiply and divide by $1-\cos x$ and expand; (v) idem with $1+\sin x$; (vii) alternatively $t=\sqrt{1-\sqrt{x}}$; (viii) $\cos^3 x = (1-\sin^2 x)\cos x$ and expand; (ix) write $x^3 = x(x^2-1)+x$ and expand.

Problem 9.2 Obtain the primitives of the following rational functions:

(i) $\int \frac{x^2}{(x-1)^3}\,dx$; (ii) $\int \frac{dx}{(x-1)^2(x^2+x+1)}$; (iii) $\int \frac{2x^2+3}{x^2(x-1)}\,dx$; (iv) $\int \frac{2}{x^2-2x+2}\,dx$; (v) $\int \frac{4x^2-x^3-46x^2-20x+153}{x^3-2x^2-9x+18}\,dx$; (vi) $\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$.

HINTS: (ii) $x^2+x+1=(x+1/2)^2+3/4$; (v) $x^3-2x^2-9x+18=(x-2)(x-3)(x+3)$; (vi) $x^4-2x^2+1=(x-1)^2(x+1)^2$.

Problem 9.3 Obtain the following primitives doing an appropriate change of variable:

(i) $\int x^2\sqrt{x-1}\,dx$; (ii) $\int x^2\sin\sqrt{x^3}\,dx$; (iii) $\int \cos(\log x)\,dx$; (iv) $\int \sin(\log x)\,dx$; (v) $\int \cos^2(\log x)\,dx$; (vi) $\int \frac{\sqrt{x}+1}{x+3}\,dx$; (vii) $\int \frac{(x+1)^3}{\sqrt{1-(x+1)^2}}\,dx$; (viii) $\int \frac{x^3}{(1+x^2)^3}\,dx$; (ix) $\int \frac{dx}{(2+x)\sqrt{1+x}}$; (x) $\int \frac{dx}{1+\sqrt[3]{1-x}}$; (xi) $\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$; (xii) $\int \frac{dx}{\sqrt{e^{2x}-1}}\,dx$; (xiii) $\int \sqrt{e^x-1}\,dx$; (xiv) $\int \frac{\sin^2 x\cos^5 x}{\tan^3 x}\,dx$; (xv) $\int \frac{dx}{3+\sqrt{2x+5}}$; (xvi) $\int \sqrt{\frac{x-1}{x+1}}\,dx$; (xvii) $\int \sqrt{\sqrt{x}+1}\,dx$; (xviii) $\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\,dx$; (xix) $\int \sqrt{2+e^x}\,dx$; (xx) $\int \frac{\sin x+3\cos x}{\sin x+2\cos x}\,dx$; (xxi) $\int \frac{\sin x+3\cos x}{\sin x\cos x+2\sin x}\,dx$; (xxii) $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$; (xxiii) $\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$; (xxiv) $\int \frac{dx}{e^x-4e^{-x}}\,dx$.

HINTS: (i) $t=\sqrt{x-1}$ (or int. by parts twice); (ii) $t^2=x^3$; (iii)–(v) $t=\log x$; (vi) $t=\sqrt{x}$; (vii) $t=\sqrt{1-(x+1)^2}$; (viii) $t=1+x^2$; (ix) $t^2=1+x$; (x) $t^3=1-x$; (xi) $t=e^x$; (xii) $t^2=e^{2x}-1$; (xiii) $t^2=e^x-1$; (xiv) $t=\cos x$; (xv) $t=3+\sqrt{2x+5}$; (xvi) $t=\sqrt{(x-1)/(x+1)}$; (xvii) $t=\sqrt{\sqrt{x}+1}$; (xviii) $t=\sqrt{x+2}$; (xix) $t=\sqrt{2+e^x}$; (xx) $t=\tan x$; (xxi) $t=\tan(x/2)$; (xxii) $t=\sqrt{1+x^{1/3}}$; (xxiii) $t^3=x+2$; (xxiv) $t=e^x$.

Problem 9.4 Obtain the following primitives with the help of some trigonometric identity:

(i) $\int \sin^2 x\,dx$; (ii) $\int \cos^2 x\,dx$; (iii) $\int \sin^4 x\,dx$; (iv) $\int \cos^4 x\,dx$; (v) $\int \cos^6 x\,dx$;

(vi) $\int \sin^2 x \cos^2 x\,dx$; (vii) $\int \tan^2 x\,dx$; (viii) $\int \tan^4 x\,dx$; (ix) $\int \frac{dx}{\cos^4 x}$; (x) $\int \sin^5 x\,dx$;

(xi) $\int \cos^3 x \sin^2 x\,dx$; (xii) $\int \sec^6 x\,dx$; (xiii) $\int \sin^3 x \cos^2 x\,dx$; (xiv) $\int \tan^3 x\,dx$; (xv) $\int \tan^3 x \sec^4 x\,dx$.

HINTS: Identities to use: $2\cos^2 x = 1+\cos 2x$; $2\sin^2 x = 1-\cos 2x$; $\cos^2 x + \sin^2 x = 1$; $\sec^2 x = 1+\tan^2 x$.

Problem 9.5 Integrate by parts to obtain the following primitives:

(i) $\int x\tan^2(2x)\,dx$; (ii) $\int e^x \sin \pi x\,dx$; (iii) $\int e^x \cos 2x\,dx$; (iv) $\int \sec^3 x\,dx$;

(v) $\int \tan^2(3x)\sec^3(3x)\,dx$; (vi) $\int e^{\sin x}\cos^3 x\,dx$; (vii) $\int x^2 \log x\,dx$; (viii) $\int x^m \log x\,dx$;

(ix) $\int (\log x)^3\,dx$; (x) $\int x(\log x)^2\,dx$; (xi) $\int \frac{x\log x}{(1+x^2)^2}\,dx$; (xii) $\int \arctan \sqrt[3]{x}\,dx$.

Problem 9.6 Obtain the following primitives by performing a trigonometric substitution:

(i) $\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx$; (ii) $\int \frac{x^2}{(x^2+1)^{5/2}}\,dx$; (iii) $\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$; (iv) $\int \frac{dx}{x^2\sqrt{1-x^2}}$; (v) $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Problem 10.2

(a) Prove that if f is odd then $\int_{-a}^{a} f(x)\,dx = 0$.

(b) Prove that if f is even then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(c) Calculate the integral $\int_6^{10} \sin\Big(\sin\big((x-8)^3\big)\Big)\,dx$.

Problem 10.4 Calculate $F(x) = \int_{-1}^{x} f(t)\,dt$, with $-1 \leqslant x \leqslant 1$, for the following functions:

(i) $f(x) = |x|e^{-|x|}$;

(ii) $f(x) = |x-1/2|$;

(iii) $f(x) = \begin{cases} -1, & -1 \leqslant x < 0, \\ 1, & 0 \leqslant x \leqslant 1; \end{cases}$

(iv) $f(x) = \begin{cases} x^2, & -1 \leqslant x < 0, \\ x^2-1, & 0 \leqslant x \leqslant 1; \end{cases}$

(v) $f(x) = \begin{cases} 1, & -1 \leqslant x \leqslant 0, \\ x+1, & 0 < x \leqslant 1; \end{cases}$

(vi) $f(x) = \begin{cases} 1+x, & -1 \leqslant x \leqslant -\frac{1}{2}, \\ \frac{1}{2}, & -\frac{1}{2} < x < \frac{1}{2}, \\ 1-x, & \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$

(vii) $f(x) = \max\{\sin(\pi x/2), \cos(\pi x/2)\}$.

Problem 10.5 Calculate the following integrals:

(i) $\int_0^{\log 2} \sqrt{e^x - 1}\,dx$; (ii) $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$.

Problem 10.12 Given the function

$$f(x) = \begin{cases} \dfrac{e^x - 1 - x}{x^2}, & x < 0, \\ a + b\displaystyle\int_0^x e^{-t^4}\,dt, & x \geqslant 0, \end{cases}$$

calculate a and b so that it is continuous and differentiable

Problems

Problem 11.1 Calculate the area delimited by the following curves:
(i) $y = x^2, y = (x-2)^2, y = (2-x)/6$;
(ii) $x^2+y^2 = 1, x^2+y^2 = 2x$;
(iii) $y = \dfrac{1-x}{1+x}, y = \dfrac{2-x}{1+x}, y = 0, y = 1$;
(iv) one loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

Problem 11.2 Determine the area between the curve $f(x) = \dfrac{x(x^2-1)}{(x^2+1)^{3/2}}$ and the X axis.

Problem 11.3 Calculate the area delimited by the following curves:
(i) $r = a\theta$ (Archimedes's spiral), $0 \leqslant \theta \leqslant 2\pi$, and the segment $\{(x,0) : 0 \leqslant x \leqslant 2\pi a\}$;
(ii) a petal of the three-petal rose $r = a\cos 3\theta$, $-\pi/6 \leqslant \theta \leqslant \pi/6$;
(iii) half a *lemniscata* $r = a\sqrt{\cos 2\theta}$, $-\pi/4 \leqslant \theta \leqslant \pi/4$.

Problem 11.4 Let A the plane figure limited by the curves $y = x^2$ and $y = \sqrt{x}$. Determine:
(a) the area of A;

Problem 12.1 Discuss the convergence of the following improper integrals:

(i) $\displaystyle\int_0^\infty e^{-t}t^{p-1}\,dt$;
(ii) $\displaystyle\int_0^a e^{1/x}x^p\,dx$;
(iii) $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx$;
(iv) $\displaystyle\int_{-\infty}^\infty e^{-x}\,dx$;
(v) $\displaystyle\int_1^\infty \frac{dx}{x^\beta\sqrt{1+x^2}}$;
(vi) $\displaystyle\int_0^1 \log x\,dx$;
(vii) $\displaystyle\int_1^2 \frac{\log t + t - 1}{(t-1)^{3/2}}\,dt$;
(viii) $\displaystyle\int_{-\infty}^\infty \frac{\cos ax}{k^2+x^2}\,dx$, with $k \neq 0$;
(ix) $\displaystyle\int_1^\infty \left(\frac{1}{\sqrt{x}} - \arctan\frac{1}{\sqrt{x}}\right)dx$.

Problem 12.2 Prove that the following integrals converge for the specified range of parameters:

(i) $\displaystyle\int_0^1 x^p(1-x)^q\,dx$, for $p,q > -1$;
(ii) $\displaystyle\int_0^\infty \log\left(1+\frac{a^2}{x^2}\right)dx$, for $a \in \mathbb{R}$;
(iii) $\displaystyle\int_0^{\pi/2} \log\left(\frac{1+s\cos x}{1-s\cos x}\right)\frac{dx}{\cos x}$, for $|s| \leqslant 1$;
(iv) $\displaystyle\int_0^1 \frac{x^p-1}{\log x}\,dx$, for $p > -1$.

Problem 12.3 Consider the improper integral of the first kind

$$\int_0^\infty \left(\frac{1}{\sqrt{1+x^2}} - \frac{\alpha}{x+1}\right)dx.$$

(a) Find the only value of α for which this integral converges.
(b) Calculate the integral for that value of α.