# Sequences and series

Michael Stich

January 13, 2021

## 11 Numerical sequences and series

In this part of calculus we will define finite and infinite *sequences* of numbers and finite and infinite *series* of numbers. This part of calculus is relatively independent of the study of functions that we have considered in some detail earlier, in particular when compared to continuity, differentiability and integrability of functions. However, there is one pivotal point that connects methodologically these areas, and it is the definition of limits, a topic that we have already used extensively.

### 11.1 Numerical sequences: definition and convergence

Let us start with a few examples:

$$\{1, 2, 3, 4, \dots\}, \quad \{-1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1\}, \quad \{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$$

are all *numerical sequences*. The dots indicate that the sequence proceeds according to the rule given by the previous terms, and it is therefore clear that the first example indicates the sequence of natural numbers, which is an infinite sequence. The order of the elements of the sequence is important: The sequence that starts as $\{1, 3, 2, 4, 5, 7, 6, 8, 9, 11, 10, 12, \dots\}$ also yields the same set of numbers, but in a different order and the sequence is different.

The second sequence is a finite sequence created with a specific rule, but we will be focus our interest on infinite sequences (in fact, many books do not consider finite sequences). Unless stated otherwise, all sequences from now on will be infinite sequences.

Finally, the third sequence is created by the rule $a_k = a_{k-1} + a_{k-2}$. This introduces several important properties of sequences: first, an infinite sequence is generally given by

$$\{a_0, a_1, a_2, \dots\} = \{a_k\}_{k=0}^{\infty} = \{a_k\} = (a_k) = a_k,$$

where $a_k$ is called the *general term* and where we show alternative notations in increasing degree of simplicity (the specific choice depends on the context). Obviously, the symbols indicating the general term (here $a$) and the index (here $k$) are arbitrary.

Second, the rule $a_k = a_{k-1} + a_{k-2}$ uses the values $a_{k-1}$ and $a_{k-2}$, which have to be known before. But to determine $a_{k-2}$, we need $a_{k-3}$ and $a_{k-4}$, and so on. Such kind of rule is called a *recurrence relation* (at least one previous value is needed to determine the next one). However, the statement of the rule is not enough: we need to know the starting value(s) of the sequence, in this case two: $a_0 = 1$ and $a_1 = 1$. The specific sequence defined in this way is called the Fibonacci sequence.

Third, the sequence typically starts with the element indexed with 0, but we may also use initializations with the first index being 1.

For the specific infinite sequences considered above, the $k$-th element becomes larger and larger and – due to the unboundedness of natural numbers – we observe

$$\lim_{k\to\infty} a_k = \infty.$$

Let us now consider the sequences

$$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \quad \text{and} \quad \{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots\}.$$

It is important to recognize the rule with which a sequence is created, and for these sequences it is quite simple to establish

$$a_k = \frac{1}{k+1}, \quad k = 0, 1, 2, \dots \qquad \text{or simpler} \qquad a_k = \frac{1}{k}, \quad k = 1, 2, 3, \dots$$

as general term for the first sequence and

$$a_k = \frac{1}{k^2}, \quad k = 1, 2, 3, \dots$$

as general term for the second sequence. For these examples, we see that

$$\lim_{k\to\infty} a_k = 0.$$

Note that $a_k \neq 0$ for all $k$, so we observe – like in the chapter on limits – that the limit value need not be part of the sequence. We define:

**Definition 1 (limit point of a sequence):**
Let $(a_k)$, $k = 0, 1, 2, \dots$ define an infinite sequence. Then, $a$ is the limit (or limit point) of the sequence $(a_k)$, i.e.,

$$\lim_{k\to\infty} a_k = a,$$

if and only if for all $\epsilon > 0$, there exist a number $N$ such that

$$|a_k - a| < \epsilon \quad \text{for all } k > N.$$

Example:

Prove, using the definition, that the limit point of $(a_k)$, $k = 0, 1, \ldots$, with $a_k = \dfrac{k+4}{k+1}$ is 1.

We proceed by using

$$|a_k - 1| < \epsilon \quad \text{for all } k > N.$$

This converts into

$$\left|\frac{k+4}{k+1} - 1\right| = \frac{3}{k+1} < \epsilon \quad \text{or} \quad k > \frac{3}{\epsilon} - 1.$$

Essentially, the right-hand side of the last inequality defines the value of $N$. Of course, the important range of $\epsilon$ is when it is a small value, but the theorem only assumes that it is positive. Therefore, we set $N = max(0, \lceil \frac{3}{\epsilon} - 1 \rceil)$. In this way, $N$ is now a non-negative integer value that is – no matter how small $\epsilon$ – large enough to ensure that all sequence elements with $k > N$ are within the range $(1-\epsilon, 1+\epsilon)$ and in this way proving that the limit is 1.

Comments:

(1) This definition is analogous to the definition of a limit of a function as $x \to \infty$, with the difference that $k$ can only take integer values and that $(a_k)$ is a sequence, not a function (see also Theorem 1).

(2) If the limit exists (and this implies that $a$ is finite), then we say that the sequence $(a_k)$ *converges* to $a$.

(3) If the limit does not exist, the sequence can either *diverge* to $\pm\infty$ or it neither converges nor diverges to $\pm\infty$, like for the infinite sequence $1, -1, 1, -1, 1, -1, \ldots$.

(4) There is an ambiguous use of notation in the literature: Depending on the context, divergence may either refer to all sequences that do not converge, or only to those that tend to $\pm\infty$.

**Theorem 1 (limit of a sequence and function):**

If $\lim\limits_{x\to\infty} f(x)$ exists, then the sequence given by $a_k = f(k)$ converges to the same limit: $\lim\limits_{k\to\infty} a_k = \lim\limits_{x\to\infty} f(x)$.

Comment: This theorem specifies what has been said in Comment 1 of Definition 1.

Example:

Find the limit of the sequence given by

$$\{\frac{2^2-2}{2^2}, \frac{3^2-2}{3^2}, \frac{4^2-2}{4^2}, \frac{5^2-2}{5^2}, \ldots\}$$

This sequence has a general term

$$a_k = \frac{k^2-2}{k^2} = 1 - \frac{2}{k^2}.$$

We apply Theorem 1 using $f(x) = 1 - \dfrac{2}{x^2}$ (so we simply replace $k$ by $x$) and write

$$\lim_{k\to\infty} a_k = \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(1 - \frac{2}{x^2}\right) = 1 - \lim_{x\to\infty}\frac{2}{x^2} = 1 - 0 = 1.$$

In order to find some more specific criteria on the convergence of sequences, it is useful to enlarge our repertoire of definitions.

**Definition 2 (accumulation point of a sequence):**
$a$ is an accumulation point of a sequence $(a_k)$ if in any $\epsilon$-neighbourhood of $a$ there is an infinite number of sequence elements.

Comments:
(1) This definition generalizes slightly Definition 1 (it is less strict on the hypotheses). The concept $\epsilon$-neighbourhood refers to the sequence elements that fulfil $|a_k - a| < \epsilon$. For example, the infinite sequence $1, -1, 1, -1, 1, -1, \ldots$ has no limit point, but it has two accumulation points, at $-1$ and 1, since there is an infinite number of elements of the sequence in any neighbourhood of 1, and also an infinite number of elements of the sequence in any neighbourhood of $-1$. We can extract two infinite *subsequences* (or *partial sequences* from $1, -1, 1, -1, 1, -1, \ldots$, namely, $-1, -1, -1, \ldots$ and $1, 1, 1, \ldots$ (in what follows, we always assume that subsequences of an infinite sequence are also infinite).
(2) Every limit point is also an accumulation point. Therefore, if a sequence has a limit point, it has only one accumulation point, which coincides with the limit point. The converse is not true: not all accumulation points are limit points. Remember: there can be only at most one limit point.
(3) If there is an accumulation point, there are an infinite number of sequence elements around it. This does not mean that *almost all* elements are in that neighbourhood (which would mean that there is at most a finite number of elements outside it), because there may still be an infinite number of sequence elements outside that neighbourhood. Take, for example, the sequence $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots$. There is a partial sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ which converges to 0 (and there is an infinite number of sequence elements in any $\epsilon$-neighbourhood of 0), but there is also another partial sequence $1, 2, 3, 4, \ldots$ which diverges (to $\infty$) and that sequence also has an infinite number of elements.

Why are accumulation points useful? Because of the following theorem:

**Theorem 2 (Bolzano-Weierstrass):**
Every bound sequence $(a_k)$ has at least one accumulation point.

Comments:
(1) Boundedness of a sequence is defined analogously to that of a function: there must exist a finite value $M$ such that all sequence elements are smaller than $M$, and a finite

value $m$ such that all sequence elements are larger than $m$.
(2) While this theorem does not guarantee the existence of a limit point (only an accumulation point), it has been shown to be very important in the development of other results for sequences and series – and calculus in general. This is because in many instances it is relatively easy to check for boundedness, and because it is a relatively weak hypothesis.
(3) An alternative formulation is that every bound sequence has at least one convergent subsequence.

**Theorem 3 (convergent sequence):**
A sequence $(a_k)$ is convergent if and only if it is bound and if the largest accumulation point is identical to the smallest accumulation point.

Comment:
Therefore, for a bound sequence, we may look for suitable subsequences and investigate them separately, before making a conclusion about convergence.

**Theorem 4 (sandwich theorem for sequences):**
Suppose that $(a_k)$, $(b_k)$ and $(c_k)$ are sequences with $\lim\limits_{k\to\infty} b_k = \lim\limits_{k\to\infty} c_k = L$ and that there exist a number $M$ such that $b_k \leq a_k \leq c_k$ if $k > M$. Then, $\lim\limits_{k\to\infty} a_k = L$.

Theorem 4 is analogous to the sandwich theorem for the limit of functions. Before we consider some examples, we give some properties of convergent sequences.

**Theorem 5 (properties of convergent sequences):**
Suppose that $(a_k)$ and $(b_k)$ are convergent sequences with limit points $a$ and $b$. Then,
(i) $(a_k \pm b_k)$ has limit point $a \pm b$;
(ii) $(a_k\, b_k)$ has limit point $ab$;
(iii) $(a_k/b_k)$ has limit point $a/b$ (for $b_k \neq 0$, $b \neq 0$);
(iv) $(c\, a_k)$ has limit point $ca$ (for all $c \in \mathbb{R}$).

Example: Show that if $\lim\limits_{k\to\infty} |a_k| = 0$, then $\lim\limits_{k\to\infty} a_k = 0$.

We know that $-|a_k| \leq a_k \leq |a_k|$. By hypothesis of this example, $\lim\limits_{k\to\infty} |a_k| = 0$ and also $\lim\limits_{k\to\infty} -|a_k| = -\lim\limits_{k\to\infty} |a_k| = 0$ (Theorem 5(iv) with $c = -1$). Therefore, we have two sequences that converge to the same limit point and can use the sandwich theorem. We conclude that also $\lim\limits_{k\to\infty} a_k = 0$.

Example: Show that $\lim\limits_{k\to\infty} \dfrac{R^k}{k!} = 0$ for $R \in \mathbb{R}$.

For $R = 0$, the limit is zero trivially. We now assume that $R > 0$ and define $M$ a positive integer such that

$$M \leq R < M + 1.$$

Since we are later interested in the limit $k \to \infty$, it is reasonable to consider the case where $k > M$ and express the term $\frac{R^k}{k!}$ as product of its $k$ factors:

$$\frac{R^k}{k!} = \left(\frac{R}{1}\frac{R}{2}\cdots\frac{R}{M}\right)\cdot\left(\frac{R}{M+1}\frac{R}{M+2}\cdots\frac{R}{k}\right).$$

From this factorization we make several conclusions. First, for all fixed $R$ and $M$, the first parenthesis represents a finite number of factors and can be represented by a constant $C$. Second, for $k$ finite, the second parenthesis represents a finite number of factors, each of which is *smaller* than unity and we have

$$\frac{R}{M+1}\frac{R}{M+2}\cdots\frac{R}{k} < \frac{R}{k},$$

and we get overall

$$0 < \frac{R^k}{k!} < C\frac{R}{k}.$$

The limit of 0 is trivially zero, and $\lim_{k\to\infty} C\frac{R}{k} = 0$, and therefore using the sandwich theorem we find $\lim_{k\to\infty}\frac{R^k}{k!} = 0$.

For $R < 0$, the limit is also zero as, following the previous example, $\lim_{k\to\infty}\left|\frac{R^k}{k!}\right| = 0$.

**Theorem 6 (function of a convergent sequence):**
If $f(x)$ is a continuous function and $\lim_{k\to\infty} a_k = a$, then:

$$\lim_{k\to\infty} f(a_k) = f(\lim_{k\to\infty} a_k) = f(a).$$

This Theorem goes beyond Theorem 1 because now we assume that $f$ is continuous. The theorem expresses the fact that if $a_k$ is a convergent sequence, the limit point of the new sequence $f(a_k)$ is identical to the function evaluated at the limit point.

Example: Determine $\lim_{k\to\infty} f(a_k)$ for $f(x) = e^x$ and $a_k = \frac{3k}{k+1}$.

First, we calculate

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{3k}{k+1} = \lim_{k\to\infty}\frac{3}{1+k^{-1}} = 3.$$

Therefore, we have

$$\lim_{k\to\infty} f(a_k) = \lim_{k\to\infty} e^{\frac{3k}{k+1}} = f\left(\lim_{k\to\infty}\frac{3k}{k+1}\right) = f(3) = e^3.$$

**Theorem 7 (convergent sequence is bound):**
If $(a_k)$ converges, then $(a_k)$ is bound.

Comment:
The converse is not true: if a sequence is bound, it is not necessarily convergent, as the example $1, -1, 1, -1, 1, -1, \ldots$ shows. However, we have the following property.

**Theorem 8 (monotonic sequence):**
If $(a_k)$ is strictly monotonic and bound, $(a_k)$ converges.

Comments:
(1) Again, monotonicity is defined in complete analogy to the monotonicity of functions. It can be tested via the first derivative after a transfer to functions or by evaluating the sign of $a_{k+1} - a_k$.
(2) If the sequence is strictly increasing, it has to be bound to above, if it is strictly decreasing, it has to be bound to below for the conclusion to be valid.

Example: Show that $a_k = \sqrt{k+1} - \sqrt{k}$ is strictly decreasing and bound to below. Does $\lim\limits_{k\to\infty} a_k$ exist?

The function $f(x) = \sqrt{x+1} - \sqrt{x}$ is strictly decreasing because its derivative is negative:

$$f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0 \quad \text{if } x > 0.$$

Therefore, $a_k = f(k)$ is strictly decreasing. Furthermore, $a_k > 0$ and the sequence is bound to below (with lower bound 0). Theorem 9 ensures that $\lim\limits_{k\to\infty} a_k = a$ exists and that $a \geq 0$. Actually, we can show that $a = 0$ using Theorem 1:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) = \lim_{x\to\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0.$$

In this section, we have identified five strategies for checking for convergence of a sequence: (a) we use the Definition 1 (often a tedious process); (b) we investigate using the limit of the associated function (we need all tools from limits of functions); (c) we check for boundedness and find the accumulation point(s) of any subsequence (if appropriate); (d) check whether the sequence is a sum/product/fraction of known convergent sequences or whether the sandwich teorem is applicable (you need to know other convergent sequences); (e) check for boundedness and monotonicity.

## 11.2 Numerical series: definition and convergence

We have actually used series already, however, in this section we will study them in their own right.

In Chapter 8, we considered Taylor's Theorem. There, we approximated a function (with appropriate properties) by a polynomial of degree $n$ and a remainder term. However, we could also try to let $n \to \infty$ and therefore disregard the remainder term. In this way, we create a *power series*, for example for $\sin(x)$ with development around $x_0 = 0$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} \cdots .$$

But we want to focus on numerical series, and we set, for example, $x = 1$:

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!} \cdots .$$

Let us know define *partial sums* $S_N$ simply as sums of all terms up to the $N$-th one. In our example:

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 - \frac{1}{3!} \approx 0.833 \\
S_3 &= 1 - \frac{1}{3!} + \frac{x^5}{5!} \approx 0.841667 \\
S_4 &= 1 - \frac{1}{3!} + \frac{x^5}{5!} - \frac{1}{7!} \approx 0.841468 \\
S_5 &= 1 - \frac{1}{3!} + \frac{x^5}{5!} - \frac{1}{7!} + \frac{1}{9!} \approx 0.8414709846 \\
S_N &= \sum_{k=1}^{N} \frac{(-1)^k}{(2k+1)!}
\end{aligned}$$

Comparing the values of $S_1$ to $S_5$ to the value of $\sin(1) \approx 0.8414709848$ (precise to 10 decimal places), we see that the agreement improves the more terms the partial sum contains, and that $S_5$ agrees to $\sin(1)$ to 9 d.p. This suggests that the partial sums converge to $\sin(1)$ and, indeed,

$$\sin 1 = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{(-1)^k}{(2k+1)!}.$$

Actually, this is not surprising since the series was constructed from a Taylor polynomial and of course we expected that the right-hand side should converge to the finite value corresponding to $\sin(1)$. In this way, we interpret the sum of an infinite series as the limit of the sequence of partial sums.

**Definition 3 (convergent infinite series):**
An infinite series $\sum_{k=1}^{\infty} a_k$ converges to the sum $S$ if the sequence of its partial sums $S_N = \sum_{k=1}^{N} a_k$ converges to $S$:

$$\lim_{N\to\infty} S_N = S,$$

and we write $S = \sum_{k=1}^{\infty} a_k$.

Comments:
(1) A sum may start from any finite index $m$: $S = \sum_{k=m}^{\infty} a_k$.
(2) If the limit exists, we say that the series converges. If the limit is $\pm\infty$, the series diverges to $\pm\infty$.
(3) This definition enables us to use the methods of sequences to study the convergence of series.

Example: Determine $S = \sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$.

First, we express $\dfrac{1}{n(n+1)}$ using the partial fractions decomposition:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Now, we write down the first partial sums, to obtain the general term of the sequence of partial sums:

$$\begin{aligned} S_1 &= \frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2} \\ S_2 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{1}{1} - \frac{1}{3} \\ S_3 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{1} - \frac{1}{4}. \end{aligned}$$

We observe that in the inner terms cancel each other (series with this property are called *telescoping* series) and we are left with

$$S_n = 1 - \frac{1}{n+1}.$$

At this point, we are able to perform the limit of the partial sums as

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1,$$

which is the result of the infinite series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$.

## 11.3 Arithmetic and geometric sequences and series

Before studying convergence tests of general series, we consider specific sequences and series, where the consecutive terms have a particular regularity. These sequences and series can be found in many applications and should be studied in their own right.

**Definition 4 (arithmetic sequence/series):**
An arithmetic sequence (or progression) is formed by a sequence of numbers where the *difference* between consecutive terms is constant. The sum of the terms of an arithmetic sequence is an arithmetic series.

For example, the sequences $\{2, 5, 8, 11, 24\}$ or $\{2, 0, -2, -4, -6, \cdots\}$ represent arithmetic sequences, a finite one in the former case, and an infinite one in the latter case. In the general case, the sequence element $a_k$ is given by

$$a_k = s + k\, d, \quad k = 0, 1, \ldots,$$

where $s$ stands for the starting value and $d$ for the distance between consecutive elements. The value of the finite series with $N$ terms of the sequence is then

$$S_N = \sum_{k=0}^{N-1} (s + k\, d).$$

Since the sum starts with $k = 0$, to have $N$ terms in total, we terminate the sum at element $N - 1$. Of course, alternative formulations of sequence and series are possible.

To obtain a simple expression for $S_N$, we write the series twice, one in normal form, one in reverse order of terms, and sum each term with the corresponding one above it:

$$\begin{aligned}
S_N &= s + (s + d) + (s + 2d) + \cdots + [s + (N-1)d] \\
S_N &= [s + (N-1)d] + [s + (N-2)d] + [s + (N-3)d] + \cdots + s \\
2S_N &= [2s + (N-1)d] + [2s + (N-1)d] + [2s + (N-1)d] + \cdots + [2s + (N-1)d],
\end{aligned}$$

with $N$ terms on the right-hand side. We solve for $S_N$:

$$S_N = \frac{N}{2}[2s + (N-1)d] = \frac{N}{2}[s + s + (N-1)d] = \frac{N}{2}(\text{first term+last term}).$$

In particular, for $s = 1$ and $d = 1$,

$$S_N = 1 + 2 + \cdots + N = \sum_{k=0}^{N-1} (1 + k) = \sum_{k=1}^{N} k = \frac{N}{2}(N + 1).$$

Infinite arithmetic sequences and series diverge.

**Definition 5 (geometric sequence/series):**
A geometric sequence (or progression) is formed by a sequence of numbers where the ratio between consecutive terms is constant. The sum of the terms of a geometric sequence is a geometric series.

For example, the sequences $\{2, 4, 8, 16, 32\}$ or $\{2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, \cdots\}$ represent geometric sequences, a finite one in the former case, and an infinite one in the latter case. In the general case, the sequence element $a_k$ is given by

$$a_k = s\,r^k, \quad k = 0, 1, \ldots,$$

where $s$ stands for the non-zero starting value and $r$ for the common ratio between consecutive elements. The value of the finite series with $N$ terms of the sequence is then

$$S_N = \sum_{k=0}^{N-1} (s\,r^k).$$

To obtain a simple expression for $S_N$, we write the series twice, one in normal form, one multiplied by $r$, and then subtract both expressions:

$$\begin{aligned} S_N &= s + sr + sr^2 + sr^3 + \cdots + sr^{N-1} \\ rS_N &= sr + sr^2 + sr^3 + sr^4 + \cdots + sr^N \\ S_N - rS_N &= s - sr^N = s(1 - r^N). \end{aligned}$$

For $r \neq 1$, we now simply solve for $S_N$:

$$S_N = \sum_{k=0}^{N-1} (s\,r^k) = \frac{s(1 - r^N)}{1 - r}.$$

Let us now investigate an infinite geometric series (for $r \neq 1$):

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{s(1 - r^N)}{1 - r} = \lim_{N\to\infty} \left( \frac{s}{1 - r} - \frac{sr^N}{1 - r} \right) = \frac{s}{1 - r} - \lim_{N\to\infty} \frac{sr^N}{1 - r}$$

if the latter limits exist. We know that $\lim\limits_{N\to\infty} r^N = 0$ if $|r| < 1$ while for $|r| > 1$ the limit diverges. Therefore, the infinite geometric series converges to

$$\lim_{N\to\infty} S_N = \frac{s}{1 - r} \qquad \text{for } |r| < 1.$$

## 11.4 Numerical series: convergence criteria for positive series

**Theorem 9 (necessary criterion for convergence):**
If $\sum_{k=1}^{\infty} a_k$ converges, $\lim\limits_{k\to\infty} a_k = 0$.

Comments:
(1) However, the converse is not true, $\lim\limits_{k\to\infty} a_k = 0$ is not sufficient to ensure convergence. For a specific kind of series with additional hypotheses (see Theorem 13), $\lim\limits_{k\to\infty} a_k = 0$ implies convergence.
(2) The logical negation of this theorem can be used as a quick test for non-convergence: If $\lim\limits_{k\to\infty} a_k \neq 0$, $\sum_{k=1}^{\infty} a_k$ does not converge.

**Theorem 10 (comparison test):**
(i) If $\sum_{k=1}^{\infty} a_k$ converges (with $a_k \geq 0$ for all $k$), then the series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$ and $b_k \leq a_k$ for all $k$ converges.
(ii) If $\sum_{k=1}^{\infty} a_k$ diverges (with $a_k \geq 0$ for all $k$), then the series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$ and $b_k \geq a_k$ for all $k$ diverges.

Example: Investigate the convergence of $\sum_{k=0}^{\infty} \dfrac{1}{k!}$ (factorial series).

We write down the partial sum of the factorial series, $A_N$, and also the partial sum of a suitable geometric series, $C_N = 1 + \sum_{k=0}^{N-1}(s\, r^k)$ with $s = 1$ and $r = 1/2$:

$$\begin{aligned} A_N &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{N!}, \\ C_N &= 1 + \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N-1}}. \end{aligned}$$

Note that there are $N+1$ terms in both partial sums and that both series are composed of positive terms. The 1 is added to the usual geometric series to make a term-wise comparison: each term of $A_N$ is equal or smaller than the corresponding term of $C_N$. We obtain the sum of the series $C_N$ as:

$$C_N = 1 + \left( \frac{1\left(1 - \dfrac{1}{2^N}\right)}{1 - \dfrac{1}{2}} \right) = 1 + 2\left(1 - \frac{1}{2^N}\right) = 3 - \frac{1}{2^{N-1}}.$$

We therefore conclude that $\lim\limits_{N\to\infty} C_N = 3$, it converges. Therefore, the limit of the factorial series must converge to a limit smaller than 3 (recall that the series is entirely composed of positive terms).
Example: Investigate the convergence of $\sum_{k=1}^{\infty} \dfrac{1}{k}$ (harmonic series).

We again try to establish a term-wise comparison of partial sums, however, now with a divergent series. For this, we write the terms with brackets to indicate how they will be compared ($A$ stands for the harmonic series, $C$ for a divergent one):

$$\begin{aligned} A &= 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+(\frac{1}{9}+\cdots+\frac{1}{16})+\cdots\ , \\ C &= 1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})+(\frac{1}{16}+\cdots+\frac{1}{16})+\cdots\ . \end{aligned}$$

In this case, each term of $A$ is larger or equal to the corresponding term of $C$. However, we see that $C = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ and $C$ is clearly divergent. This also shows us that for convergence it is not sufficient that the limit of the $k$-th term tends to zero as $k \to \infty$.

**Theorem 11 (d'Alembert's test or quotient test):**
Suppose an infinite series $\sum_{k=1}^{\infty} a_k$ (with $a_k \geq 0$ for all $k$) and $\lim\limits_{k\to\infty} \dfrac{a_{k+1}}{a_k} = R$. Then, the series is convergent if $R < 1$, divergent if $R > 1$ and the criterion does not decide if $R = 1$.

Comment:
Since a series with non-positive terms can be translated into a series with non-negative terms through a multiplication with $-1$, the above-mentioned theorems hold actually for all series with terms with non-changing sign.

Example: Use d'Alembert's test to determine whether $\sum_{k=0}^{\infty} \dfrac{2^k}{(k+1)^2}$ is convergent.

We confirm that all terms $a_k = \dfrac{2^k}{(k+1)^2}$ are positive and fulfil the hypothesis of d'Alembert's test. We calculate

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2^{k+1}}{(k+2)^2} \cdot \frac{(k+1)^2}{2^k} = 2 \cdot \lim_{k\to\infty} \left(\frac{k+1}{k+2}\right)^2 = 2.$$

Since $2 > 1$, the infinite series diverges.

## 11.5 Numerical series: convergence criteria for general series

**Definition 6 (absolut convergence):**
An infinite series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

Comments:
(1) An absolutely convergent series is also convergent in the usual sense.
(2) While checking for absolute convergence, the terms of the series are non-negative and the tests mentioned above (comparison test, d'Alembert) can be used.
(3) There are convergent series that are not absolutely convergent. These are *conditionally convergent* series. Examples can be found among *alternating* series.

**Definition 7 (alternating series):**
Series of the kind $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ or $\sum_{k=1}^{\infty}(-1)^k a_k$ with $a_k \geq 0$ for all $k$ are called alternating series.

Comment: Since $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = -\sum_{k=1}^{\infty}(-1)^k a_k$, these two forms are actually equivalent (compare with the comment made after Theorem 11).

**Theorem 12 (Leibniz test):**
If $(a_k)$ is monotonically decreasing with $\lim\limits_{k\to\infty} a_k = 0$, then, the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

Comment: We have just provided an additional hypothesis under which $\lim\limits_{k\to\infty} a_k = 0$ becomes part of a criterion for convergence. As Theorem 9 stated, $\lim\limits_{k\to\infty} a_k = 0$ alone is not sufficient.

Example: Determine whether the series $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}$ converges.

We have seen that harmonic series $\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges. However, the sequence $\dfrac{1}{k}$ is monotonically decreasing with $\lim\limits_{k\to\infty} \dfrac{1}{k} = 0$ and hence the alternating series $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}$ converges conditionally.

Example: Determine whether the series $\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \cdots$ converges.

The signs are alternating and the terms tend to zero. However, the terms are not monotonically decreasing ($\dfrac{1}{\sqrt{2}-1} > \dfrac{1}{\sqrt{2}+1}$ but $\dfrac{1}{\sqrt{2}+1} < \dfrac{1}{\sqrt{3}-1}$) and the criterion cannot be applied. Actually, the series is divergent: The partial sum $S_{2k} = \dfrac{2}{1} + \dfrac{2}{2} + \dfrac{2}{3} + \cdots + \dfrac{2}{k-1}$ which is twice the sum of the divergent harmonic series.