Fundamental Concepts of Statistics

Exercise session 4

1. Let (X, Y) have the joint density

$$f(x,y) = \frac{6}{7}(x+y)^2, \quad \text{if } 0 < x, y < 1.$$

a) Find $P(X > Y)$ and $P(X < 1/2)$

b) Find the marginal densities of X and Y

c) Find the two conditional densities

2. Let

$$f(x,y) = c(x^2 - y^2)e^{-x}, \; x > 0, \; -x < y < x.$$

a) Find c.

b) Find the marginal densities.

c) Find the conditional densities.

3. Suppose that two components have independent exponentially distributed lifetimes T_1 and T_2 with parameters α and β respectively. Find

$$P(T_1 > T_2) \text{ and } P(T_1 > 2T_2)$$

4. Let X and Y be jointly continuous random variables. Develop an expression for the joint density of $X+Y$ and $X-Y$. Use the density transformation theorem for this.

5. Find the joint density of $X+Y$ and X/Y where X and Y are independent exponential random variables with parameter λ. Show that $X+Y$ and X/Y are independent.

7. Suppose that $X_1, \ldots, X_{20}$ are independent random variables with density

$$f(x) = 2x, \quad 0 < x < 1.$$

Let $S = X_1 + \ldots + X_{20}$. Use the central limit theorem to approximate $P(S \leq 10)$.

8. Suppose that a measurement has mean μ and $\sigma^2 = 25$. Let $\bar{X}$ be the average of n such independent measurements. How large should n be so that $P(|\bar{X} - \mu| < 1) = 0.95$?

9. Assume that a company ships packages that are variable in weight, with an average weight of 15 pound and a standard deviation of 10 pound. Assuming that the packages come from a large number of different customers so that it is reasonable to model their weights as independent random variables, find the probability that 100 packages will have a total weight exceeding 17 000 pound.

10. How can one approximate $\int_0^1 \cos(2\pi x)dx$ using n simulated uniform (0,1) i.i.d. random variables $U_1, \ldots, U_n$. What is the expected value and variance of this approximation denoted say by $T(U_1, \ldots, U_n)$.

11. Consider the maximum $U_{(n)}$ of n simulated uniform (0,1) i.i.d. random variables $U_1, \ldots, U_n$.
a) Show that $n(1 - U_{(n)})$ converges in distribution to a standard exponential distribution with distribution function $F(y) = 1 - e^{-y}$ $(y > 0)$ as $n \to \infty$. [To this end compute $P(n(1 - U_{(n)}) > y)$ and take the limit for $n \to \infty$.]
b) How to adapt this result if the random variables are uniformly distributed on $(0, a)$ for some $a > 0$. [Hint: how to transform the given case to the uniform (0,1) case.]