General lift Distribution

Consider the transformation

$$y = -\frac{b}{2} \cdot \cos\theta \qquad 0 \leq \theta \leq \pi$$

In terms of θ, the elliptic lift distribution is written as

$$\Gamma(\theta) = \Gamma_0 \cdot \sin\theta .$$

This equation can be approximated by a Fourier sine series. for the general circulation distribution along an arbitrary finite wing. Assume for general case.

$$\Gamma(\theta) = 2 b V_\infty \cdot \sum_{1}^{N} A_n \cdot \sin(n\theta)$$

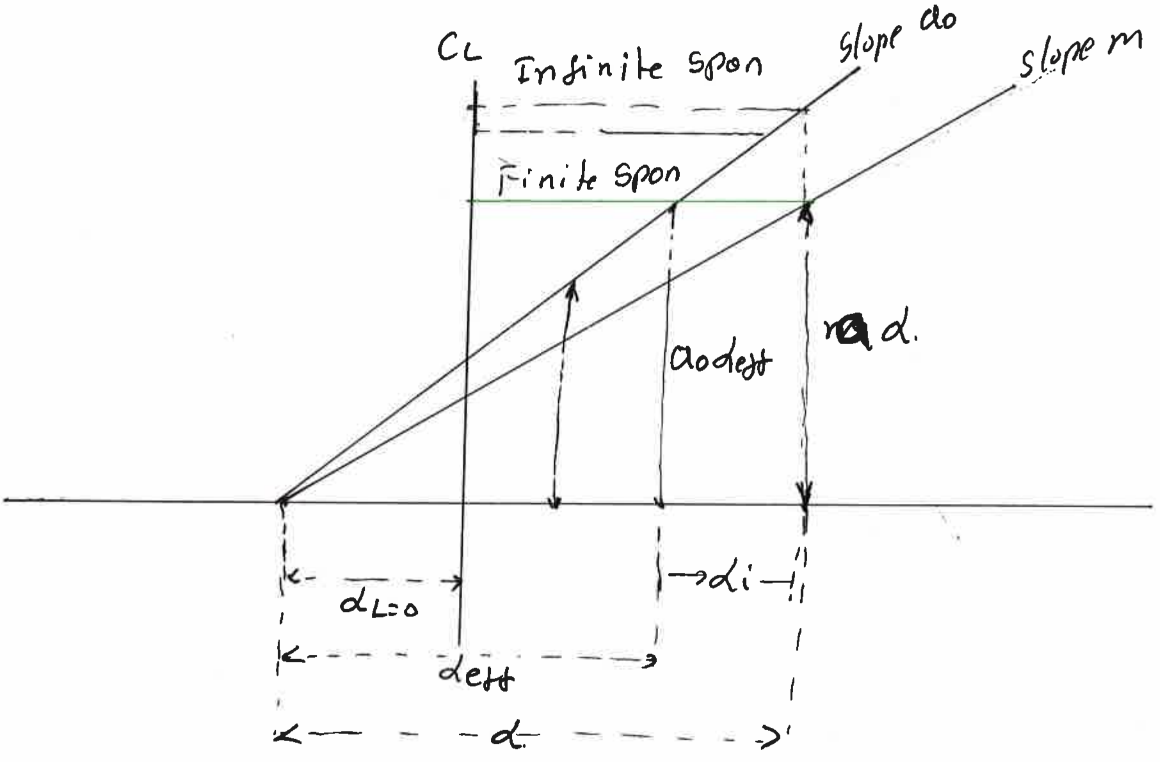
The coefficients must satisfy the fundamental equation Prandtl's Lifting-Line theory.

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy} = 2 b V_\infty \sum_{1}^{N} A_n \, n . \cos(n\theta) . \frac{d\theta}{dy}$$

substituting this equation into the equation to obtain α.

$$\alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \cdot \sum_{1}^{N} A_n \sin(n\theta) + \alpha_{L=0}(\theta_0) + \frac{1}{\pi} \int_0^{\pi} \frac{\sum_{1}^{N} A_n . n \cos(n\theta)}{\cos\theta - \cos\theta_0} \, d\theta$$

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty . c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy) . dy}{y_0 - y}$$



$$\alpha = \alpha_{eff} + \alpha_i$$

$$\frac{C_L'}{a} = \frac{C_L'}{a_0} + \alpha_i$$

$$|a| = \frac{a_0}{1 + \frac{\alpha_i \cdot a_0}{C_L}} = \frac{a_0}{1 + \frac{\alpha_i}{\alpha_{eff}}}$$

$$\left[a = \frac{a_0}{1 + \frac{0.045}{0.135}} = 0.75 a_0\right]$$

This shows that the finite wing, in this example, generates only 75% of the Lift that would be generated by the same wing if the effect of induced downwash were ignored

This integral
$$\frac{1}{\pi}\int_0^{\pi}\frac{A_n . n . \cos(n\theta)}{\cos(\theta)-\cos\theta_0}.d\theta$$

$$\frac{1}{\pi} A_n . n \int_0^{\pi}\frac{\cos(n\theta)}{\cos\theta - \cos\theta_0}\,d\theta = \pi . \frac{\sin n\theta_0}{\sin\theta_0} . \frac{1}{\pi} A_n n$$

Then we obtain that

$$\left[\alpha(\theta_0) = \frac{2b}{\pi C(\theta_0)} \sum_1^N A_n . \sin(n\theta_0) + \alpha_{L=0}(\theta_0) + \sum_1^N n . A_n . \frac{\sin(n\theta_0)}{\sin(\theta_0)}\right]$$

It is, closely, evaluated at a given spanwise location; hence, θ_0 is specified. In turn b, $C(\theta_0)$, and $\alpha_{L=0}(\theta_0)$ are known quantities from the geometry and airfoil section of the finite wing. Only unknown are the coefficients A_n. Hence, written at a given spanwise location (θ_0). is one algebraic equation with N unknowns $A_1, A_2 \dots A_N$, However, let us choose N different spanwise stations. and let's evaluate the equation. at each of these N. stations. We obtain N independent algebraic equation with N unknowns

Now that $\Gamma(\theta)$ is known via the $\Gamma(\theta) = 2bV_\infty \sum_1^N A_n \sin(n\theta)$

The lift coefficient for the finite wing follows

$$C_L = \frac{2}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)dy = \frac{2b^2}{S}\sum_1^N A_n \int_0^{\pi} \sin(n\theta) . \sin\theta\, d\theta$$

The integral is $\displaystyle\int_0^{\pi} \sin(n\theta).\sin\theta\, d\theta = \begin{cases} \pi/2 & \text{for } n=1 \\ 0 & \text{for } n\neq 1 \end{cases}$

$$C_L = A_1.\pi.\frac{b^2}{S} = A_1.\pi.AR$$

(only depends on the leading coefficient of the Fourier series expansion)

Then we must solve for all the A_n's simultaneously in order to obtain A_1.

The induced drag coefficient is obtained from

$$C_{D_i} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y).\alpha_i(y)\,dy$$

$$= \frac{2b^2}{S}\int_0^{\pi} \left(\sum_1^N A_n.\sin(n\theta)\right).\alpha_i(\theta).\sin\theta\, d\theta$$

where $\alpha_i(\theta)$ is obtained by

$$\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy).\,dy}{y_0 - y}$$

$$\alpha_i(\theta_0) = \frac{1}{\pi}.\sum_1^N n.A_n.\int_0^{\pi} \frac{\cos(n\theta)}{\cos\theta - \cos\theta_0}\, d\theta$$

$$\alpha_i(\theta_0) = \sum_1^N n.A_n.\frac{\sin(n\theta_0)}{\sin\theta_0}$$

can be written as

$$d_i(\theta) = \sum_{1}^{N} n.A_n.\frac{\sin(n\theta)}{\sin\theta}$$

We obtain C_{Di} as

$$C_{Di} = \frac{2b^2}{S}\int_0^{\pi}\sum_{1}^{N} A_n.\sin(n\theta).\sum_{1}^{N} A_n.n.\sin(n\theta)\,d\theta$$

From the standard integral.

$$\int_0^{\pi}\sin(n\theta).\sin(n\theta) = \int_0^{\pi}\sin(m\theta).\sin(k\theta)\,d\theta \begin{cases} 0 & m\neq k \\ \frac{\pi}{2} & m=k \end{cases}$$

$$C_{Di} = \frac{2b^2}{S}\sum_{1}^{N}(n A_n^2).\frac{\pi}{2} = 2.AR.\sum_{1}^{N}(n A_n^2)$$

$$C_{Di} = \pi AR\left(A_1^2 + \sum_{2}^{N} n A_n^2\right)$$

$$= \pi AR.A_1^2.\left[1 + \sum_{2}^{N} n\left(\frac{A_n}{A_1}\right)^2\right]$$

$$C_{Di} = \frac{C_L^2}{\pi AR}(1+\delta) \qquad \text{where } \delta = \sum_{2}^{N} n\left(\frac{A_n}{A_1}\right)^2 \geq 0$$

Let us define a span efficiency factor, e, as $e = (1+\delta)^{-1}$

$$\boxed{C_{Di} = \frac{C_L^2}{\pi.e AR}}$$

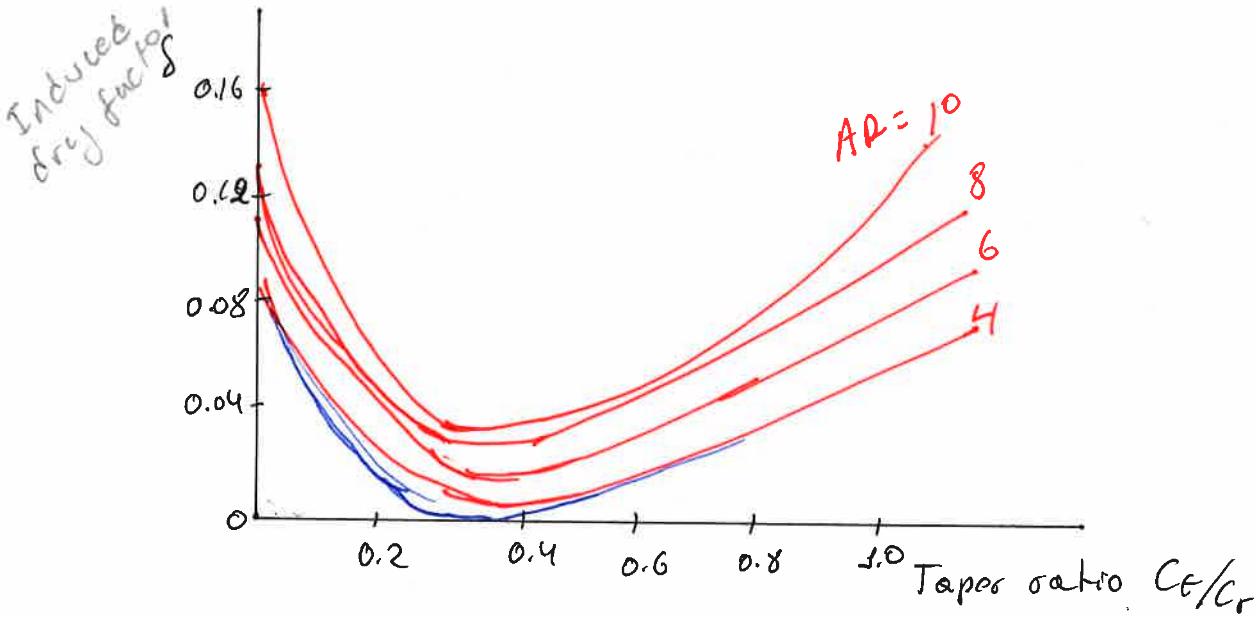
$e \leq 1$ for general lift distribution,
For elliptical lift distribution
$\delta = 0$ and $e = 1$

Hence the lift distribution which yields minimum induced drag is the elliptical lift distribution. this is why we have a practical interest in the elliptical lift distribution.

<u>Elliptic wing</u> : Elliptical planform, are more expensive to manufacture.

<u>Rectangular wing</u>: Generates a lift distribution far from optimum.

<u>Tapered wing</u>: Can be designed with taper ratio, that is Tip chord/root chord $\equiv C_t/C_r$, such that the lift distribution closely approximates the elliptic case.



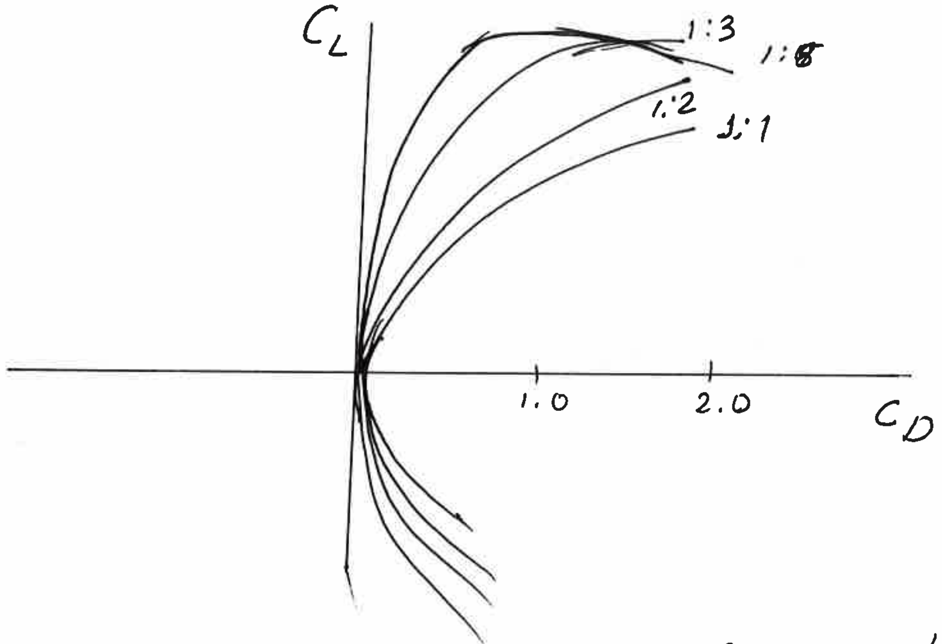
The variation of δ as function of taper ratio for wings of different A.R. with this graph a tapered wing can be designed with an induced drag coefficient reasonably close to the minimum value. The manufacture is considerable easier to manufacture than elliptic plantform.

Effect of Aspect Ration

The AR. varies from 6 to 22 for standard subsonic airplanes and sailplanes. has much. stronger effect on C_{Di} than the value of δ. which from the above graph varies only by about 10 percent over the practical range of taper ratio. So that the primary design factor for minimizing induced drag is not the closeness to an elliptical lift distribution, but rather, the ability to make the aspect ratio as larger as possible.

Recall that. the total drag of finite wing is giving by

$$C_D = C_d + \frac{C_L^2}{\pi e AR}$$



Rectangular wing. (Drag polar)

Consider two wings with different aspect ratio AR_1 and AR_2. The Drag Coefficients C_{D1} and C_{D2} for the two wings as

$$C_{D_1} = C_d + \frac{C_L^2}{\pi e AR_1}$$

$$C_{D2} = C_d + \frac{C_L^2}{\pi e . AR_2}$$

Assume that the wings are at the same C_L. Also since the airfoil section is the same for both wings. C_d. is essencially the Same.

Also the varriation of e between the wing is only a few percent and can be ignored.

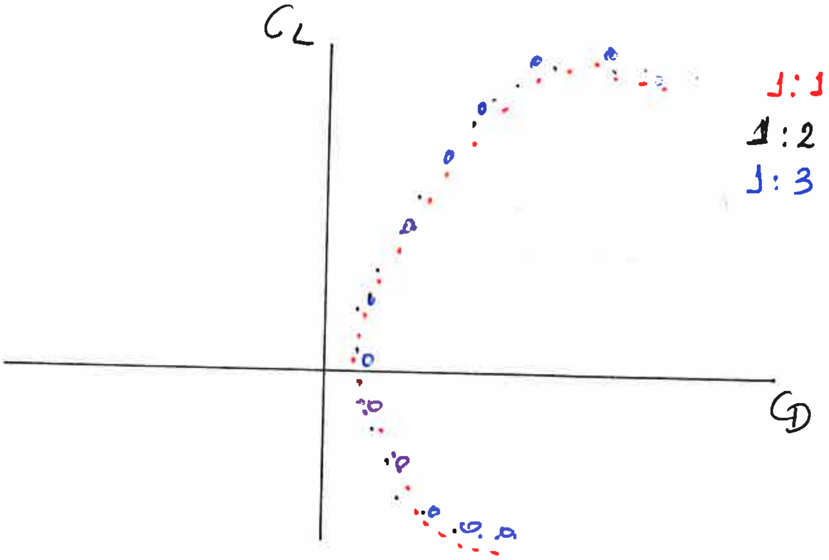
$$C_{D_1} = C_{D_2} + \frac{C_L^2}{\pi e}\left(\frac{1}{AR_1} - \frac{1}{AR_2}\right)$$

This equation can be used to scale the data of a wing with aspect ratio AR_2 to correspond to the case of another aspect ratio AR_1.

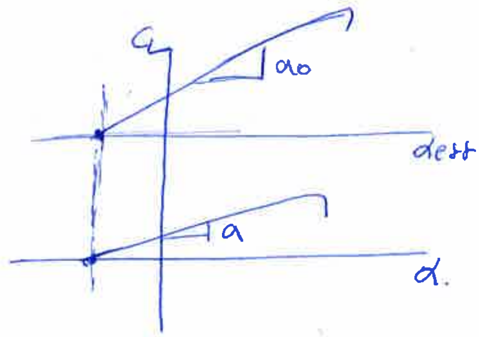
Ex.

$$C_{D1} = C_{D2} + \frac{C_L^2}{\pi e}\left(\frac{1}{5} - \frac{1}{AR_2}\right)$$

Inserting the value of C_{D2} and AR_2 from the above graph. The resulting data for C_{D1} vs. C_L collopsed to essentially the same curve.



Airfoil and finite wing have some differences. Finite wing produces a induced drag. Second major difference appears in the lift slope.



The slope of airfoil $a_0 = \frac{dC_L}{d\alpha}$

The slope of wing $a = \frac{dC_L}{d\alpha}$

where $a < a_0$. the values of a_0 and a are related as follows.

$$\frac{dC_L}{d(\alpha-\alpha_i)} = a_0 \quad \text{Integrating}$$

$$C_L = a_0(\alpha-\alpha_i) + \text{Constant}$$

$\alpha_i = \frac{C_L}{\pi AR}$ into

$$C_L = a_0(\alpha-\alpha_i) + \text{Constant}$$

$$C_L = a_0\left(\alpha - \frac{C_L}{\pi AR}\right) + \text{Constant}$$

Differentiating.

$$\frac{dC_L}{d\alpha} = a = \frac{a_0}{1 + a_0/\pi AR}$$

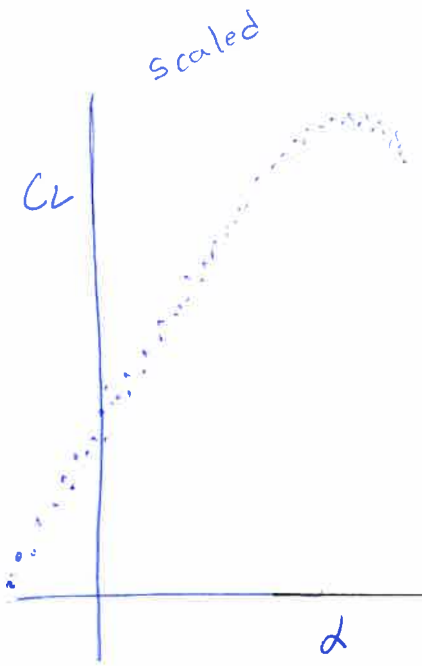
This equation gives the desired relation between a_0 and a for an elliptic finite wing.

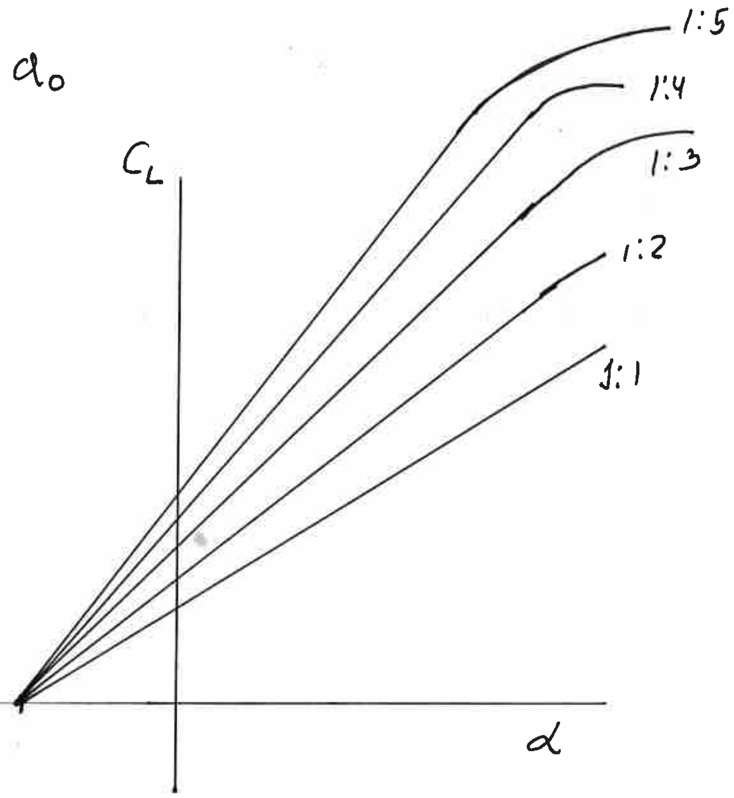
$$a = \frac{a_0}{1 + (a_0/\pi AR)(1+\tau)}$$

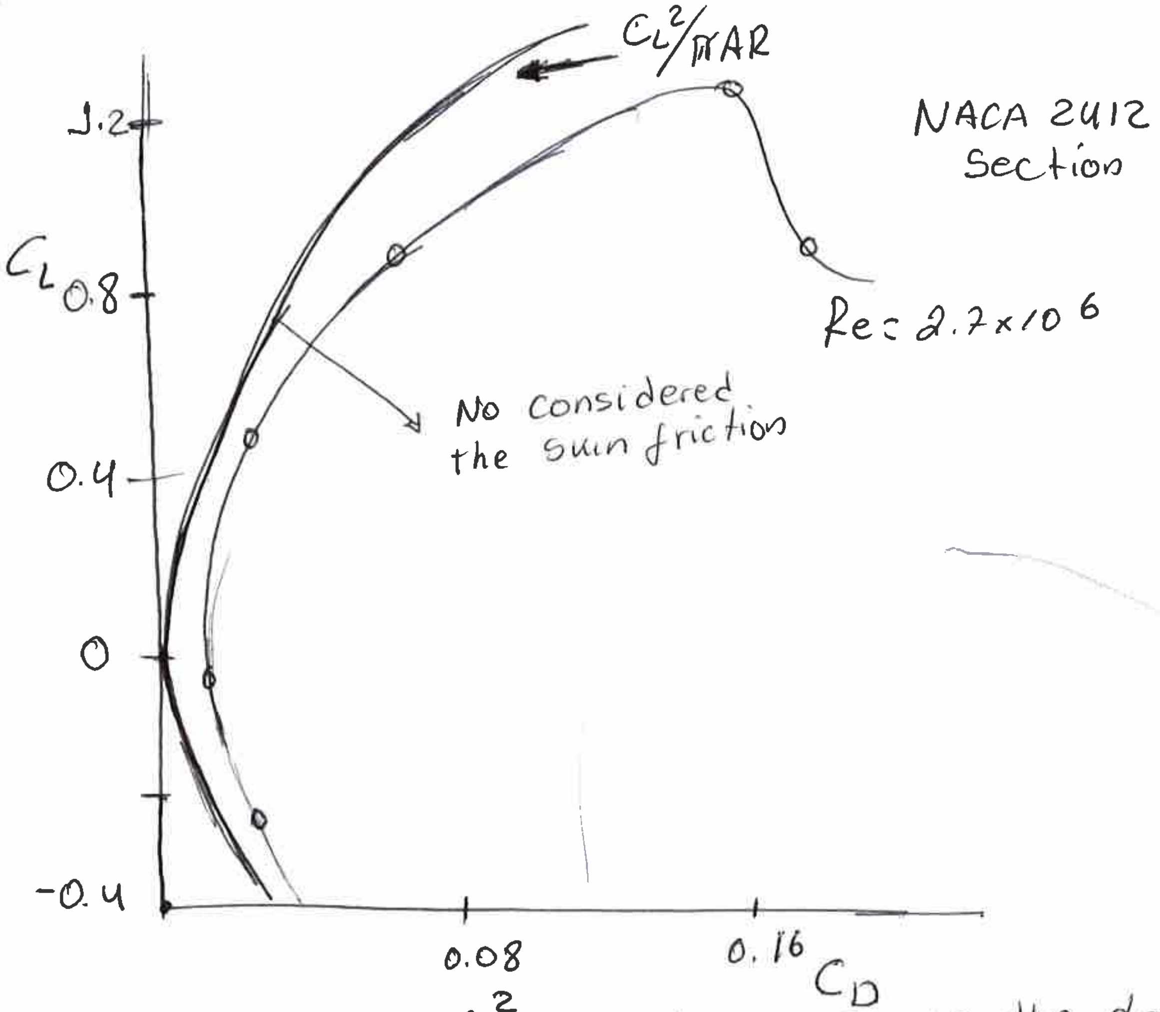
This equation is for general planform. where τ is a function of Fourier Coefficients A_n. The values of τ range between 0.05 and 0.25.

$AR \rightarrow 0$, a substantial difference can exit between a_0 and a.

$AR \rightarrow \infty \quad a \rightarrow a_0$







$C_D = C_{D0} + KC_L^2$ where C_{D0} is the drag coefficient at zero lift and KC_L^2 is lift-dependent drag coefficient, which ~~depends~~ includes tha part of viscous drag and of the form drag.

These relation describing the influence of the AR have been verified experimentally by Prandtl and Betz. If we compare the drag polars for two wings which have aspect ratios of AR_1 and AR_2 for a given value of lift coefficient.

$$C_{D2} = C_{D1} + \frac{C_L^2}{\pi}\left(\frac{1}{AR_2} - \frac{1}{AR_1}\right)$$

where $C_{D0_1} = C_{D0_2}$

Universidad Europea de Madrid

Aerodynamics and aeroelasticity

Homework 1 (Exam)

Wednesday, April 22nd 2015

Problem 1

Consider two different flows over geometrically similar airfoil shapes, one airfoil being twice the size of the other. The flow over the smaller airfoil has a freestream properties given by $T_\infty = 200K$, $\rho_\infty = 1{,}23\frac{Kg}{m^3}$, $v_\infty = 100\frac{m}{s}$. The flow over the larger airfoil is described by $T_\infty = 800K$, $\rho_\infty = 1{,}739\frac{Kg}{m^3}$, $v_\infty = 200\frac{m}{s}$. Assume that both μ and c (speed of sound) are proportional to $T^{1/2}$ Are the two flows dynamically similar?

Example

The analysis of this section is now applied to compute the characteristics of an untwisted rectangular wing $AR = 6$ flying with α. Assume that the airfoil is uncambered so that the absolute of attack α_a is equal to α everywhere along the span. For $\alpha = 0°, 5°, 10°, 15°$

$C = C_s$ and $a = a_0 = 2\pi$, the equation of α is reduced to

$$\alpha = \sum_{n=1}^{\infty} A_n \sin(n\theta) \left(1 + \frac{n\pi}{2AR\sin\theta}\right)$$

For symmetrically loaded wing, the coefficients A_n vanish for even values of n.

$$\alpha = A_1 \sin\theta \left(1 + \frac{\pi}{6\times 2 . \sin\theta}\right) + A_3 . \sin 3\theta \left(1 + \frac{\pi}{4\sin\theta}\right)$$

$$+ A_5 \sin(5\theta) . \left(1 + \frac{5\pi}{12\sin\theta}\right) + A_7 \sin 7\theta \left(1 + \frac{7\pi}{12 . \sin\theta}\right)$$

We take only one-half of the span because of the symmetry of the rectangular wing. So we take only the four stations of the wing

$\theta = \frac{\pi}{8}$, $\theta = \frac{\pi}{4}$; $\theta = \frac{3\pi}{8}$ and $\theta = \frac{\pi}{2}$

We obtain a set of equations.

$$0.6444 A_1 + 2.8200 A_3 + 4.0841 A_5 + 2.2153 A_7 = \alpha$$

$$0.9689 A_1 + 1.4925 A_3 + 2.0161 A_5 + 2.5397 A_7 = \alpha$$

$$1.1857 A_1 + 0.7080 A_3 - 0.9249 A_5 + 2.7565 A_7 = \alpha$$

$$1.261 A_1 - 1.7854 A_3 + 2.3090 A_5 - 2.8326 A_7 = \alpha$$

$A_1 = 0.9174\alpha$ $\quad A_3 = 0.1104\alpha$ $\quad A_5 = 0.0218\alpha$ $\quad A_7 = 0.0038\alpha$

The wing-lift coefficient is

$$C_L = \pi^2 \frac{A_1}{2} = 4.5273\alpha.$$

Based on the values.

$$(C_{Di})_{el} = \frac{C_L^2}{\pi AR} = 1.0874\alpha^2$$

and $$\delta = \frac{3 A_3^2 + 5 A_5^2 + 7 A_7^2}{A_1^2} = 0.0464$$

Induced Drag for the wing.

$$C_{Di} = C_{Di\,el}(1+\delta) = 1.1378\alpha^2$$

this higher about 5% of $(C_{Di})_{el}$

Calculating the lift coefficient

$$(C_L)_{el} = \frac{2\pi \cdot \alpha}{1 + \frac{2\pi}{\pi AR}} = 1.5\pi\alpha = 4.7124\alpha.$$

So the slope of the lift coefficient curves, $dC_L/d\alpha$ of rectangular wing is approximately 4% lower.

The result verifies the properties of arbitrary planforms are close to those of elliptical wing.

Ex. 2

Consider a finite wing with an aspect ratio of 8 and taper ratio of 0.8. The airfoil section is thin and symmetric. Calculate the lift and induced drag coefficients for the wing when it is an angle of attack of 5°. Assume that $\delta = \tau$

<u>Solution</u>

from graph $\delta = 0.055$ and $\tau = \delta = 0.055$ and $a_0 = 2\pi$.

$$C_L = \alpha a \Rightarrow a = \frac{a_0}{1 + (a_0/\pi AR)(1+\tau)}$$

$a = 4.97\ rad^{-1}$

$a = 0.0867\ Degree^{-1}$

$C_L = 0.4335$

Since airfoil is symmetric $\alpha_L = 0 = 0°$

Then $C_L = a.\alpha = (0.0867 \frac{\ }{\text{Degree}} \times 5 \text{ Degree}) = 0.4335$

$C_{Di} = \frac{C_L^2}{\pi AR e} = 0.00789$ $\quad e = \frac{1}{1+\delta} = 0.947$

Ex. Jet transport patterned after Cesna 560 citation V. shown a drag coefficient of cruise 0.015. An Addition the zero-lift angle of attack is $-2°$, the lift slope of the airfoil section is 0.1 per degree. The lift efficienty factor $\tau = 0.04$ and the wing aspect ratio is 7.96. Calculate the angle of attack of the airplane.

Solution.

The lift. slope of the airfoil is in radian

$a_0 = 0.1$ Per dgree $= 0.1 \times (57.3) = 5.73$ rad.

$$a = \frac{a_0}{1 + (a_0/\pi AR)(1+\tau)}$$

$$a = \frac{5.73}{1 + \left(\frac{5.73}{\pi \cdot 7.96}\right)(1+0.04)} = 4.627 \text{ per rad.}$$

$$a = \frac{4.627}{57.3} = 0.0808 \text{ per degree}$$

$$C_L = a(\alpha - \alpha_{L=0})$$

Cruising Data is : $V = 790$ Km/h $\quad \rho = 0.41$ kg/m³

$W = 68$ KN $\quad$ planform area $= 31.8$ m²

Then $$C_L = \frac{2W}{\rho_\infty V_\infty^2 S} = \frac{2 \cdot 68000}{(0.41)\cdot(219.4)^2\cdot(31.8)} = 0.21$$

$$\alpha = \frac{C_L}{a} + \alpha_{L=0} = \frac{0.21}{0.0808} + (-2) = 0.6^\circ$$

Ex. Consider a rectangular wing on aspect ratio of 6. an induced drag factor $\delta = 0.055$ and $\alpha_{L=0} = -2^\circ$. and angle of attack of 3.4°.
The induced drag coefficient for this wing is 0.01. Calculate the induced drag coefficient for a similar wing. at the same angle of attack but with an AR. 10. Induced factors for drag and lift slope δ and τ are equal to each other AR 10 , $\delta = 0.105$

Solution

$$C_{Di} = \frac{C_L^2}{\pi AR\, e}$$ we need to determine C_L

For AR. 6 the C_L is

$$C_L^2 = \frac{\pi\ AR.\ C_{Di}}{(1+\delta)} = \frac{\pi.\ 6\times(0.01)}{(1+0.055)} = 0.1787$$

$$C_L = 0.423$$

The lift slope of this wing is therefore

$$\frac{dC_L}{d\alpha} = \frac{0.423}{(3.4^\circ-(-2^\circ))} = 0.078/\text{degree} = 4.485/\text{rad}$$

The lift slope for the airfoil (the infinite wing).

$$\frac{dC_L}{d\alpha} = a = \frac{a_0}{1+(a_0/\pi AR)(1+\tau)}$$

$$4.485 = \frac{a_0}{1+\left(\frac{(1.055).\ a_0}{\pi.\ 6}\right)} = \frac{a_0}{1+0.056a_0}$$

$$a_0 = \frac{5.989}{\text{rad}}$$

The second wing has the same airfoil section

Then a_0 is the same.

The lift slope for the second wing.

$$a = \frac{a_0}{1+\left(\frac{(1.105\times a_0)}{\pi.\ 10}\right)} = 4.95/\text{rad}$$

$$a = 0.086/\text{degree}.$$

The lift coefficient for the second wing is therefore

$$C_L = a(\alpha - \alpha_{L=0}) = 0.086 [3.4° - (-2°)$$

$$C_L = 0.464$$

In turn, the induced drag coefficient is

$$C_{Di} = \frac{C_L^2}{\pi AR}(1+\delta) = \frac{0.464(1+0.105)}{\pi \cdot 10} = 0.0076$$

Consider same C_L rather than α. for both wing.

$$C_{Di} = \frac{C_L^2}{\pi AR}(1+\delta) = \frac{(0.423)^2(1+0.105)}{\pi \cdot 10} = 0.0063.$$

One observation about induced Drag D_i, and contrast to the induced drag coefficient C_{Di}, that C_{Di} decreases by increases the AR. The Drag (D_i) is governed by other parameter.

$$D_i = q_\infty S . C_{Di} = q_\infty . S . \frac{C_L^2}{\pi e AR} \quad \text{for steady flight}$$

$$C_L^2 = \left(\frac{L}{q_\infty S}\right)^2 = \left(\frac{W}{q_\infty S}\right)^2$$

$$D_i = q_\infty . S . \frac{W^2}{q_\infty^2 S^2} \cdot \frac{1}{\pi e AR}$$

$$D_i = \frac{1}{\pi e}\frac{1}{q_\infty}\left(\frac{W^2}{b^2}\right)$$ For steady flight induced Drag does not depend on the aspect ratio; but rather other parameter $\left(\frac{W}{b}\right)$ called the span loading: span loading $= \frac{W}{b}$. Drag (induced) can be reduced by increasing b. span.

How much of the total drag of an airplane is induced drag?

The parasite drag for a generic subsonic jet transport is the sum of the drag due to skin friction and pressure drag due to flow separation associated with the complete airplane including wing. At cruise about 25% is induced drag but at take off can be 60% or more of the total drag.