(10)

$$(\partial_t - D\nabla^2)\, g(\bar r, t) = \delta(\bar r)\,\delta(t)$$

$$g(\bar r,t) = \frac{1}{2\pi}\frac{1}{(2\pi)}\int d^2k \int d\omega\, e^{i\bar k \bar r - i\omega t} \qquad g(k,\omega) = \frac{1}{(2\pi)^2}\int d^2k\, e^{i\bar k \bar r}$$

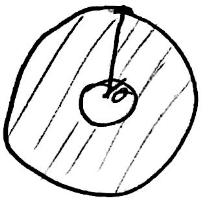
2D

$$= \frac{1}{2\pi}\int d\omega \frac{e^{-i\omega t}}{Dk^2 - i\omega} = \frac{\theta(t)}{2\pi}\int d^2k\, e^{i\bar k \bar r}\, e^{-Dtk^2} =$$

$$= \frac{1}{(4\pi D t)^{1/2}}\, e^{-r^2/4Dt} \qquad (k^2 = k_x^2 + k_y^2) \quad (\bar k \bar r = k_x \cdot x + k_y \cdot y)$$

3D $\quad g(\bar r, t) = \dfrac{\theta(t)}{(4\pi D t)^{3/2}}\, e^{-r^2/4Dt} \qquad \langle r^2\rangle \sim t$

(11)



$$(\partial_t^2 - c^2\nabla^2)\,\psi(\bar r, t) = 0$$

$$\psi(\bar r, t) = \phi(\bar r)\, e^{\pm i\omega t}$$

$$-\nabla^2 \phi_{nm}(\bar r) = k_{nm}^2\, \phi(\bar r) \qquad \phi_{nm}(\bar r) = R_{nm}(r)\, e^{im\varphi}$$

$$R(r) = A_{nm} J_m(k_{nm} r) + B_{nm} N_m(k_{nm} r)$$

$$A_{nm} J_m(k_{nm} r_1) + B_{nm} N_m(k_{nm} r_1) = 0$$

$$A_{nm} J_m(k_{nm} r_2) + B_{nm} N_m(k_{nm} r_2) = 0$$

buscamos $\det = 0$

$$J_m(k_{nm} r_1)\, N_m(k_{nm} r_2) = J_m(k_{nm} r_2)\, N_m(k_{nm} r_1)$$

↑ ordenador

(13)

$$(\partial_t - D\nabla^2)\, g(\bar{r},t) = \delta(\bar{r})\,\delta(t)$$

$$g(\bar{r},t) = \frac{1}{2\pi}\frac{1}{(2\pi)^2}\int d^2k \int d\omega\, e^{i\bar{k}\bar{r} - i\omega t} \qquad g(\bar{k},\omega) = \frac{1}{(2\pi)^2}\int d^2k\, e^{i\bar{k}\bar{r}}$$

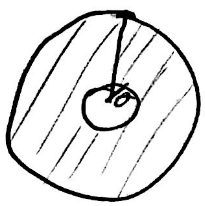
2D

$$\cdot \frac{1}{2\pi}\int d\omega \frac{e^{-i\omega t}}{Dk^2 - i\omega} = \frac{\theta(t)}{2\pi}\int d^2k\, e^{i\bar{k}\bar{r}}\, e^{-Dtk^2} =$$

$$= \frac{1}{(4\pi D t)^{1/2}}\, e^{-r^2/4Dt} \qquad (k^2 = k_x^2 + k_y^2) \quad (\bar{k}\bar{r} = k_x \cdot x + k_y \cdot y$$

3D $\quad g(\bar{r},t) = \frac{\theta(t)}{(4\pi D t)^{3/2}}\, e^{-r^2/4Dt} \qquad \langle r^2 \rangle \sim t$

(11)



2a

$$(\partial_t^2 - c^2\nabla^2)\,\psi(\bar{r},t) = 0$$

$$\psi(\bar{r},t) = \phi(\bar{r})\, e^{\pm i\omega t}$$

$$-\nabla^2 \phi_{nm}(\bar{r}) = k_{nm}^2\, \phi(\bar{r}) \qquad \phi_{nm}(\bar{r}) = R_{nm}(r)\, e^{im\varphi}$$

$$R(r) = A_{nm} J_m(k_{nm} r) + B_{nm} N_m(k_{nm} r)$$

$$A_{nm} J_m(k_{nm} r_1) + B_{nm} N_m(k_{nm} r_1) = 0$$

$$A_{nm} J_m(k_{nm} r_2) + B_{nm} N_m(k_{nm} r_2) = 0$$

buscamos $\det = 0$

$$- J_m(k_{nm} r_1) N_m(k_{nm} r_2) = J_m(k_{nm} r_2)\, N_m(k_{nm} r_1)$$

$\uparrow$ ordenador

22

$$(\partial_t^2 - c^2\nabla^2)\, g(\bar r, t) = \delta(\bar r)\,\delta(t)$$

$$g(\bar r, t) = \frac{1}{(2\pi)^{3/2}} \int d^2k \int d\omega\, \tilde g(\bar k, \omega)\, e^{i(\bar k \bar r - \omega t)}$$

$$g(\hat k, \omega) = \frac{1}{(2\pi)^{3/2}} \int d^3 r \int dt\, g(\bar r, t)\, e^{-i(\bar k \bar r - \omega t)}$$

2

$$\tilde g(\hat k, \omega) = \frac{1}{(2\pi)^{3/2}} \frac{1}{c^2k^2 - \omega^2}$$

$$g(\bar r, t) = \frac{1}{4\pi^2} \int d^2k\, e^{i\bar k \bar r} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{c^2k^2 - \omega^2}$$

$$g(t) = \frac{1}{2\pi} \int d\omega \frac{e^{i\omega t}}{c^2k^2 - \omega^2} = \begin{cases} 0 & t<0 \\ \dfrac{\operatorname{sen}(ckt)}{ck} & t>0 \end{cases}$$

$$g(\bar r, t) = \frac{1}{4\pi^2} \int d^2k\, e^{i\bar k \bar r}\, g(k,t) = \frac{1}{2\pi} \int_0^\infty k\,dk\, g(k,t) \underbrace{\int \frac{d\varphi}{2\pi} e^{ikr\cos\varphi}}_{J_0(kr)} =$$

$$= \frac{1}{2\pi c} \int_0^\infty dk\, J_0(kr)\, \operatorname{sen}(kct) = \begin{cases} 0 & ct<r \quad r = |\bar r| > 0 \\ \dfrac{1}{2\pi c} \dfrac{1}{\sqrt{c^2t^2 - r^2}} & ; ct > r \end{cases}$$

$$G(\bar r, t, \bar r_0, t_0) \qquad r \to |\bar r - \bar r_0| \qquad t \to t - t_0$$

22

$$(\partial_t^2 - c^2 \nabla^2) g(\bar{r},t) = \delta(\bar{r})\,\delta(t)$$

$$g(\bar{r},t) = \frac{1}{(2\pi)^{3/2}} \int d^2k \int d\omega\, \tilde{g}(\bar{k},\omega)\, e^{i(\bar{k}\bar{r}-\omega t)}$$

$$g(\hat{k},\omega) = \frac{1}{(2\pi)^{3/2}} \int d^3r \int dt\, g(\bar{r},t)\, e^{-i(\bar{k}\bar{r}-\omega t)}$$

2

$$\tilde{g}(\hat{k},\omega) = \frac{1}{(2\pi)^{3/2}} \frac{1}{c^2k^2-\omega^2}$$

$$g(\bar{r},t) = \frac{1}{4\pi^2} \int d^2k\, e^{i\bar{k}\bar{r}} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{c^2k^2-\omega^2}$$

$$g(t) = \frac{1}{2\pi} \int d\omega \frac{e^{i\omega t}}{c^2k^2-\omega^2} = \begin{cases} 0 & t<0 \\ \dfrac{\operatorname{sen}(ckt)}{ck} & t>0 \end{cases}$$

$$g(\bar{r},t) = \frac{1}{4\pi^2} \int d^2k\, e^{i\bar{k}\bar{r}}\, g(k,t) = \frac{1}{2\pi} \int_0^\infty k\,dk\, g(k,t) \underbrace{\int \frac{d\varphi}{2\pi} e^{ikr\cos\varphi}}_{J_0(kr)} =$$

$$= \frac{1}{2\pi c} \int_0^\infty dk\, J_0(kr)\, \operatorname{sen}(kct) = \begin{cases} 0 & ct<r \quad r=|\bar{r}|\geq 0 \\ \dfrac{1}{2\pi c} \dfrac{1}{\sqrt{c^2t^2-r^2}} ; & ct>r \end{cases}$$

$$G(\bar{r},t,\bar{r}_0,t_0) \qquad r \to |\bar{r}-\bar{r}_0| \qquad t \to t-t_0$$

(21) H4

$$\left(\left(\frac{\omega}{c}\right)^2+\nabla^2\right)G(\bar{r},\bar{r}_0)=-\delta(\bar{r}-\bar{r}_0)$$

$$g(\bar{r})=G(\bar{r},\bar{r}_0=0)$$

$$\left(\left(\frac{\omega}{c}\right)^2+\nabla^2\right)g(\bar{r})=-\delta(\bar{r})$$

• $g(\bar{r})=\frac{1}{2\pi}\int d^2k\, e^{i\bar{k}\bar{r}}\,\tilde{g}(k)$; $\tilde{g}(k)=\frac{1}{2\pi}\int d^2r\, e^{-i\bar{k}\bar{r}}\, g(r)$

$$\tilde{g}(\bar{k})=\frac{1}{(2\pi)^2}\frac{1}{k^2-\tilde{\omega}^2} \qquad \tilde{\omega}=\omega/c$$

$$g(r)=\frac{1}{(2\pi)^2}\int d^2k\,\frac{e^{i\bar{k}\bar{r}}}{k^2-\tilde{\omega}^2}=\frac{1}{(2\pi)^2}\int_0^\infty k\,dk\int_0^{2\pi} d\varphi\,\frac{e^{ikr\cos\varphi}}{k^2-\omega^2}=$$

$$=\int_0^\infty k\,dk\,\frac{J_0(kr)}{k^2-\omega^2}\frac{1}{2\pi} \qquad \frac{J_0(kr)}{k^2-\omega^2}$$

$\omega\to\omega^+=\omega+i0^+$ 10.27.46

$$g(r)=\frac{K_0}{2\pi}(-i\hat{\omega}^+r)=\begin{cases}\frac{+\pi i}{2}H_0^{(1)}(|\hat{\omega}|r) & \mathrm{sgn}(\hat{\omega})>0\\ \frac{-\pi i}{2}H_0^{(2)}(|\hat{\omega}|r) & \mathrm{sgn}(\hat{\omega})<0\end{cases}$$

DLMF 10.27.9

DLMF 10.27.8

$$\boxed{g(\bar{r})=\begin{cases}i/4\,H_0^{(1)}(|\tilde{\omega}|r) & \mathrm{sgn}(\tilde{\omega})>0\\ -i/4\,H_0^{(2)}(|\tilde{\omega}|r) & \mathrm{sgn}(\tilde{\omega})<0\end{cases}}$$

$$H_0^{(1,2)}\sim J_0\pm iN_0$$

$$g(r\to 0)=-\frac{1}{4}N_0(|\tilde{\omega}|r)\to-\frac{1}{2\pi}\ln(r)$$

$$g(r)e^{-i\omega t}\xrightarrow[r\to\infty]{}\begin{cases}\frac{1}{\sqrt{r}}\,e^{i\frac{|\omega|}{c}(r-ct)}\,e^{-i\pi/4} & \omega>0\\ \frac{1}{\sqrt{r}}\,e^{-i\frac{|\omega|}{c}(r-ct)}\dots\end{cases}$$

21 H4

$$\left(\left(\frac{\omega}{c}\right)^2+\nabla^2\right)G(\bar r,\bar r_0)=-\delta(\bar r-\bar r_0)$$

$$g(\bar r)=G(\bar r,\bar r_0=0)$$

$$\left(\left(\frac{\omega}{c}\right)^2+\nabla^2\right)g(\bar r)=-\delta(\bar r)$$

$$\bullet\ g(\bar r)=\frac{1}{2\pi}\int d^2k\, e^{i\bar k\bar r}\,\tilde g(k);\quad \tilde g(k)=\frac{1}{2\pi}\int d^2r\, e^{-i\bar k\bar r}\, g(r)$$

$$\tilde g(\bar k)=\frac{1}{(2\pi)^2}\frac{1}{k^2-\tilde\omega^2}\qquad \tilde\omega=\omega/c$$

$$g(r)=\frac{1}{(2\pi)^2}\int d^2k\,\frac{e^{i\bar k\bar r}}{k^2-\tilde\omega^2}=\underbrace{\frac{1}{(2\pi)^2}}\int_0^\infty k\,dk\,\underbrace{\int_0^{2\pi}d\varphi\,\frac{e^{ikr\cos\varphi}}{k^2-\omega^2}}=$$

$$=\int_0^\infty k\,dk\,\frac{J_0(kr)}{k^2-\omega^2}\frac{1}{2\pi}\qquad \frac{J_0(kr)}{k^2-\omega^2}$$

$\omega\to\omega^+=\omega+i0^+$ 10.27.46

$$g(r)=\frac{K_0(-i\tilde\omega^+r)}{2\pi}=\begin{cases}\frac{+\pi i}{2}H_0^{(1)}(|\tilde\omega|r) & \mathrm{sgn}(\tilde\omega)>0\\ \frac{-\pi i}{2}H_0^{(2)}(|\tilde\omega|r) & \mathrm{sgn}(\tilde\omega)<0\end{cases}$$

DLMF 10.27.9

DLMF 10.27.8

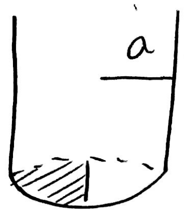
$$\boxed{g(\bar r)=\begin{cases}i/4\, H_0^{(1)}(|\tilde\omega|r) & \mathrm{sgn}(\tilde\omega)>0\\ -i/4\, H_0^{(2)}(|\tilde\omega|r) & \mathrm{sgn}(\tilde\omega)<0\end{cases}}$$

$$H_0^{(1,2)}\sim J_0\pm iN_0$$

$$g(r\to 0)=-\frac{1}{4}N_0(|\tilde\omega|r)\to-\frac{1}{2\pi}\ln(r)$$

$$g(r)e^{-i\omega t}\xrightarrow[r\to\infty]{}\begin{cases}\frac{1}{\sqrt r}e^{i\frac{|\omega|}{c}(r-ct)}e^{-i\pi/4} & \omega>0\\ \frac{1}{\ }\ e^{-i\frac{|\omega|}{c}\ldots}\ldots\end{cases}$$

(16)



$$u(r,\varphi) = \frac{i V_0}{2} g(r,\varphi)\, e^{-i\Omega t} + c.c.$$

$$g(r,\varphi) = \begin{cases} 1 & \varphi \in [0, \pi) \\ -1 & \varphi \in (\pi, 2\pi] \end{cases}$$

$$\nabla^2 \psi = \left[\left(\frac{\omega}{c}\right)^2 - k_{nm}^2\right] \psi$$

$$\psi(r,\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} J_m(k_n r) \qquad J_m'(k_n a) = 0$$

$$\partial_z^2 C_n(z) = \underbrace{\left[\left(\frac{\omega}{c}\right)^2 - k_{nm}^2\right]}_{\Omega} C_{nm}(z)$$

$$C_n(z) \begin{cases} \operatorname{Sen}(\Omega z) & (k_{nm} < \omega/c) \\ \operatorname{Senh}(\Omega z) & (k_{nm} > \omega/c) \end{cases}$$

$$\rho_0\, \partial_t^2\, u(r,\varphi,t) = (-\partial_z \psi)\,(r,\varphi,z=0,t)$$

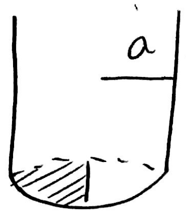
$$-\frac{i \rho V_0}{2} g(r,\varphi)\, \omega^2 = \Omega^2\, C_n(z)\, \phi_{nm}(r,\varphi)$$

$$D_{nm} = \frac{\langle \phi(r,\varphi) \mid g_m \rangle}{b_{nm}} \; ; \; b_{nm} = \int_0^a r\,dr \left[J_m(k_{nm} r)\right]^2$$

$$C_{nm} = \frac{\rho_0\, \omega^2\, u_0\, D_{nm}}{2\Omega}$$

$$g_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{im\varphi} g(r,\varphi) = -\frac{4i}{m\sqrt{2\pi}} \qquad m = 1, 3, 5$$

(16)



$$u(r,\varphi) = \frac{i V_0}{2} g(r,\varphi)\, e^{-i\Omega t} + c.c.$$

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$$\nabla^2 \psi = \left[\left(\frac{\omega}{c}\right)^2 - k_{nm}^2\right] \psi$$

$$\psi(r,\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} J_m(k_n r) \qquad J'_m(k_n a) = 0$$

$$\partial_z^2 C_n(z) = \underbrace{\left[\left(\frac{\omega}{c}\right)^2 - k_{nm}^2\right]}_{\Omega} C_{nm}(z)$$

$$C_n(z) \begin{cases} \text{Sen}(\Omega z) & (k_{nm} < \omega/c) \\ \text{Senh}(\Omega z) & (k_{nm} > \omega/c) \end{cases}$$

$$\rho_0 \partial_t^2 u(r,\varphi,t) = (-\partial_z \psi)(r,\varphi,z=0,t)$$

$$-\frac{i \rho V_0}{2} g(r,\varphi)\, \omega^2 = \Omega^2 C_n(z)\, \phi_{nm}(r,\varphi)$$

$$D_{nm} = \frac{\langle \phi(r,\varphi) | g_m \rangle}{b_{nm}} \ ; \ b_{nm} = \int_0^a r\,dr \left[J_m(k_{nm} r)\right]^2$$

$$C_{nm} = \frac{\rho_0 \omega^2 u_0 D_{nm}}{2\Omega}$$

$$g_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{im\varphi} g(r,\varphi) = -\frac{4i}{m\sqrt{2\pi}} \qquad m = 1, 3, 5$$

13 cont

T_i

T_0

$T(\bar{r},t) = T_0 + \tilde{T}(\bar{r},t)$

$\tilde{T} \propto e^{-\lambda_n t}$ $\quad t \to \infty \quad \tilde{T} = 0 \quad T(r,\infty) = T_0$

$m=0 \to$ invariante frente a traslaciones

$\cancel{\nabla^2 \tilde{T}(\bar{r},t)} = \quad (\partial_t - \chi \nabla_r^2)\,\tilde{T}(\bar{r},t) = 0$

$\tilde{T}(\bar{r},t) = \underbrace{\psi(\bar{r},t)}_{\text{apartado anterior}}\, C_n(t)$

$\chi K_n^2 = \lambda_n \qquad \partial_t C_n(t) = \chi C_n(t) K_n^2$

$C_n(t) = C_n(0)\, e^{-\lambda_n t} \qquad C_n(0) = \langle \psi \,|\, T_i \rangle$

$$\psi_{m=0}(r,\varphi) = \sum_n \sqrt{\frac{2}{b}}\, \mathrm{Sen}\left(\frac{n\pi}{b}z\right) \frac{1}{\sqrt{2\pi}}\, J_{0n}(K_{0n} r)$$

$$\langle \tilde{T} | T_i' \rangle = \frac{T_i'}{\sqrt{b\pi}} \int_0^b dz\, \mathrm{Sen}\left(\frac{n\pi}{b}z\right) \int_0^r r J_{0n}(K_{0n} r) \int_0^{2\pi} d\varphi\, b_{nm}^{-1}$$

$T_i' = (T_i - T_0)$

$n =$ impar

$K_{0n} r = x \qquad dr = \frac{dx}{K_{0n}}$

$$\downarrow = \frac{T_i'}{\sqrt{b\pi}} \frac{b}{n\pi}\, 2 \cdot 2\pi \int_0^{K_0 a} \frac{x J_0(x)\, dx}{b_{nm}} = \frac{T_i' b}{\sqrt{b\pi}\, n\pi}\, 2\, Ka\, J_1(Ka)\, 2\pi\, b_{nm}^{-1}$$

$$C_n(0) = \sqrt{\frac{b}{\pi}}\, \frac{T_i'}{n\pi}\, 2 K_{0n} a\, J_1(K_{n0} a)\, 2\pi\, b_{nm}^{-1}$$

$$T(\bar{r},t) = \sum \psi_{m=0}(\bar{r},t)\, C_n(0)\, e^{-\lambda_n t} + T_0$$

$$\boxed{\quad T_i' \,\mathrm{Sen}\left(\frac{n\pi}{}z\right) \frac{4}{} J_0(K_n a) J_1(K_n A)\, e^{-\lambda_n t} \quad K_n a}$$

$x(t)\, p(1) \qquad p(1)\, x(t) - \chi\, \ddot{p}(1)\, x(t)$

(13) cont

T_i

T_0

$T(\bar r, t) = T_0 + \tilde T(\bar r, t)$

$\tilde T \propto e^{-\lambda_n t} \quad t \to \infty \quad \tilde T = 0 \quad T(r, \infty) = T_0$

$m=0 \to$ invariante frente a traslaciones

$\cancel{\nabla^2 \tilde T(\bar r, t)} = \quad (\partial_t - \chi \nabla_r^2)\tilde T(\bar r, t) = 0$

$\tilde T(\bar r, t) = \underbrace{\psi(\bar r, t)}_{\text{apartado anterior}} C_n(t)$

$\chi K_n^2 = \lambda_n \qquad \partial_t C_n(t) = \chi C_n(t) K_n^2$

$C_n(t) = C_n(0)\, e^{-\lambda_n t} \qquad C_n(0) = \langle \psi | T_i \rangle$

$$\psi_{m=0}(r, \varphi) = \sum_n \sqrt{\frac{2}{b}} \operatorname{Sen}\left(\frac{n\pi}{b} z\right) \frac{1}{\sqrt{2\pi}} J_{0n}(K_{0n} r)$$

$$\langle \tilde T | T_i' \rangle = \frac{T_i'}{\sqrt{b\pi}} \int_0^b dz \operatorname{Sen}\left(\frac{n\pi}{b} z\right) \int_0^r r J_{0n}(K_{0n} r) \int_0^{2\pi} d\varphi \; b_{nm}^{-1}$$

$T_i' = (T_i - T_0)$

$n =$ impar

$K_{0n} r = x \qquad dr = \frac{dx}{K_{0n}}$

$$= \frac{T_i'}{\sqrt{b\pi}} \frac{b}{n\pi} 2 \cdot 2\pi \int_0^{K_0 a} \frac{x J_0(x)\, dx}{b_{nm}} = \frac{T_i' b}{\sqrt{b\pi}\, n\pi} 2 K a\, J_1(Ka)\, 2\pi b_{nm}^{-1}$$

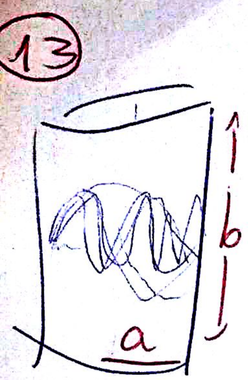
$$C_n(0) = \sqrt{\frac{b}{\pi}} \frac{T_i'}{n\pi} 2 K_{0} a\, J_1(K_{n0} a) 2\pi\, b_{nm}^{-1}$$

$$T(\bar r, t) = \sum_{m=0} \psi(\bar r, t) C_n(0) e^{-\lambda_n t} + T_0$$

$T_i' \operatorname{Sen}\left(\frac{n\pi}{b} z\right) \frac{4}{\quad} J_0(K_n a) J_1(K_n A) e^{-\lambda_n t} \quad K_n a$

$x(t) P(l) \qquad P(l) \dot x(t) - \chi \ddot P(l)\, x(t)$

(13) $V(z=0)=V(z=b)=V(r=a)=0$



a) $\nabla^2\psi = 0 \Rightarrow$ autovaloes son 0

$$\nabla^2 = \Delta_2 + \partial^2_z$$

$$\psi(r,\varphi,z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \phi_{nm}(r,\varphi)\, C_{nm}(z)$$

• $\phi_{nm}(\bar r) \rightarrow -\nabla^2 \phi_{nm}(\bar r) = k^2 \phi_{nm}(\bar r)$

$$\cancel{\phi_{nm}(\bar r) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} J_m(k_{nm} r)}$$

$C_{nm}(z)$ $\quad C_{nm}(0)=0$, $C_{nm}(b)=0$ $\quad \sqrt{\frac{2}{b}}\,\mathrm{Sen}(k_n z)$ $\quad k_n = \frac{n\pi}{b}$

$$\phi_{nm}(r,\varphi) = R_n(r)\, g_n(\varphi) \qquad \nabla^2(r,\varphi) = \underbrace{\partial^2_r + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_\varphi}$$

$$\begin{cases} g''_m(\varphi) = -m^2 g_m(\varphi) \\ R''_n(r) + \frac{1}{r} R'_m(r) - \frac{m^2}{r^2} R_n(r) = 0 \end{cases}$$

$$g_m(\varphi) \propto e^{im\varphi} \Rightarrow g_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

$$R_m(r) = a_m r^{|m|} + b_m r^{-|m|} \qquad \forall m \neq 0$$

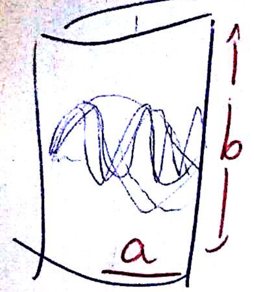
$$m=0 \rightarrow a_0 + \ln r$$

$$\psi(r,\varphi,z) = \phi\, R_m(r)\, g_m(\varphi)\, C_{nm}(z)$$

$$\boxed{\psi(r,\varphi,z) = \sum_{m,n} \sqrt{\frac{2}{b}}\,\mathrm{Sen}\left(\frac{n\pi}{b}z\right)\left[\frac{1}{\sqrt{2\pi}} e^{im\varphi}\left(a_m r^{|m|} + b_m r^{-|m|}\right) + a_0 + \ln r\right]}$$

b) (regular en $r=0$) $\rightarrow J_m(k \ldots$

(13) $V(z=0)=V(z=b)=V(r=a)=0$



a) $\nabla^2\psi=0 \Rightarrow$ autovalores son 0

$\nabla^2=\Delta_2+\partial^2_z$

$$\psi(r,\varphi,z)=\sum_{n=-\infty}^{\infty}\sum_{1}^{\infty}\phi_{nm}(r,\varphi)\,C_{nm}(z)$$

• $\phi_{nm}(\bar r)\rightarrow -\nabla^2\phi_{nm}(\bar r)=k^2\phi_{nm}(\bar r)$

$\cancel{\phi_{nm}(\bar r)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}J_m(k_{nm}r)}$

$C_{nm}(z)$ $\quad C_{nm}(0)=0$ $\quad C_{nm}(b)=0$ $\quad \sqrt{\frac{2}{b}}\,\mathrm{Sen}(k_n z)$ $\quad k_n=\frac{n\pi}{b}$

$\phi_{nm}(r,\varphi)=R_n(r)\,g_n(\varphi)$ $\qquad \nabla^2(r,\varphi)=\underbrace{\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial^2_\varphi}$

$$\begin{cases} g''_m(\varphi)=-m^2 g_m(\varphi) \\ R''_m(r)+\frac{1}{r}R'_m(r)-\frac{m^2}{r^2}R_m(r)=0 \end{cases}$$

$g_m(\varphi)\propto e^{im\varphi} \Rightarrow g_m(\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}$

$R_m(r)=a_m r^{|m|}+b_m r^{-|m|}$ $\quad \forall m\neq 0$

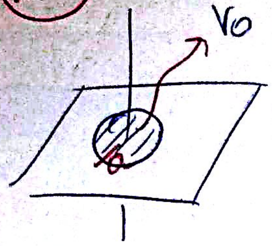
$m=0\rightarrow a_0+\ln r$

$\psi(r,\varphi,z)=\phi\, R_m(r)\,g_m(\varphi)\,C_{nm}(z)$

$$\boxed{\psi(r,\varphi,z)=\sum_{m,n}\sqrt{\frac{2}{b}}\,\mathrm{Sen}\left(\frac{n\pi}{b}z\right)\underbrace{\left[\frac{1}{\sqrt{2\pi}}e^{im\varphi}\left(a_m r^{|m|}+b_m r^{-|m|}\right)+a_0+\ln r\right]}}$$

b) (regular en $r=0$) $\rightarrow J_m(k\ \ \ r)$

(12)



probar $V(r,z) = V_0 a \int_0^\infty dk\, J_1(ka) J_0(kr) e^{-kz}$

$$V = \begin{cases} V_0 & r < a \\ 0 & r > a \end{cases}$$

como no depende de $\varphi \Rightarrow \underline{m=0}$ → invariante frente a rotaciones

$\nabla^2 \psi(r,z) = 0$ $\quad \partial_z^2 C_m(k,z) = k^2 C_m(k,z)$

$$\boxed{\psi(r,\varphi,z) = \sum_{-\infty}^{\infty} \int_0^\infty k\,dk\, C_m(kz)\phi(r,\varphi)}$$

$C_m(k,z) = C_m(k) e^{-kz}$ $\quad (z>0)$

$\quad\quad = \underbrace{C_m(k)}_{V_m(k)} e^{kz}$ $\quad (z<0)$

sólo queremos $z>0$

$$\phi_{km}(r,\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} J_m(kr)$$

$$V_m(k) = \langle \phi_{km} | V \rangle = \int_0^a r\,dr \int_0^{2\pi} d\varphi\, \phi_{km}^*(r,\varphi) \underbrace{V(r,\varphi)}_{V_0}$$

$m=0$

$$= \sqrt{2\pi}\, V_0 \int_0^a r\,dr\, J_0(kr) = \sqrt{2\pi} \frac{V_0}{k^2} \int_0^{ka} x\,dx\, J_0(x) = \sqrt{2\pi} \frac{a V_0}{k} J_1(ka)\, \varphi) =$$

tablas

$(x = kr \quad dx = k\,dr)$

$$\boxed{\psi(r,\varphi) = \int_0^\infty V_0 a\, e^{-kz} J_1(ka) J_0(kr)}$$

$$\begin{cases} J_m(0) = \frac{1}{m!}\left(\frac{x}{2}\right)^m \\ J_0(0) = 1 \end{cases}$$

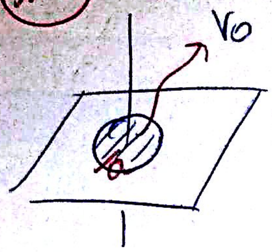
• $r=0$

$$\psi(r=0,z) = \int_0^\infty V_0 a\, e^{-kz} J_1(ka) J_0(k\cdot 0)\, dk =$$

tablas

$\psi(0,z) = V_0 [\,\ldots\; z \quad]$

(12)



probar $V(r,z) = V_0 a \int_0^\infty dk\, J_1(ka) J_0(kr) e^{-kz}$

$$V = \begin{cases} V_0 & r < a \\ 0 & r > a \end{cases}$$

como no depende de $\varphi \Rightarrow \underline{m=0} \rightarrow$ invariante frente a rotaciones

$\nabla^2 \psi(r,z) = 0 \qquad \partial_z^2 C_m(k,z) = k^2 C_m(k,z)$

$$\boxed{\psi(r,\varphi,z) = \sum_{-\infty}^{\infty} \int_0^\infty k\,dk\, C_m(k,z)\,\phi(r,\varphi)}$$

$$C_m(k,z) = \begin{cases} C_m(k) e^{-kz} & (z>0) \\ \underbrace{C_m(k)}_{\hookrightarrow V_m(k)} e^{kz} & (z<0) \end{cases}$$

sólo queremos $z>0$

$$\phi_{km}(r,\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} J_m(kr)$$

$$V_m(k) = \langle \phi_{km} | V \rangle = \int_0^a r\,dr \int_0^{2\pi} d\varphi\, \phi^*_{km}(r,\varphi)\, V(r,\varphi) =$$

$m=0$

$$= \sqrt{2\pi}\, V_0 \int_0^a r\,dr\, J_0(kr) = \sqrt{2\pi}\, \frac{V_0}{k^2} \int_0^{ka} x\,dx\, J_0(x) = \sqrt{2\pi}\, \frac{a V_0}{k} J_1(ka)\, e) =$$

$(x = kr \quad dx = k\,dr)$ tablas

$$\boxed{\psi(r,\varphi) = \int_0^\infty V_0 a\, e^{-kz} J_1(ka) J_0(kr)}$$

$$\begin{cases} J_m(0) = \frac{1}{m!}\left(\frac{x}{2}\right)^m \\ J_0(0) = 1 \end{cases}$$

• $r=0$

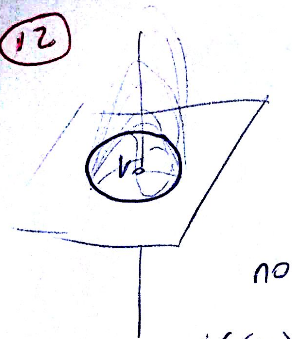
$$\psi(r=0,z) = \int_0^\infty V_0 a\, e^{-kz} J_1(ka) J_0(k\cdot 0)\, dk =$$

tablas

$\psi(0,z) = V_0 \Big[1 \quad z \quad \Big]$

$x_{n=6} \approx 18'1$ $\qquad x_{n=7} \approx 24'4$

12



$$V(r,z) = V_0\, a \int_0^\infty dk\, J_1(ka)\, J_0(kr)\, e^{-kz}$$

$x = kr$

$dx = k\,dr$

$V_1(r>a, z=0) = 0$

$V_2(r<a, z=0) = V_0$

invariante frente a rotaciones

no depende de $\varphi \to V_{m=0}(k)$

$$V_m(k) = \int_0^r dr \int_0^{2\pi} d\varphi\, \frac{e^{-im\varphi}}{\sqrt{2\pi}}\, J_m(kr)\, V(r,\varphi)$$

$\to V_0$ todas

$m=0$

$$V_0(k) = \sqrt{2\pi}\int_0^r r\,dr\, J_0(kr) = \sqrt{2\pi}\,\frac{V_0}{k^2}\int_0^{ka} x\,dx\, J_0(x) = \sqrt{2\pi}\,\frac{V_0}{k}\,a\, J_1(kb)$$

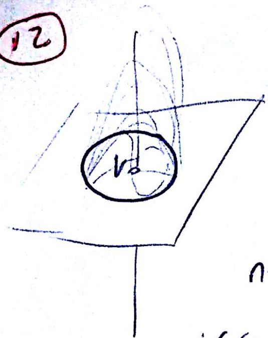
$$\boxed{\Psi_m(r,z) = \sum_{-\infty}^{\infty}\int_0^\infty k\,dk\, V_m(k)\, e^{-kz}\, \phi_{k_m}(r)} \qquad \boxed{= V_0\, a\int_0^\infty dk\, J_1(ka)\, J_0(kr)\, e^{-kz}}$$

en $r=0$ (eje z)

$$V(r=0, z) = V_0\, a\int_0^\infty dk\, e^{-kz}\, J_1(kb) = V_0\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]$$

$x_{n=6} \approx 18'1$ $\quad x_{n=7} \approx 24'4$

(12)



$$V(r,z) = V_0 a \int_0^\infty dk\, J_1(ka)\, J_0(kr)\, e^{-kz}$$

$x = kr$

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$V_1(r>a, z=0) = 0$

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invariante frente a rotaciones

no depende de $\varphi \to V_{m=0}(k)$

$$V_m(k) = \int_0^r dr \int_0^{2\pi} d\varphi\, \frac{e^{-im\varphi}}{\sqrt{2\pi}}\, J_m(kr)\, V(r,\varphi)$$

$\to V_0$ todas

$m=0$

$$V_0(k) = \sqrt{2\pi} \int_0^r r\,dr\, J_0(kr) = \sqrt{2\pi}\, \frac{V_0}{k^2} \int_0^{ka} x\,dx\, J_0(x) = \sqrt{2\pi}\, \frac{V_0}{k}\, a\, J_1(kb)$$

$$\boxed{\Psi_m(r,z) = \sum_{-\infty}^{\infty} \int_0^\infty k\,dk\, V_m(k)\, e^{-kz}\, \phi_{k_m}(r)}$$

$$\boxed{= V_0 a \int_0^\infty dk\, J_1(ka)\, J_0(kr)\, e^{-kz}}$$

en $r=0$ (eje z)

$$V(r=0, z) = V_0 a \int_0^\infty dk\, e^{-kz}\, J_1(kb) = V_0 \left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]$$

$$J_{m-1}(x) = \left(\frac{x}{2}\right)^{m-1} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} \left(\frac{x}{2}\right)^{2j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} \left(\frac{x}{2}\right)^{2j+m-1}$$

$$J_m'(x) = \frac{m}{x} J_m(x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} 2j \frac{x^{2j+m-1}}{2^{2j+m}} = J_{m-1}(x) - \frac{m}{x} J_m(x)$$

$$J_{m-1}(x) = 2\frac{m}{x} J_m(x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \frac{x^{2j+m-1}}{2^{2j+m}} 2j =$$

$$= 2\frac{m}{x} J_m(x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j-1)!(m+j)!} \left(\frac{x}{2}\right)^{2j+m-1}$$

$$J_{m-1}(x) = \frac{2}{x}\left(\frac{x}{2}\right)^m \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} \left(\frac{x}{2}\right)^{2j} \overset{j \to j+1}{=} \cancel{\frac{2}{x}\left(\frac{x}{2}\right)^m \sum_{j=2}^{\infty}}$$

$$\cancel{\frac{(-1)^{j+1}}{(j+1)!(m+j)!} \left(\frac{x}{2}\right)^{2j+2}} = \cancel{\frac{2}{x}\left(\frac{x}{2}\right)^m \sum_{j=2}^{\infty} \frac{(-1)^{j+1}(m+j)}{(j+1)!(m-1+j)!} \left(\frac{x}{2}\right)^{2j+2}}$$

$$= \cancel{2\frac{m}{x}\left(\frac{x}{2}\right)^m \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{(j+1)!(m-1+j)!} \left(\frac{x}{2}\right)^{2j+2}}$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} 2 \frac{x^{2j+m-1}}{2^{2j+m}} = \sum_{j=1}^{\infty} \frac{(-1)^j (j+m) 2}{j!(m+j)!} \frac{x^{2j+m-1}}{2^{2j+m}} =$$

$$= \frac{m}{x} J_m(x) + \underbrace{\sum_{j=1}^{\infty} \frac{(-1)^j (2j+m)}{j!(m+j)!} \frac{x^{2j+m-1}}{2^{2j+m}}}_{J_m'(x)}$$

$$J_{m-1}(x) + J_{m+1}(x) = \frac{m}{x} J_m(x) + \cancel{J_m'(x)} + \frac{m}{x} J_m(x) +$$

$$J_{m-1}(x) = \left(\frac{x}{2}\right)^{m-1} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} \left(\frac{x}{2}\right)^{2j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} \left(\frac{x}{2}\right)^{2j+m-1}$$

$$J_m'(x) = \frac{m}{x} J_m(x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} 2j \frac{x^{2j+m-1}}{2^{2j+m}} = J_{m-1}(x) - \frac{m}{x} J_m(x)$$

$$J_{m-1}(x) = 2\frac{m}{x} J_m(x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \frac{x^{2j+m-1}}{2^{2j+m}} 2j =$$

$$= 2\frac{m}{x} J_m(x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j-1)!(m+j)!} \left(\frac{x}{2}\right)^{2j+m-1}$$

$$J_{m-1}(x) = \frac{2}{x}\left(\frac{x}{2}\right)^m \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} \left(\frac{x}{2}\right)^{2j} \quad j \to j+1 \quad = \cancel{\frac{2}{x}\left(\frac{x}{2}\right)^m \sum_{j=2}^{\infty}}$$

$$\cancel{\frac{(-1)^{j+1}}{(j+1)!(m+j)!} \left(\frac{x}{2}\right)^{2j+2}} = \cancel{\frac{2}{x}\left(\frac{x}{2}\right)^m \sum_{j=2}^{\infty} \frac{(-1)^{j+1}(m+j)}{(j+1)!(m-1+j)!} \left(\frac{x}{2}\right)^{2j+2}} =$$

$$= \cancel{2\frac{m}{x}\left(\frac{x}{2}\right)^m \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{(j+1)!(m-1+j)!} \left(\frac{x}{2}\right)^{2j+2}}$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m-1+j)!} 2 \frac{x^{2j+m-1}}{2^{2j+m}} = \sum_{j=1}^{\infty} \frac{(-1)^j (j+m) 2}{j!(m+j)!} \frac{x^{2j+m-1}}{2^{2j+m}} =$$

$$= \frac{m}{x} J_m(x) + \underbrace{\sum_{j=1}^{\infty} \frac{(-1)^j (2j+m)}{j!(m+j)!} \frac{x^{2j+m-1}}{2^{2j+m}}}_{J_m'(x)}$$

$$J_{m-1}(x) + J_{m+1}(x) = \frac{m}{x} J_m(x) + \cancel{J_m'(x)} + \frac{m}{x} J_m(x) +$$

(14) Hoja 4 HMIII

$$\begin{cases} J_{m+1}(x) = \frac{m}{x} J_m(x) - J_m'(x) \\ J_{m-1}(x) = \frac{m}{x} J_m(x) + J_m'(x) \end{cases}$$

1

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j}$$

$$J_{m\pm1}(x) = J_m(x)\frac{m}{x} \mp J_m'(x)$$

$$-J_m'(x) = \frac{\partial J_m(x)}{\partial x} = \left[\sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} (m+2j) \frac{x^{2j+m-1}}{2^{2j+m}}\right] J_m'(x)$$

$$-\mp J_m'(x) = J_{m\pm1}(x) - J_m(x)\frac{m}{x}$$

$$J_{m+1}(x) = \left(\frac{x}{2}\right)^{m+1} \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+1+j)!} \left(\frac{x}{2}\right)^{2j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+1+j)!} \left(\frac{x}{2}\right)^{2j+m+1}$$

$$J_m(x)\frac{m}{x} = \frac{m}{x}\left(\frac{x}{2}\right)^m \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m-1} \frac{m}{2}$$

$$J_{m+1}(x) + J_m(x) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m+1} - \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m-1} \frac{m}{2}$$

$$J_m'(x) = \underbrace{\frac{m}{x} \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m}}_{\frac{m}{x} J_m(x)} + \underbrace{\sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} 2j \frac{x^{2j+m-1}}{2^{2j+m}}}_{\text{tiene que ser } -J_{m+1}(x)}$$

$j \to j+1$

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} 2j \frac{x^{2j+m-1}}{2^{2j+m}} \Rightarrow \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(j+1)!(m+1+j)!} (2j+2) \frac{x^{2j+m+1}}{2^{2j+m+2}} =$$

$$= -\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+1+j)!} \left(\frac{x}{2}\right)^{2j+m+1} \quad //$$

(14) Hoja 4 MMIII

$$\begin{cases} J_{m+1}(x) = \frac{m}{x} J_m(x) - J_m'(x) \\ J_{m-1}(x) = \frac{m}{x} J_m(x) + J_m'(x) \end{cases}$$

1

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j}$$

$$J_{m\pm 1}(x) = J_m(x)\frac{m}{x} \mp J_m'(x)$$

$$-J_m'(x) = \frac{\partial J_m(x)}{\partial x} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} (m+2j) \frac{x^{2j+m-1}}{2^{2j+m}} \Bigg] J_m'(x)$$

$$-\mp J_m'(x) = J_{m\pm 1}(x) - J_m(x)\frac{m}{x}$$

$$J_{m+1}(x) = \left(\frac{x}{2}\right)^{m+1} \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+1+j)!} \left(\frac{x}{2}\right)^{2j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+1+j)!} \left(\frac{x}{2}\right)^{2j+m+1}$$

$$J_m(x)\frac{m}{x} = \frac{m}{x}\left(\frac{x}{2}\right)^m \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m-1} \frac{m}{2}$$

$$J_{m+1}(x) + J_m(x) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m+1} - \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m-1} \frac{m}{2}$$

$$J_m'(x) = \underbrace{\frac{m}{x} \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m}}_{\frac{m}{x} J_m(x)} + \underbrace{\sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} 2j \frac{x^{2j+m-1}}{2^{2j+m}}}_{\text{tiene que ser } -J_{m+1}(x)}$$

$j \to j+1$

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j!(m+j)!} 2j \frac{x^{2j+m-1}}{2^{2j+m}} \Rightarrow \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(j+1)!(m+1+j)!} (2j+2) \frac{x^{2j+m+1}}{2^{2j+m+2}} =$$

$$= -\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+1+j)!} \left(\frac{x}{2}\right)^{2j+m+1} \quad //$$

(5)

$$|k,m\rangle \longrightarrow \psi_{k,m}(\vec r) = \frac{e^{im\varphi}}{\sqrt{2\pi}} J_m(kr)$$

$$|\vec k\rangle \Rightarrow \phi_{\vec k}(\vec r) = \frac{1}{2\pi} e^{i\vec k \vec r} \qquad \langle \vec k | k', m\rangle = C_m(k)\,\delta(k-k')\, e^{im\varphi_{\vec k}}$$

$$|k,m\rangle = \int d^3k \langle \vec k | k', m\rangle |\vec k\rangle ;$$

$$\frac{1}{\sqrt{2\pi}} J_m(k_1) = V\, C_m(k) \int \frac{d\varphi'}{2\pi} e^{ik_1 \cos\varphi'} e^{im\varphi'} \qquad \lim_{r\to 0} J_m(k_1) = \left(\frac{k_1}{2}\right)^m \frac{1}{m!}$$

$$e^{ik_1\cos(\varphi-\varphi_k)} = \sum_m i^m e^{-im(\varphi_k-\varphi)} J_m(k_1) \qquad \lim_{r\to 0} \int \frac{d\varphi'}{\sqrt{2\pi}} e^{ik_1\cos\varphi'} e^{im\varphi'} = \left(\frac{ik_1}{2}\right)^m \frac{1}{m!}$$

$$\downarrow$$

$$e^{i\vec k \vec r} = \sum_m i^m J_m(k_1)\, e^{im(\varphi-\varphi_k)}$$

$$\downarrow$$

$$\boxed{C_m(k) = \frac{(-i)^m}{k} \frac{1}{\sqrt{2\pi}}}$$

3d

$$|k,\ell m\rangle \longrightarrow \psi_{k\ell m}(\vec r) = j_\ell(k_1)\, Y_{\ell m}(\theta,\varphi)$$

$$|\vec k\rangle \longrightarrow \phi_{\vec k}(\vec r) = \frac{1}{(2\pi)^{3/2}} e^{i\vec k \vec r}$$

$$\langle \vec k | k', \ell m\rangle \propto \delta(k-k')\, Y_{\ell m}(\theta_{\vec k}, \varphi_{\vec k})$$

$$e^{i\vec k \vec r} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_\ell(k)\, Y_{\ell m}(\theta_k, \varphi_k)\, j_\ell(k_1)\, Y_{\ell m}(\theta,\varphi)$$

$$C_\ell(k) \frac{2\ell+1}{2} j_\ell(k_1) = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, e^{ik_1 x} P_\ell(x)$$

$$r \to 0 ; \quad j_\ell(k_1) \to \frac{(k_1)^\ell}{(2\ell+1)!!}$$

$$\lim_{r\to 0} \int dx\, e^{ik_1 x} P_\ell(x) = \frac{(ik_1)^\ell}{(2\ell+1)!!}\, 2$$

$$\boxed{C_\ell(k) = 4\pi i^\ell} \qquad e^{i\vec k \vec r} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell\, Y^*_{\ell m}(\theta_{\vec k}, \varphi_{\vec k})\, j_\ell(k_1)\, Y_{\ell m}(\theta,\varphi)$$

m dado

$$\downarrow$$

$$= 4\pi \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, P_\ell(\cos\gamma_{\vec k \vec r})\, j_\ell(k_1)$$

(5)

$|k,m\rangle \longrightarrow \psi_{k,m}(\vec r) = \frac{e^{im\varphi}}{\sqrt{2\pi}} J_m(kr)$

$|\vec k\rangle \Rightarrow \phi_{\vec k}(\vec r) = \frac{1}{2\pi} e^{i\vec k \vec r}$ $\quad \langle \vec k | k', m\rangle = C_m(k)\,\delta(k-k')\, e^{im\varphi_{\vec k}}$

$|k,m\rangle = \int d^3k \langle \vec k | k', m\rangle |\vec k\rangle$;

$\frac{1}{\sqrt{2\pi}} J_m(k_1) = V\, C_m(k) \int \frac{d\varphi'}{2\pi} e^{ik_1 \cos\varphi'} e^{im\varphi'}$ $\quad \lim_{r\to 0} J_m(k_1) = \left(\frac{k_1}{2}\right)^m \frac{1}{m!}$

$e^{ik_1 \cos(\varphi-\varphi_k)} = \sum_m i^m e^{-im(\varphi_k-\varphi)} J_m(k_1)$ $\quad \lim_{r\to 0} \int \frac{d\varphi'}{\sqrt{2\pi}} e^{ik_1\cos\varphi'} e^{im\varphi'} = \left(\frac{ik_1}{2}\right)^m \frac{1}{m!}$

$\downarrow$

$e^{i\vec k \vec r} = \sum_m i^m J_m(k_1)\, e^{im(\varphi-\varphi_k)}$

$\downarrow$

$\boxed{C_m(k) = \frac{(-i)^m}{k} \frac{1}{\sqrt{2\pi}}}$

3d

$|k,\ell,m\rangle \rightarrow \psi_{k\ell m}(\vec r) = j_\ell(k_1)\, Y_{\ell m}(\theta,\varphi)$

$|\vec k\rangle \rightarrow \phi_{\vec k}(\vec r) = \frac{1}{(2\pi)^{3/2}} e^{i\vec k \vec r}$

$\langle \vec k | k', \ell m\rangle \propto \delta(k-k')\, Y_{\ell m}(\theta_{\vec k}, \varphi_{\vec k})$

$e^{i\vec k \vec r} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_\ell(k)\, Y_{\ell m}(\theta_k, \varphi_k)\, j_\ell(k_1)\, Y_{\ell m}(\theta,\varphi)$

$C_\ell(k)\frac{2\ell+1}{2} j_\ell(k_1) = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, e^{ik_1 x} P_\ell(x)$

$r \to 0$; $j_\ell(k_1) \to \frac{(k_1)^\ell}{(2\ell+1)!!}$

$\lim_{r\to 0} \int dx\, e^{ik_1 x} P_\ell(x) = \frac{(ik_1)^\ell}{(2\ell+1)!!} 2$

$\boxed{C_\ell(k) = 4\pi i^\ell}$ $\quad e^{i\vec k \vec r} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell\, Y^*_{\ell m}(\theta_{\vec k}, \varphi_{\vec k})\, j_\ell(k_1)\, Y_{\ell m}(\theta,\varphi)$

m dado

$\downarrow$

$= 4\pi \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, P_\ell(\cos\gamma_{\vec r \vec k})\, j_\ell(k_1)$

$$\int_0^{2\pi} mx \cos\varphi \, e^{ix\sin\varphi - im\varphi} d\varphi =$$

$$\cancel{u = \cos x\, e^{ix\sin\varphi - im}} \quad ; \quad \cancel{dv = \cos\varphi / mx}$$

$$\cancel{du = i(x\sin\varphi - m) e^{ix\sin\varphi - im}} \quad \cancel{v = \sin\varphi / mx}$$

$$\cancel{\sin\varphi\, mx\, e^{ix\sin\varphi - im}\Big|_0^{2\pi} + \int \sin\varphi\, mx\, i(x\sin\varphi - m) e^{ix\sin\varphi - im}} =$$

$$\cancel{= i\int (x^2 \sin^2\varphi\, m - m^2 x \sin\varphi) e^{ix\sin\varphi - im} d\varphi} =$$

$$u = e^{-im\varphi} \quad ; \quad dv = \cos\varphi\, e^{ix\sin\varphi}$$

$$du = -im\, e^{-im\varphi} \qquad v = \frac{e^{ix\sin\varphi}}{ix}$$

$$= \frac{\cancel{e^{-im\varphi}} e^{ix\sin\varphi}}{ix}\Bigg|_0^{2\pi} + \int_0^{2\pi} \left[\frac{e^{ix\sin\varphi - im\varphi}}{\cancel{x}}\, m \, d\varphi \right] m\cancel{x} =$$

$$e^{-im2\pi} = 1$$

$$= \int_0^{2\pi} m^2 (e^{ix\sin\varphi - im\varphi})\, d\varphi$$

(4)

$$\underbrace{e^{ikr\cos\varphi}} = \sum_m i^m J_m(kr) e^{im\varphi}$$

$$\cos(\varphi - \pi/2) = \sin(\varphi + \pi/2) \qquad \varphi \pm \pi/2 = \theta$$

$$e^{ix\sin\theta} = \sum_m i^m J_m(kr) e^{im\theta} = \sum_m J_m(kr) e^{im\varphi + \frac{im\pi}{2}} =$$

$$= \sum_m J_m(kr) e^{im\varphi} i^m \qquad i^m = e^{im\pi/2}$$

$$\int_0^{2\pi} mx \cos\varphi \, e^{ix\sin\varphi - im\varphi} d\varphi =$$

$$\cancel{u = \cos t \, e^{ix\sin\varphi - im}} \; ; \; \cancel{dv = \cos\varphi / mx}$$

$$\cancel{du = i(x\sin\varphi - m) e^{ix\sin\varphi - im}} \quad \cancel{v = \sin\varphi / mx}$$

$$\cancel{\sin\varphi \, mx \, e^{ix\sin\varphi - im}\Big|_0^{2\pi} + \int \sin\varphi \, mx \, i(x\sin\varphi - m) e^{ix\sin\varphi - im}} =$$

$$\cancel{= i\int (x^2 \sin^2\varphi \, m - m^2 x \sin\varphi) e^{ix\sin\varphi - im} d\varphi} =$$

$$u = e^{-im\varphi} \; ; \; dv = \cos\varphi \, e^{ix\sin\varphi}$$

$$du = -im \, e^{-im\varphi} \qquad v = \frac{e^{ix\sin\varphi}}{ix}$$

$$= \frac{e^{-im\varphi} e^{ix\sin\varphi}}{ix}\Big|_0^{2\pi} + \int_0^{2\pi} \left[\frac{e^{ix\sin\varphi - im\varphi}}{\cancel{x}} m \, d\varphi \right] m\cancel{x} =$$

$$e^{-im2\pi} = 1$$

$$= \int_0^{2\pi} m^2 (e^{ix\sin\varphi - im\varphi}) d\varphi$$

(4)

$$\underbrace{e^{ik_r\cos\varphi}} = \sum_m i^m J_m(k_r) e^{im\varphi}$$

$$\cos\varphi + \pi/2 = \sin\varphi + \pi/2 \qquad \varphi \mp \pi/2 = \theta$$

$$e^{ix\sin\theta} = \sum_m i^m J_m(k_r) e^{im\theta} = \sum_m J_m(k_r) e^{im\varphi + \frac{im\pi}{2}} =$$

$$= \sum_m J_m(k_r) e^{im\varphi} i^m \qquad i^m = e^{im\pi/2}$$

③

$$g(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi} e^{ix \operatorname{Sen}\varphi}$$

$$x^2 y'' + xy' + (x^2 - m^2)y = 0$$

$$g'(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi} e^{ix \operatorname{Sen}\varphi} \operatorname{Sen}\varphi$$

$$g''(x) = -\frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi} e^{ix\operatorname{Sen}\varphi} \operatorname{Sen}^2\varphi$$

$$-g'(x) = \frac{1}{2\pi}\int_0 i e^{-im\varphi} e^{ix \operatorname{Sen}\varphi} \operatorname{Sen}\varphi$$

$$u = e^{i(x\operatorname{Sen}\varphi - m)} \quad ; \; dv = \operatorname{Sen}\varphi$$

$$du = i(x\cos\varphi - im)e^{i(x\operatorname{Sen}\varphi - im)} \quad ; \; v = -\cos\varphi$$

$$g'(x) = \frac{1}{2\pi}\left[e^{i(x\operatorname{Sen}\varphi - im)}(\cancel{-\cos\varphi})\right]_0^{2\pi} + \int_0^{2\pi} i\cos\varphi(x\cos\varphi - im)e^{i(x\operatorname{Sen}\varphi - im)}$$

$$g'(x) = \frac{1}{2\pi}\int_0^{2\pi} -i\cos\varphi(x\cos\varphi - im)e^{i(x\operatorname{Sen}\varphi - im)}$$

$$-\frac{x^2}{2\pi}\int e^{-im\varphi} \cancel{e^{ix\operatorname{Sen}\varphi}} \operatorname{Sen}^2\varphi + -\int \cos^2(\cancel{1} - \operatorname{Sen}^2\cancel{\varphi}) x^2 - mx\cos\varphi) e^{-im\varphi + ix\operatorname{Sen}\varphi} d\varphi$$

$$+\frac{x^2}{2\pi}\int \cancel{e^{-im\varphi}} e^{ix\operatorname{Sen}\varphi} - \frac{m^2}{2\pi}\int e^{-im\varphi} e^{ix\operatorname{Sen}\varphi} d\varphi = 0$$

$$\frac{1}{2\pi}\int^{2\pi} mx\cos\varphi\, e^{im\varphi + ix\operatorname{Sen}\varphi} d\varphi = \frac{m^2}{2\pi}\int^{2\pi} e^{-im\varphi} e^{ix\operatorname{Sen}\varphi} d\varphi$$

③

$$g(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi} e^{ix\operatorname{Sen}\varphi}$$

$$x^2y'' + xy' + (x^2-m^2)y = 0$$

$$g'(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi} e^{ix\operatorname{Sen}\varphi}\operatorname{Sen}\varphi$$

$$g''(x) = -\frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi} e^{ix\operatorname{Sen}\varphi}\operatorname{Sen}^2\varphi$$

$$-g'(x) = \frac{1}{2\pi}\int_0^{2\pi} i e^{-im\varphi} e^{ix\operatorname{Sen}\varphi}\operatorname{Sen}\varphi$$

$$u = e^{i(x\operatorname{Sen}\varphi - \varphi m)} \quad ; \; dv = \operatorname{Sen}\varphi$$

$$du = i(x\cos\varphi - im)e^{i(x\operatorname{Sen}\varphi - im)} \quad ; \; v = -\cos\varphi$$

$$g'(x) = \frac{1}{2\pi}\left[e^{i(x\operatorname{Sen}\varphi - im)}\cancel{(-\cos\varphi)}\right]_0^{2\pi} + \int_0^{2\pi} i\cos\varphi\,(x\cos\varphi - im)\, e^{i(x\operatorname{Sen}\varphi - im)}$$

$$g'(x) = \int_0^{2\pi} -i\cos\varphi\,(x\cos\varphi - im)e^{i(x\operatorname{Sen}\varphi - im)} \quad \frac{1}{2\pi}$$

$$-\frac{x^2}{2\pi}\int e^{-im\varphi}\cancel{e^{ix\operatorname{Sen}\varphi}}\operatorname{Sen}^2\varphi - \int \cancel{\cos^2}(\cancel{1} - \cancel{\operatorname{Sen}^2\varphi}\, x^2 - mx\cos\varphi)e^{-im\varphi + ix\operatorname{Sen}\varphi}\, d\varphi$$

$$+\frac{x^2}{2\pi}\int \cancel{e^{-im\varphi}} e^{ix\operatorname{Sen}\varphi} - \frac{m^2}{2\pi}\int e^{-im\varphi} e^{ix\operatorname{Sen}\varphi}\, d\varphi = 0$$

$$\frac{1}{2\pi}\int^{2\pi} mx\cos\varphi\, e^{im\varphi + ix\operatorname{Sen}\varphi}\, d\varphi = \frac{m^2}{2\pi}\int^{2\pi} e^{-im\varphi} e^{ix\operatorname{Sen}\varphi}\, d\varphi$$

$$\int^{a} r\,dr\left[J_m(K_{nm} r)\right]^2$$

$$\int_0^a \left[J_m(x)\right]^2 x\,dx = \frac{1}{2}a^2\left[J_m'(a)\right]^2 + \frac{1}{2}(a^2-m^2)\left[J_m(a)\right]^2$$

usando la demostración anterior

$$\int_0^a \left[J_m(x)\right]^2 x\,dx = \int_0^a \left[(x^2 J_m'(x))^2 + (x^2-m^2)\,J_m^2(x)\right]' dx\,\frac{1}{2} =$$

$$= \left[(x^2 J_m'(x))^2 + (x^2-m^2)\,J_m^2(x)\right]\Big|_0^a = \frac{1}{2}\left(a^2 J_m'^2(a) + (a^2-m^2)\,J_m^2(a)\right)$$

$J_m(0) = J_m'(0) = 0$

Dirichlet

$$b_{nm} = \frac{1}{2}a^2\left[J_{m+1}(x_{nm})\right]^2 \qquad (J_m(x_{nm})) = 0$$

$$b_{nm} = \int_0^a r\,dr\left[J_m(K_{nm} r)\right]^2 \qquad k^2 a^2 = x$$

$J_m(x) = 0$

$dr = K\,dx$

$$b_{nm} = \int_0^a \frac{x\,dx}{K_{nm}^2}\left[J_m(x_{nm})\right]^2 = \frac{1}{2}a^2\left[J_m'(x)\right]^2 + \frac{1}{2}(a^2-m^2)\left[\cancel{J_m(0)}\right]^2$$

$$J_m'(x_{nm})^2 = -J_{m+1} + \frac{m}{x}\cancel{J_m(x)} = \left[J_{m+1}(x_{nm})\right]$$

Neumann

$$b_{nm} = \frac{1}{2}a^2\left(1-\frac{m^2}{x_{nm}^2}\right)\left[J_m(x_{nm})\right]^2 \qquad J_m'(x_{nm}) = 0$$

$$\int_0^a \frac{x_{nm}\,dt}{K_{nm}^2} = \frac{1}{2}a^2\left[\cancel{J_m'(x_{nm})}\right]^2 + \frac{1}{2K_{nm}^2}(x_{nm}^2-m^2)\left[J_m(x_{nm})\right]^2 =$$

$$= \frac{a^2}{2}\left(1-\frac{m^2}{\ldots}\right)\left[J_m(x_{nm})\right]^2$$

$$\int_0^a [J_m(x)]^2 x\,dx = \frac{1}{2}a^2[J_m'(a)]^2 + \frac{1}{2}(a^2-m^2)[J_m(a)]^2$$

usando la demostración anterior

$$\int_0^a [J_m(x)]^2 x\,dx = \int_0^a \left[(x^2 J_m'(x))^2 + (x^2-m^2) J_m^2(x)\right]' dx\,\frac{1}{2} =$$

$$= \left[(x^2 J_m'(x))^2 + (x^2-m^2) J_m^2(x)\right]\Big|_0^a = \frac{1}{2}\left(a^2 J_m'^2(a) + (a^2-m^2) J_m^2(a)\right)$$

$J_m(0) = J_m'(0) = 0$

Dirichlet

$$b_{nm} = \frac{1}{2}a^2\left[J_{m+1}(x_{nm})\right]^2 \quad (J_m(x_{nm}) = 0)$$

$$b_{nm} = \int_0^a r\,dr\left[J_m(k_{nm} r)\right]^2 \qquad k a = x$$

$J_m(x) = 0$

$dr = k\,dx$

$$b_{nm} = \int_0^a \frac{x\,dx}{k_{nm}^2}\left[J_m(x_{nm})\right]^2 = \frac{1}{2}a^2\left[J_m'(\cancel{x})\right]^2 + \frac{1}{2}(a^2-m^2)\left[\cancel{J_m}(0)\right]^2$$

$$J_m'(x_{nm})^2 = -J_{m+1} + \frac{m}{x}\cancel{J_m}(x) = \left[J_{m+1}(x_{nm})\right]$$

Newmann

$$b_{nm} = \frac{1}{2}a^2\left(1-\frac{m^2}{x_{nm}^2}\right)\left[J_m(x_{nm})\right]^2 \qquad J_m'(x_{nm}) = 0$$

$$\int_0^a \frac{x_{nm}\,dt}{k_{nm}^2} = \frac{1}{2}a^2\left[\cancel{J_m'}(x_{nm})\right]^2 + \frac{1}{2k_{nm}^2}(x_{nm}^2-m^2)\left[J_m(x_{nm})\right]^2 =$$

$$= \frac{a^2}{2}\left(1-\frac{m^2}{\ldots}\right)\left[J_m(x_{nm})\right]^2$$

②

$$\left[(xy')^2 + (x^2 - m^2) y^2 \right]' = 2xy^2$$

$$-[xy'] + \frac{m^2}{x} y' = xy \qquad \text{Ec. Bessel SL}$$

$$\cancel{-(xy')^2 + \frac{m^2}{x} x y y' = (xy')^2 \quad y = \frac{2x}{m^2}}$$

$$\cancel{m^2 y y'^2 = 2(xy')^2}$$

$$-[xy'][xy'] + m^2 y (y') = y y' x^2$$

$$\int (xy')^2 \rightarrow \begin{matrix} xy' = u \\ d[xy'] = du \end{matrix} \qquad \int u\,du = \frac{u^2}{2} = \frac{(xy')^2}{2}$$

$$\int m^2 y (y') \rightarrow \begin{matrix} y = u \\ y' = du \end{matrix} \qquad \int u\,du = \frac{u^2}{2} = \frac{(y)^2}{2} m^2$$

$$\int y y' x^2 \qquad \begin{matrix} u = x^2 & du = 2x \\ dv = y y' & v = \frac{y^2}{2} \end{matrix} \rightarrow \frac{x^2 y^2}{2} - \int y^2 x$$

$$\frac{1}{2}\left[(xy')^2 + (x^2 - m^2) y^2 \right] = \int y^2 x$$

derivando:

$$\boxed{\left[(xy')^2 + (x^2 - m^2) y^2 \right]' = 2y^2 x}$$

②

$$\left[(xy')^2 + (x^2 - m^2)y^2 \right]' = 2xy^2$$

$$-[xy'] + \frac{m^2}{x} y' = xy \quad \text{Ec. Bessel SL}$$

$$\cancel{-(xy')^2 + \frac{m^2}{x} xy y' = (xy')^2 \quad y = \frac{2x}{m^2}}$$

$$\cancel{m^2 y y'^2 = 2(xy')^2}$$

$$-[xy'][xy'] + m^2 y(y') = yy'x^2$$

$$\int (xy')' \rightarrow \begin{matrix} xy' = u \\ d[xy'] = du \end{matrix} \qquad \int u\,du = \frac{u^2}{2} = \frac{(xy')^2}{2}$$

$$\int m^2 y(y') \rightarrow \begin{matrix} y = u \\ y' = du \end{matrix} \qquad \int u\,du = \frac{u^2}{2} = \frac{(y)^2}{2} m^2$$

$$\int yy'x^2 \qquad \begin{matrix} u = x^2 & du = 2x \\ dv = yy' & v = \frac{y^2}{2} \end{matrix} \rightarrow \frac{x^2y^2}{2} - \int y^2 x$$

$$\frac{1}{2}\left[(xy')^2 + (x^2 - m^2)\, y^2 \right] = \int y^2 x$$

derivando:

$$\boxed{\left[(xy')^2 + (x^2 - m^2)\, y^2 \right]' = 2y^2 x}$$

$$= \int_0 m^2 \left(e^{ix \operatorname{Sen}\varphi - im\varphi}\right) d\varphi$$

(4)

$$\underbrace{e^{ikr \operatorname{Cos}\varphi}} = \sum_m i^m J_m(kr) e^{im\varphi}$$

$$\operatorname{Cos}\varphi + \pi/2 = \operatorname{Sen}\varphi + \pi/2 \qquad \varphi \pm \pi/2 = \theta$$

$$e^{ix \operatorname{Sen}\theta} = \sum_m i^m J_m(kr) e^{im\theta} = \sum_m J_m(kr) e^{im\varphi + \frac{im\pi}{2}} =$$

$$= \boxed{\sum_m J_m(kr) e^{im\varphi} i^m} \qquad i^m = e^{im\pi/2}$$