Types of medium

- Homogeneous medium : properties described are always the same
- Heterogeneous medium : properties vary w/ location of sampling.

CONTINUUM APROACH : In modeling media flow, it's important to considerate different length scales.
↳ valid if the mean free path lenght of the fluid molec. is much smaller than the physical domain of interest.

(a) macroscopic (inhomogeneity) (b) microscopic (c) molecular

Fluid prop. like viscosity, density, diffusion coeff. & miscibility are determined on the molecular scale by the individual prop. of the molecules.

(b) The flow of a single newtonian fluid in the void space of a porous medium is described on the microscopic level by the Navier-Stokes system of equations with appropiate boundary conditions.

The Creeping Flow interface approximates the Navier-Stokes equations for the case when the Reynolds number is significantly less than 1. This is often called Stokes flow and it is appropiate for use when viscous flow is dominant.

(a) In order to derive a mathematical model on the macroscopic level another continuum is considered.
Each point in the continuum on the macroscopic level is assigned average values over elementary volumes of quantities on the microscopic levels.

"Representative elementary volume": the averaging process used for passing from microscopic to macroscopic level is illustrated for the porosity, a simple geometric property of the porous medium. $\boxed{1 << diam(\Omega_0(x_0)) << L}$

The porous medium is supposed to fill the domain Ω w/ volume meas (Ω)
Let $\Omega_0(x_0)$ belongs to Ω be a subdomain of Ω centered at the point x_0 on the macroscopic level.

Further we define the void space indicator function on the microscopic level:

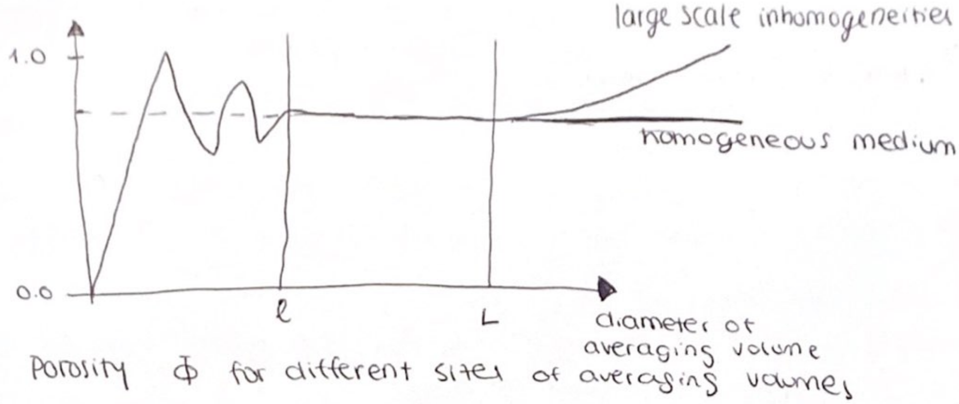
$$\gamma(x) = \begin{cases} 1 & x \in \text{void space} \\ 0 & x \in \text{solid matrix} \end{cases} \quad \forall x \in \Omega$$

the porosity is given at x_0

$$\phi(x_0) = \frac{1}{meas(\Omega_0(x_0))} \int_{\Omega_0(x_0)} \gamma(x)\, dx$$

The macroscopic quantity porosity is obtained by averaging over the microscopic void space indicator function.

Plotting the value of $\Omega(x_0)$ at a fixed position x_0 for different diameters d of the averaging volume $\Omega_0(x_0)$.



Porosity Φ for different sizes of averaging volumes

POROUS MEDIA

- SINGLE - PHASE FLUID FLOW AND TRANSPORT
  ↳ one fluid

  consider macroscopic eqs for flow & transport in porous media when the void space is filled w/ a single fluid

- FLUID MASS CONSERVATION

  Suppose that the porous medium fills the domain $\Omega \subset \mathbb{R}^3$ then the macroscopic fluid mass conservation is expressed by the partial differential equation is given by:

$$\frac{\partial(\Phi \rho)}{\partial t} + \nabla \cdot \{\rho \cdot u\} = \rho \cdot q \quad \text{in } \Omega.$$

$\Phi(x)$: Porosity

$\rho(x;t)$: Density of the fluid (kg/m3)
↳ either constant if incompressible or assumed to be ideal gas : $p = \rho \cdot RT$.

$u(x;t)$: Macroscopic apparent velocity (m/s)

$q(x;t)$: Specific source/sink term with dimensions (s^{-1})

- DARCY'S LAW (1856)

  It can be shown that under appropiate assumptions the momentum conservation of the Navier-Stokes equation reduces to the Darcy Law on the macroscopic level which is given by:

$$u = -\frac{K}{\mu}(\nabla p - \rho g)$$

$p(x;t)$: Fluid pressure in [Pa] = [N/m²] this will be the unknow function

g : gravity vector pointing in the direction of gravity with size g [m/s²]

Inserting equation (18) in (17) yields a single eq for fluid pressure.

$$(19) \qquad \frac{\partial(\phi\rho)}{\partial t} - \nabla\cdot\left\{\rho\cdot\frac{K}{\mu}(\nabla p - \rho\cdot g)\right\} = \rho\cdot g \quad \text{in } \Omega.$$

with initial boundary conditions :

$$p(x,0) = p_0(x) \quad \text{in } \Omega$$

- $p(x,t) = p_d(x,t)$ on I_d : DIRICHLET BOUNDARY

$$\frac{\partial^2 f}{\partial x_i} + x_i = 0 \quad ; [0,1] \text{ domain}, \quad \begin{array}{l} f(0) = \alpha_1 \text{ Numeric value} \\ f(1) = \alpha_2 \text{ Numeric value} \end{array}$$

- $\rho\cdot u\cdot n = \phi(x,t)$ or I_n : NEUMANN BOUNDARY

$$\frac{\partial f}{\partial x_i}(0) = \alpha_1 \quad ; \quad \frac{\partial f}{\partial x_i}(1) = \alpha_2$$

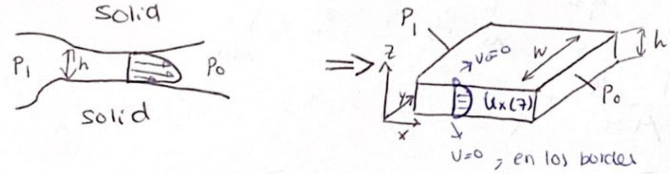
In case of a compressible fluid (eq. 19) $\Rightarrow$ Parabolic type
of an incompressible fluid $\Rightarrow$ elliptic type $\Rightarrow$ (initial cond. not necessary)

- MICROSCOPIC CONSIDERATIONS OF MULTIPHASE SYSTEMS

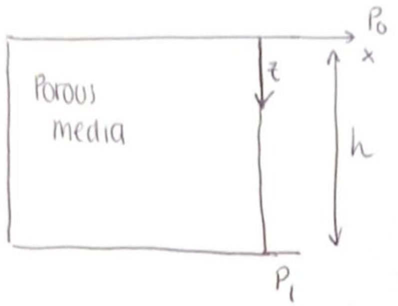
Single-phase flow is governed by pressure forces arising from pressure differences within the reservoir & the exterior gravitational force. In multiphase flows the sharp interfaces between fluid phases on the microscopic level give rise to a capillary force that plays an important role in this flows.

* NOTE :
- Darcy's law is valid for slow flow of Newtonian fluids through a porous medium with rigid solid matrix.
- No slip boundary conditions are assumed at the fluid-solid boundary on the microscopic level.

- VISCOUS FLOW BETWEEN PARALLEL PLATES



Problema de porous media



Assumptions

1) Incompressible
2) Steady state
3) Constant properties (μ and k constant)

Solution:

$$\frac{\partial \cancel{\phi \rho}}{\partial t} - \nabla \left(\frac{k \cancel{\rho}}{\cancel{\mu}} \left(\nabla P - \rho \cdot \cancel{g}\right)\right) = \cancel{\rho} q$$

Since it's incompressible, ρ is out of the eq. $\rho = 0$
therefore

$$-\nabla \cdot \frac{k}{\mu} \left(\nabla P - \rho \cdot g\right) = q$$

$$-\frac{d}{dz} \cdot \frac{k}{\mu} \left(\frac{dP}{dz} - \rho \cdot g\right) = q$$

$$\frac{k}{\mu} \left(\frac{dP}{dz} - \rho \cdot g\right) = -qz + C_1$$

$$\frac{dP}{dz} - \rho \cdot g = (-qz + C_1)\frac{\mu}{k} \; ; \; \frac{dP}{dz} = (-qz + C_1)\frac{\mu}{k} + \rho g$$

$$\left[P(z) = -q\frac{z^2}{2} \cdot \frac{\mu}{k} + C_1 \cdot z \cdot \frac{\mu}{k} + \rho \cdot g \cdot z + C_2 \right]$$

Boundary conditions

$Z(0) = P_0$; $Z(h) = P_i$

$Z(0) = P_0 = C_2$

$$Z(h) = P_i = -\frac{q}{2} h^2 \frac{\mu}{k} + C_1 \cdot h \cdot \frac{\mu}{k} + \rho \cdot g \cdot h + P_0$$

$$C_1 = \left(\frac{\cancel{k}}{\cancel{\mu}} \cdot \frac{q}{2} \cdot \frac{h^2}{h} \cdot \frac{\cancel{\mu}}{\cancel{k}}\right) - \left(\frac{k}{\mu} \frac{\rho g h}{h}\right) - \left(\frac{P_0 k}{h \mu} - \frac{P_i k}{h \mu}\right)$$

$$C_1 = \frac{qh}{2} - \frac{k}{\mu} \rho g + (P_1 - P_0)\frac{k}{\mu h}$$

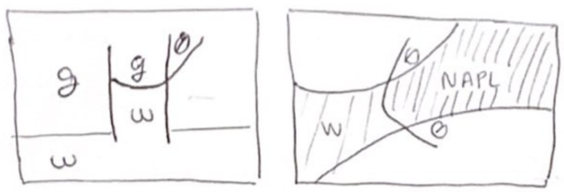
$$P(z) = \frac{-q \cdot z^2 \mu}{2 \cdot k} + \left(\frac{qh}{2} - \frac{k \cancel{\rho g}}{\cancel{\mu}} + (P_i - P_0)\frac{k}{\mu h}\right) \frac{z \mu}{k} + \cancel{\rho g z} + P_0$$

Pressure distribution:

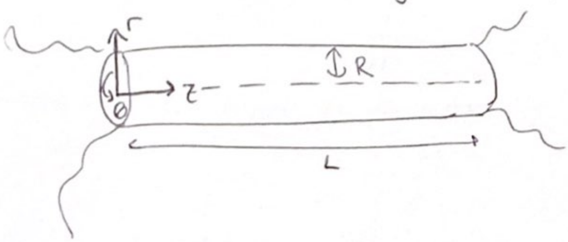
$$\boxed{P(z) = \frac{-q \cdot z^2 \mu}{2 k} + \frac{q \cdot h \cdot z \mu}{2 k} + (P_i - P_0)\frac{z}{h} + P_0}$$

- CAPILLARITY

Next figure shows the interface between two phases in more detail. On the molecular level adhesive forces are attracting fluid molecules to the solid and cohesive forces are attracting molecules of one fluid to each other.
At the fluid - fluid interface these forces are not balanced leading to the curved form of the interface.



★ Viscous flow through a capillarity



In cylindrical coordinates, the stokes momentum equation becomes

$$\mu_B \cdot \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \cdot \frac{\partial V_z}{\partial r} \right) = \frac{\Delta P}{L}$$

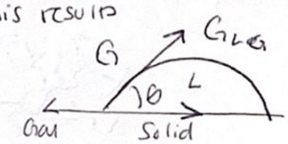
- WETTABILITY

The magnitude of the adhesive forces is decreasing rapidly with distance to the wall. The interaction w/ the cohesive forces leads to a specific contact angle θ between the solid surface & the fluid - fluid interface that depends on the properties of the fluid.

- SURFACE TENSION

The cohesive forces are not balanced at a fluid - fluid interface. Molecules of the wetting phase fluid at the interface experience a net attraction towards the interior of the wetting phase fluid body. This results in the curved form of the interface.

$$\sigma = \frac{\Delta W}{\Delta A} \quad \left[\frac{J}{m^2} \right] \left\{ \frac{N \cdot m}{m^2} = \frac{N}{M} \right\}$$



CAPILLARITY PRESSURE

In order to derive a relation, it considers a tube with radius diameter $2R$ that is filled w/ a wetting phase and a non-wetting phase.

$R = r \cdot \cos\theta$. The work required to increase the area of the interface is given by:

$$\Delta W = \sigma \cdot \Delta A = \sigma \left(A(r+dr) - A(r) \right)$$
$$\Delta W = r \left(1/2 - \theta/\pi \right) 8\pi r \, dr + O(dr^2)).$$

This work is done by capillarity pressure which is assumed to be uniform over the entire interface: $\Delta W = F \cdot dr = p_c \cdot A(r) \cdot dr = p_c \left(1/2 - \frac{\theta}{\pi} \right) 4\pi r^2 dr$

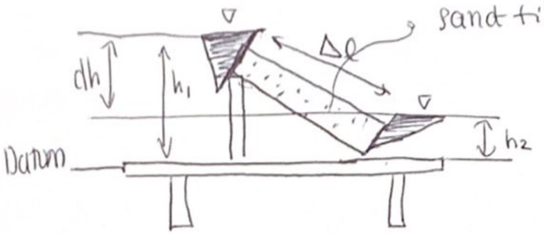
Equating these 2 expressions → $\boxed{p_c = \frac{2\sigma \cdot \cos\theta}{R}}$ R = ~~radius~~ parameter of porous medium.

According to $P_c = \frac{2\sigma \cdot \cos\theta}{R}$, capillary pressure increases with decreasing pore size diameter.

Similar arguments relate capillary pressure (Laplace's eq^n)

$$P_c = \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

DARCY'S LAW



* constant head tanks at each end

sand-filled column with bulk cross-sectional area A

Q = Rate of discharge $[L^3/T]$

* water moves d/o TO gravity

For a particular sand : $Q = -K \cdot A \cdot \frac{dh}{dl}$

if we invert it we express conductivity in terms of discharge, area & gradient

$$K = \frac{Q}{A}\left[\frac{-1}{dh/dl}\right]$$

we will look at 3 major topics important to Darcy's Law:

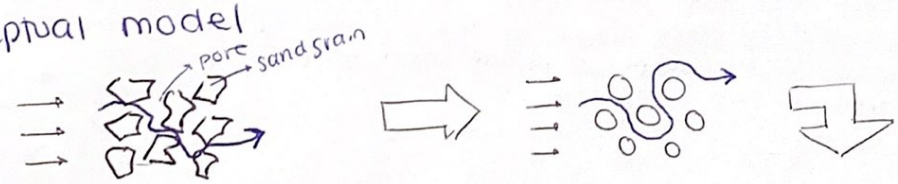
$Q = K A \cdot \frac{dh}{dl}$ → Hydraulic Head Gradient

Bulk cross-sectional area of flow

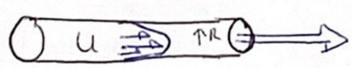
Hydraulic conductivity (next time)

class explanation: $P = \rho \cdot g h$; $\frac{P}{g \cdot \rho} = h$ (m)

Conceptual model



There are two standart, simple models used to explain Darcy's Law, and thus to explore the Reynolds number:

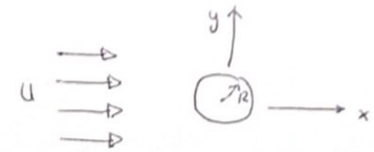
→ FLOW in a tube  mean flow

→ FLOW around and object (cylinder, sphere)  mean flow

+ Darcy's law

- FLOW in the vicinity of a sphere

- Inertial force per unit volume at any location : Rate of change of momentum
- viscous force per unit volume.
- Dynamic similarity



ρ = density $[M/L^3]$
u = local fluid velocity $[L/T]$
U = mean approach velocity $[L/T]$

Using a dimensional analysis approach, assume that functions vary with characteristic quantities U and R, thus

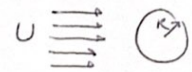
$u \propto$ $\quad \frac{\partial^2 u}{\partial y^2} \propto$

$\frac{\partial u}{\partial x} \propto$

μ = fluid dynamic viscosity $[M/LT]$
ν = fluid kinematic viscosity $[L^2/T]$
x, y = cartesian coordinates $[L]$
x in the direction of free stream velocity, U.

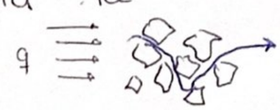
x For a fluid flow past a sphere : Re

$$Re = \frac{\rho \cdot U R}{\mu} = \frac{UR}{\nu}$$



x For a flow in porous media : Re

$$Re = \rho q L / \mu$$



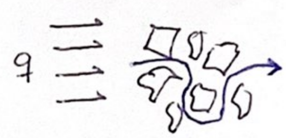
where
q ≡ specific discharge
L ≡ characteristic pore dimension
* Sand : $L = d_{50}$

When does Darcy's law apply in a porous media?

- For $Re < 1$ to 10
  - Flow is laminar and linear
  - Darcy's law applies

$$Re = \frac{\rho \cdot q\, d_{50}}{\mu} = \frac{q\, d_{50}}{\nu}$$

- For $Re \geq 1$ to 10
  - Flow is still laminar but not linear
  - Inertial forces becoming important
  - Linear Darcy's Law no longer applies



$$Re = \frac{\rho \cdot q\, d_{50}}{\mu} = \frac{q\, d_{50}}{\nu}$$

* Notice that Darcy's Law start to fail when inertial effects are important, even through the flow is still laminar.

Specific Discharge, q $\qquad Q = -KA \cdot \frac{dh}{dl}$

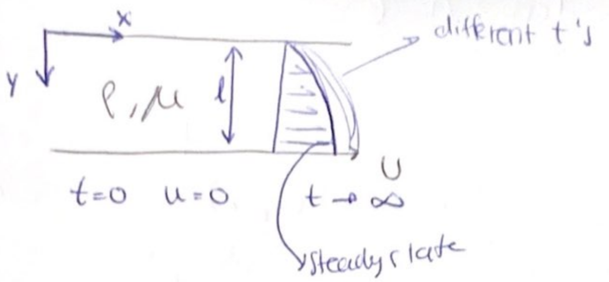
Supose we want to know water "velocity". Divide Q by A to get volumetric flux density (specific discharge), often called the Darcy velocity.

$$\frac{Q}{A} = \frac{L^3 T^{-1}}{L^2} = \frac{L}{T} \longrightarrow \text{units of velocity}$$

$$\frac{Q}{A} = \boxed{q = -K \cdot \frac{dh}{dl}}$$

By definition, this is the discharge per unit bulk cross-sectional area.

HOWK : Problem # 3



Assumptions

- Incompressible
- 1D flow
- Pressure gradient is negligible
- No gravity

Continuity eq : $\frac{\partial u}{\partial x} = 0$ $u(y,t)$

Momentum eq : $\frac{\partial u}{\partial t} = \nu \cdot \frac{\partial^2 u}{\partial y^2}$

Supersition principle :

$$\frac{0}{u(y,t)} \xrightarrow{} u \quad = \quad \frac{0}{U_y(y)} \xrightarrow{} u \quad + \quad \frac{0}{U_t(y,t)} \; \frac{}{0}$$

$$u(y,t) = U_y + U_t$$

Boundary conditions $\longrightarrow$ $\begin{cases} u(0,t) = 0 \\ u(-l,t) = U \\ u(y,0) = 0 \end{cases}$

U_y the boundary conditions (steady state)

$U_y(0) = 0$

$U_y(-1) = U$

$$\boxed{U_y = -\frac{U}{l} y}$$

$U_t(y,t)$ BC's

$u(t,0) = \cancel{U_y(0)}^{0} + U_t(0,t) = 0$; $U_t(0,t) = 0$

$u(-l,t) = U = \underbrace{U_y(-l)}_{"U} + U_t(-l,t)$; $U_t(-l,t) = 0$

Initial condition :

$u(y,0) = U_y(y) + U_t(y,0) = 0$; $\boxed{U_t(y,0) = -U_y(y)}$

Solution for U_t can be : $U_t(y,t) = \Gamma_{(t)} \cdot Y_{(y)}$

$\downarrow$

$\frac{\partial U_t}{\partial t} = \dot{\Gamma} \cdot Y$

$\frac{\partial^2 U_t}{\partial y^2} = \Gamma \cdot Y''$

$\frac{1}{\nu} \cdot \frac{\partial U_t}{\partial t} = \frac{\partial^2 U_t}{\partial y^2}$ $\rightarrow$ $\frac{1}{\nu} \cdot Y \cdot \dot{\Gamma} = \Gamma \cdot Y''$

$$\frac{\dot{\Gamma}}{\nu \cdot \Gamma} = \frac{Y''}{Y} = \frac{\lambda^2}{l^2}$$

$\searrow$ 18

$$\begin{cases} \dot{\Gamma}_{(t)} - \frac{\lambda^2}{\ell^2} \nu \Gamma_{(t)} = 0 \\ Y''_{(y)} - \frac{\lambda^2}{\ell^2} \cdot Y_{(y)} = 0 \end{cases} \Rightarrow \Gamma_{(t)} = A \cdot e^{\left(\overset{\swarrow \text{negativo}}{-}\frac{\lambda^2}{\ell^2} \cdot \nu \cdot t\right)}$$

$$\Rightarrow \left[B \cdot e^{-\zeta_1 y} + C \cdot e^{+\zeta_2 y}\right]$$

$(y'' + ay' + b \cdot y)$

resolución de una función

$$\zeta^2 = a\zeta + b = 0$$

$$\zeta_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} = +\frac{\lambda}{\ell}$$

$$\zeta_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} = -\frac{\lambda}{\ell}$$

$$Y(y) = B \cdot \cos\left(\frac{\lambda}{\ell} y\right) + C \cdot \sin\left(\frac{\lambda}{\ell} y\right)$$

→ esto es repaso de ecuaciones diferenciales!!

Analizamos:

$$\Gamma_{(t)} = A \cdot e^{\left(\frac{\lambda^2}{\ell^2} \nu t\right)}$$

$\lambda^2 < 0$ ☑

$\lambda^2 = 0$ ✗ → no son válidos para

$\lambda^2 > 0$ ✗ → este problema.

así es como descubrimos que λ es negativo.

entonces

$$\zeta^2 = a\zeta + b = 0$$

complejo p/q es negativo.

$$\zeta_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} = +\frac{\lambda}{\ell} i$$

$$\zeta_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} = -\frac{\lambda}{\ell} i$$

$$Y(y) = B \cdot \cos\left(\frac{\lambda^2 y}{\ell^2}\right) + C \cdot \sin\left(\frac{\lambda^2 y}{\ell^2}\right)$$

Volvemos a que:

$$\left[\underbrace{U(y,t) = -\frac{U}{\ell} y}_{\text{esta es la solucion steady}} + A e^{\left(-\frac{\lambda^2 \cdot \nu \cdot t}{\ell^2}\right)} \left[B \cdot \cos\left(\frac{\lambda^2 y}{\ell^2}\right) + C \cdot \sin\left(\frac{\lambda^2}{\ell^2} \cdot y\right)\right]\right]$$

Bound

$y = 0 \quad u = 0$

$$A e^{-\frac{\lambda^2 \nu t}{\ell^2}} \left(B \cdot \cos(0)(0) + C \cdot \sin(0)\right) = 0$$

$B = 0$

$$A e^{-\frac{\lambda^2 \nu t}{\ell^2}} \left(C \cdot \sin\left(\frac{\lambda^2 y}{\ell^2}\right)\right)$$

$u_t(-\ell, t) = 0 \quad y = -\ell \quad u = 0$

$$e^{-\frac{\lambda^2 \nu t}{\ell^2}} \left[A_1 \cdot \sin\left(\frac{\lambda^2 \cancel{(-\ell)}}{\cancel{\ell}}\right)\right] = 0 \Rightarrow \boxed{u_t(y,t) = e^{-\frac{\lambda^2 \nu t}{\ell^2}} \cdot A_n \cdot \sin\left(\frac{n\pi y}{\ell}\right)}$$

$A_1 = A \cdot C$

cuando $\lambda = \cancel{0}; \pi; 2\pi; 3\pi \ldots$

$\boxed{\lambda_n = n\pi;\ n > 1}$

Initial conditions:

$$U_t(y,0) = A_n \cdot \sin\left(\frac{n\pi y}{\ell}\right) = U_y(y) = -\frac{Uy}{\ell}$$

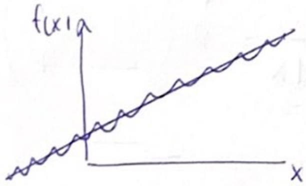
entonces, al igualarlas:

$$\boxed{A_n \cdot \sin\left(\frac{n\pi \cdot y}{\ell}\right) = -\frac{Uy}{\ell}}$$

Esto es una serie de Fourier, con la forma: $f(x) = A_0 + \sum A_n \cos x + \sum B_n \sin \theta$
en nuestro caso $A_0 = 0$ y $\sum A_n \cos x = 0$, así que tiene la forma:

$$f(x) = \sum B_n \cdot \sin(x)$$

Para poder aproximarla, lo hacemos a través de una sinosoidal.



ó en un pulso

Para hallar los coeficientes de la serie:

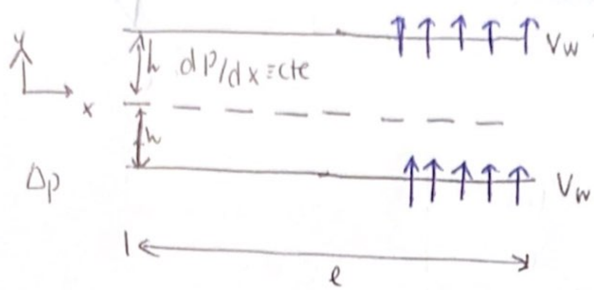
$$A_n = \frac{-2}{\ell}\int_0^\ell \frac{U}{\ell} y \cdot \sin\left(\frac{n\pi y}{\ell}\right) dy$$

$$U(y,t) = \frac{-U}{\ell} y - \frac{2U}{\pi} \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{\ell}\right)^2 \cdot t} \cdot \frac{(-1)^n}{n\pi} \cdot \sin\left(\frac{n\pi y}{\ell}\right)$$

Example (POROUS MEDIA)

$\Delta P = P_2 - P_1$

Se mueve x Δp (dif. de presiones)



Vw Insertamos un fluido por medio poroso

Assumptions

1) Steady state
2) 1D
3) No gravity
4) Incompressible
5) V_w constant
6) No heat transfer

Solution

Continuity eq:

$$\frac{\partial u}{\partial x} + \underbrace{\frac{\partial v}{\partial y}}_{=0} = 0 \quad ; \quad \frac{\partial u}{\partial x} = f(y)$$

Momentum equation:

x-component:

$$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0} + u \cdot \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0} + \overset{V_w}{v} \cdot \frac{\partial u}{\partial y} = -\frac{\partial P}{\rho \cdot dx} + \overset{=\mu/\rho}{\nu} \cdot \left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial^2 u}{\partial y^2} - \frac{V_w}{\nu} \cdot \frac{du}{dy} - \frac{1}{\mu} \cdot \frac{dP}{dx} = 0$$

$$(\lambda^2 + a\lambda + b = 0)$$

$$\left[u(y) = C_1 \cdot e^{\lambda_1 y} + C_2 e^{\lambda_2 y} \right]$$

donde

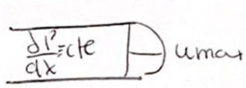
$$\lambda_{1,2} = \frac{V_w}{\nu} = \pm \sqrt{\frac{V_w^2}{\nu^2} - 4 \cdot \frac{1}{\mu} \cdot \frac{\partial P}{\partial x}}$$

Boundary conditions

$y(h)$: $u(h) = 0$
$y(-h)$: $u(-h) = 0$

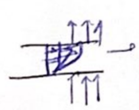
$$\left[u(y) = -\frac{h^2}{2\mu} \cdot \frac{dP}{dx} \left[\left(\frac{y}{h} - 1 \right) + \frac{e^{Re} - e^{\frac{Re\, y}{h}}}{\sinh(Re)} \right] \right]$$

SOLUCIÓN EJ. MEDIO POROSO.



} un sist sin medio poroso

en el caso de medio poroso $Re = \frac{\rho \cdot V_w \cdot h}{\mu}$ y tendría una forma distinta

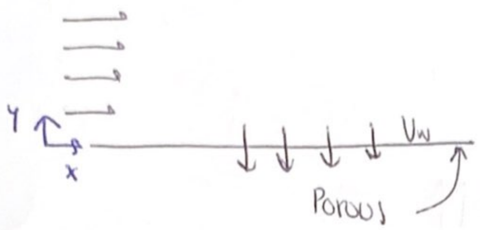


→ depende ..

* el profe está poniendo ejemplos en excel de como se graficarían

Ejemplo: Tenemos una placa porosa que está succionando

No hay ΔP



$$\frac{\partial^2 u}{\partial y^2} - \frac{V_w}{\nu} \cdot \frac{du}{dy} = 0$$

$y = 0 \; : \; u(0) = 0$

$y = \infty \; : \; u(\infty) = U_\infty$

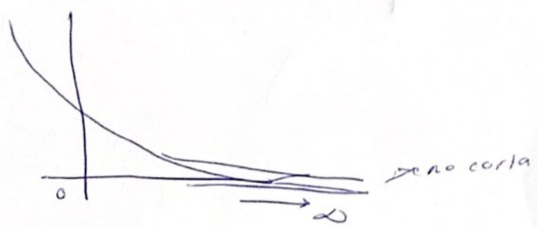
donde $\lambda^2 + a\lambda + b = 0$ donde $\begin{cases} \lambda^2 = \frac{\partial^2 u}{\partial y^2} \\ a\lambda = -\frac{V_w}{\nu} \cdot \frac{du}{dy} \end{cases}$

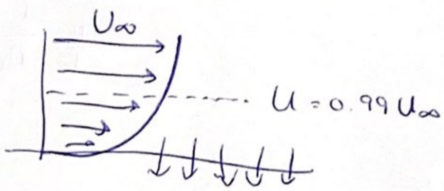
Boundary conditions:

$$u(y) = U_\infty \left(1 - e^{\frac{V_w}{\nu} \cdot y}\right)$$

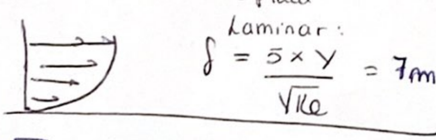
$$u(y) = 0.99 \cdot U_\infty$$

$$y = \boxed{\delta = -4.6 \frac{\nu}{V_w}}$$





* En el caso de un medio No poroso (placa)



Laminar:

$$\delta = \frac{5 \times Y}{\sqrt{Re}} = 7\,mm$$

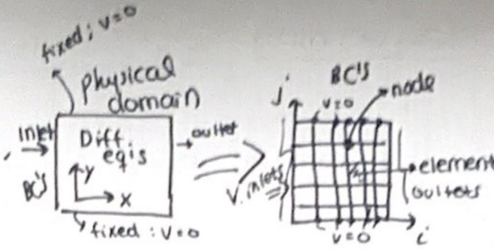
$V_w = -1$ cm/s $\quad T = 20°C$

$$\boxed{\delta = 0.0149}$$

[↓ Thickness de la capa límite ↓ viscosidad]

* El tema este de succión (medios porosos) es pq tiene que ver con capas límite y entrada en perdida, por eso la estudiamos.

COMPUTATION FLUID DYNAMICS



In Fluids I we had a physical domain, that had boundary conditions and was Solved by differential equations.

In CFD instead of solving the diff. eq's in an analytical way, we discretitide the domain to solve the diff. eq's in the nodes.

The other way of solving for fluids is through experiments.

We can use 3 different methods to solve for fluid dynamics.

Analytical / CFD / Experimental

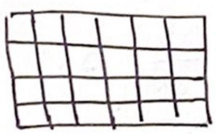
we can also combine them.

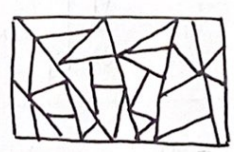
CFD can obtain closer to reality (experiment) solution than analytically, and it's cheaper than the experimental way.

* **Mesh** : Is the combination of the cells used to conduct an analysis, that come from splitting the solution domain into multiple sub-domains (called cells) in the computational structure.

Two types of Mesh :

Structured Mesh or Unstructured Mesh



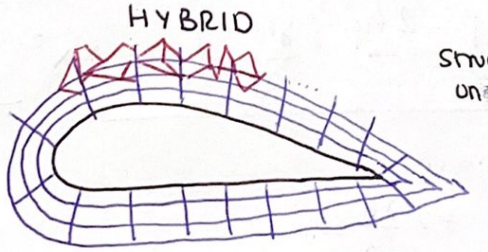


A structured grid has two possible element choices : 2D - Quadrilateral & 3D - hexahedra . This model is highly space efficient since the neighborhood relationships are defined by storage arrangement

Is identified by irregular connectivity, it cannot be easily expressed as a 2D or 3D array in computer memory.

Combined

HYBRID



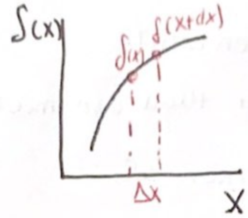
It integrates the structured meshes & the unstructured ones in an efficient manner.

The parts of the geometry that are:
regular (structured)
complex (unstructured)

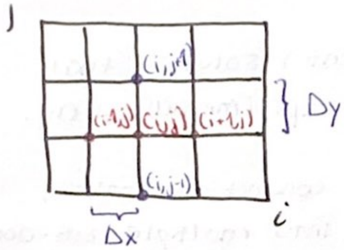
Governing eq's in CFD

- **Continuity eq's** : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (Incompressible)

- **Momentum eq** : $\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} = \frac{-\partial P}{\rho \cdot \partial x} + g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$



Taylor series expansion



× Taylor series expansion (forward)

$$f_{i+1,j} = f_{ij} + \frac{\partial f}{\partial x}\Big|_{ij} \Delta x + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\Big|_{ij} \Delta x^2 + \frac{1}{6} \frac{\partial^3 f}{\partial x^3}\Big|_{ij} \Delta x^3 =$$

$$\underbrace{\boxed{\frac{f_{i+1,j} - f_{ij}}{\Delta x} = \frac{\partial f}{\partial x}}}_{\text{this is my first derivative.}} + \underbrace{\boxed{\overset{\text{order}}{(\Delta x)}}}_{\text{first order approximation.}}$$

× Taylor series expansion (backwards)

$$f_{i-1,j} = f_{ij} - \frac{\partial f}{\partial x}\Big|_{ij} \Delta x + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\Big|_{ij} \Delta x^2 - \frac{1}{6} \frac{\partial^3 f}{\partial x^3}\Big|_{ij} \Delta x^3$$

$$\boxed{\frac{f_{ij} - f_{i-1,j}}{\Delta x} = \frac{\partial f}{\partial x}\Big|_{ij}} \; O(\Delta x)$$ first order accuracy.

× Second derivative. (forward + backward) approximation

$$f_{i+1,j} + f_{i-1,j} = 2 f_{ij} + \frac{\partial^2 f}{\partial x^2}\Big|_{ij} \Delta x^2 + \Delta x^4$$

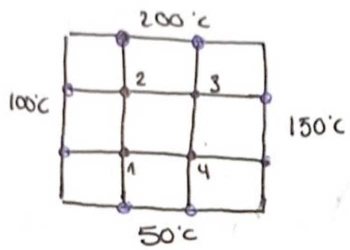
$$\boxed{\frac{f_{i+1,j} + f_{i-1,j} - 2 f_{ij}}{\Delta x^2} = \frac{\partial^2 f}{\partial x^2}\Big|_{ij} + \Delta x^2}$$

× Central approximation:

$$f_{i+1,j} - f_{i-1,j} = 2 \frac{\partial f}{\partial x} \Delta x$$

$$\frac{f_{i+1,j} - f_{i-1,j}}{2 \Delta x} = \frac{df}{dx}\Big|_{ij} + (\Delta x^2) \rightarrow$$ this is a second order approximation

Ejemplo CFD :

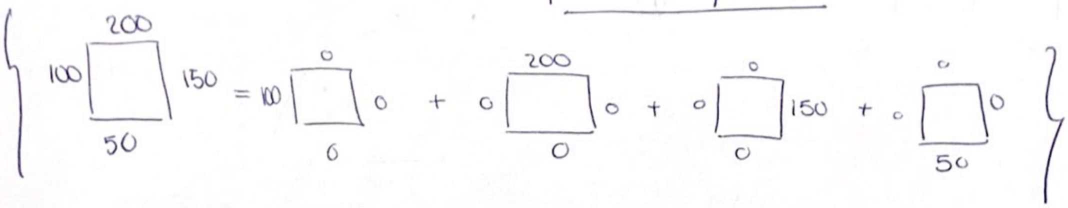


It is a plate with thermal conductivity that is constant. (k) and no source of heat.

We use the Energy equation :

$$K\left(\frac{\partial^2 T}{\partial x} + \frac{\partial^2 T}{\partial y^2}\right) = 0 \quad \text{where } T = \text{temperature}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$ La Place eq => heat transfer.



- $$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} + T_{i-1,j} - 2T_{ij}}{\Delta x^2}$$

- $$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} + T_{i,j-1} - 2T_{ij}}{\Delta y^2}$$

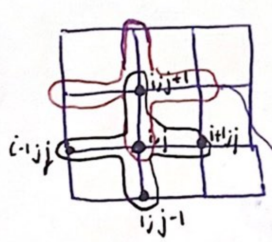
therefore :

$$\frac{T_{i+1,j} + T_{i-1,j} - 2T_{ij}}{\Delta x^2} + \frac{T_{i,j+1} + T_{i,j-1} - 2T_{ij}}{\Delta y^2} = 0$$

and :

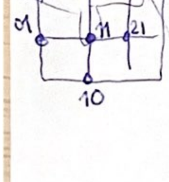
$$T_{ij} = \underbrace{\frac{\Delta x^2}{\Delta y^2}}_{\Delta x = \Delta y} \overset{\to 1}{} \cdot \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right)$$

Locating or i's and j's in the plate :



for example we can choose any node, but we could also place it in another node that will give us the info we need.

For example, fixing at node 1 where i, j :



$$T_{11} = \frac{T_{10} + T_{12} + T_{21} + T_{01}}{4}$$ → en estrella !!! where we know

{ $T_{01} = 100°C$
$T_{10} = 50°C$
$T_{12} = T_{12}$ como no tenemos valor de referencia asumimos que son =0. al principio.

$$T_{11} = \frac{50 + 0 + 0 + 100}{4} = 37{,}5°$$

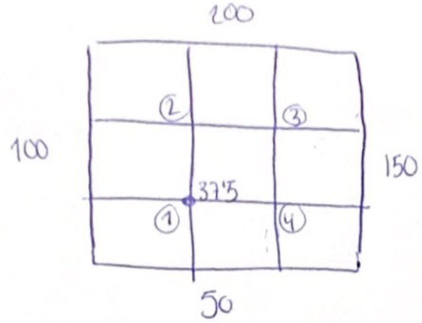
$$T_{12} = \frac{0 + 200 + 0 + 100}{4} = 75°C$$

$$T_{22} = \frac{0 + 0 + 200 + 150}{4} = 87'5°C$$

$$T_{21} = \frac{0 + 150 + 50 + 0}{4} = 50°C$$

Calculamos para todos los nodos, nombrándolos desde el nodo de referencia (11) y en estrella sumando los nodos de alrededor, pero asumiendo que los q no están en el borde son T = 0.

Hacemos una segunda vuelta, ahora sí teniendo en cuenta las temperaturas halladas en los nodos del interior., comentando por el (1)



For node 11: $T_{11} = 37'5 \ °C$

then node 12:

$$T_{12} = \frac{37'5 + 100 + 200 + 87{,}5}{4} = 106.25$$

then node 22

$$T_{22} = \frac{200 + 150 + (50) + (T_{12})}{4} = 118'75$$

then node 21

$$T_{21} = \frac{150 + 50 + 37'5 + T_{22}}{4} = 81'75$$

+ Governing equations in CFD

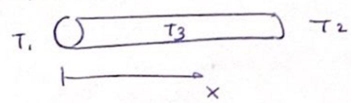
• Energy eq's:

$$C_p \cdot \rho \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \cdot \frac{\partial T}{\partial y} \right) = K \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \cancel{\Phi}^{\,0}$$

$$\frac{\partial T}{\partial t} = \frac{K}{\rho C_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$\boxed{\alpha = \frac{K}{\rho C_p}}$ → α = disipación coeff.

example:



heat transfer is in x axis so:

$$\underbrace{\frac{\partial T}{\partial t} = \frac{K}{\rho C_p} \cdot \frac{\partial^2 T}{\partial x^2}}$$

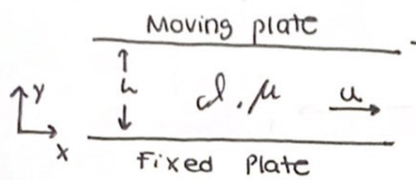
bc $\frac{\partial^2 T}{\partial y^2} = 0$

so $\left[\frac{\partial T}{\partial t} = \alpha \cdot \frac{\partial^2 T}{\partial x^2} \right]$ for cases like this

To approximate the diff eq's in CFD (parabolic Partial Diff Eq)

- Explicit Method
- Implicit Method

Considering the example from homework :



{at time t=0 / u=0}

this example could be Solved analytically, but also as CFD.

1° We take assumptions :

1) 1D flow
2) Incompressible
3) No gravity
4) No gradient of pressure
5) Newtonian

2° x - component :

$$\left[\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \right]$$

this is the diff eq' we need to solve

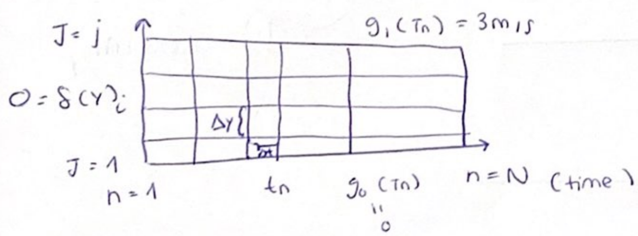
3° Boundary condition (we need the initial cond. and two B.C.'s)

- for $y = 0 \longrightarrow u(0,t) = 0$
- for $y = h \longrightarrow u(h,t) = 3\,m/s$

4° Initial condition (t = 0)

$u(y,0) = 0$

5° we draw the domain :



Forward :

$u(y,t)$

$$\frac{\partial u}{\partial t} = \frac{U_j^{n+1} - U_j^{n}}{\Delta t} \quad \text{Explicit mode}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_{j+1}^{n} + U_{j-1}^{n} - 2U_j^{n}}{\Delta y^2}$$

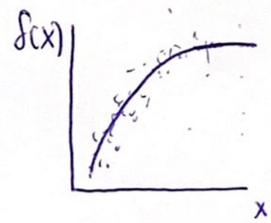
$$\frac{U_j^{n+1} - U_j^{n}}{\Delta t} = \nu \left(\frac{U_{j+1}^{n} + U_{j-1}^{n} - 2U_j^{n}}{\Delta y^2} \right)$$

$$U_j^{n+1} = \frac{\nu \Delta t}{\Delta y^2} \left(U_{j+1}^{n} + U_{j-1}^{n} - 2U_j^{n} \right) + U_j^{n} \qquad \text{where } \lambda = \frac{\nu \Delta t}{\Delta y^2}$$

The explicit method is conditional stable $\boxed{0 < \lambda < 1/2}$

Based on λ , $\lambda = \frac{1}{2} = \frac{\nu \Delta t}{\Delta y^2}$

$$\hookrightarrow \left[\Delta t = 1/2 \cdot \frac{\Delta y^2}{\nu} \right]$$



we continue solving forward

$$u_j^{n+1} = u_j^n + \lambda\left(u_{j+1}^n + u_{j-1}^n - 2u_j^n\right)$$

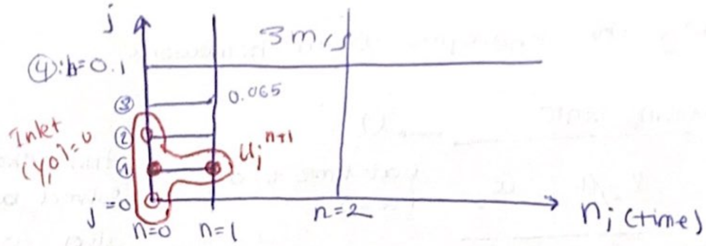
$h = 0.1m$ → separation between two plates

$u = 3 m/s$

AIR:

$\rho = 1.23 \; kg/m^3$

$\mu = 1.789 \cdot 10^{-5} \; \frac{kg}{m \cdot s}$

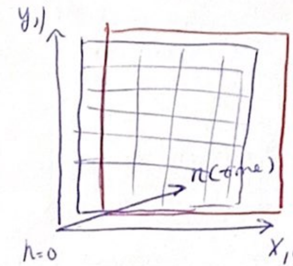


$j = 1 \; ; n = 0 \; (\Delta t = 1)$ where $\Delta y = 0.025$ (we've devided 0.1m in 4)

$\Delta t = 1$ second (in all cases)

$\boxed{\lambda = 0.023}$ bc $\lambda = \frac{\Delta t}{(\Delta y)^2} \cdot \nu$ → viscosidad cinemática (m^2/s)

$$u_1^1 = \overset{=0}{u_1^0} + \overset{\lambda}{(0.023)}\left(\overset{=0}{u_2^0} + \overset{=0}{u_0^0} - 2\overset{=0}{u_1^0}\right)$$



$j = 2 \; ; n = 0 \; (\Delta t = 1)$

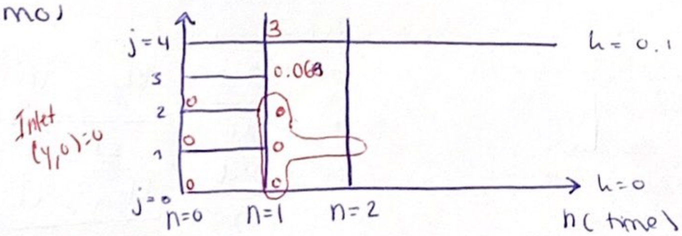
$$u_2^1 = \overset{=0}{u_2^0} + (0.023)\left(\overset{=0}{u_3^0} + \overset{=0}{u_1^0} - 2\overset{=0}{u_2^0}\right) = 0$$

$j = 3 \quad n = 0 \quad (\Delta t = 1)$

$$u_3^1 = \overset{=0}{u_3^0} + 0{,}023\left(\overset{=3m/s}{u_4^0} + \overset{=0}{u_2^0} - 2\overset{=0}{u_3^0}\right) = 0.069$$

$\rightarrow u_3^1 = 0.069 m/s$

continuamos



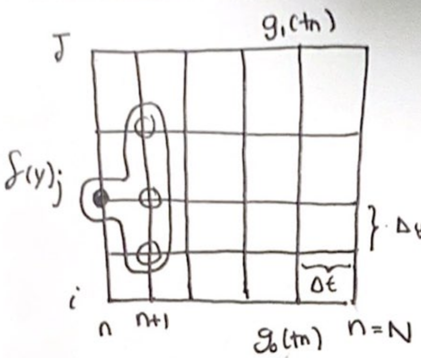
$j = 1 \quad n = 1 \quad (\Delta t = 1 \text{ sec} \Rightarrow 2 \text{ sec})$

$$u_1^2 = u_1^1 + 0.023\left(u_2^1 + u_0^1 - 2u_0^1\right) = 0$$

$j = 2 \quad n = 2 \quad (2 \text{ secs})$

$$u_2^2 = \overset{=0}{u_2^1} + 0.023\left(\overset{=0.069}{u_3^1} + u_2^1 - 2u_2^1\right)$$

Implicit Method



$$\frac{u_j^{n+1} - u_j^{n}}{\Delta t} = \frac{\nu\left(u_{j+1}^{n+1} + u_{j-1}^{n+1} - 2u_j^{n+1}\right)}{(\Delta y)^2}$$

$$u_j^{n} = -\lambda\left(u_{j+1}^{n+1} + u_{j-1}^{n+1} - 2u_j^{n+1}\right) + u_j^{n+1}$$

unconditional stable $\left[\lambda = \frac{\nu \Delta t}{\Delta y^2}\right]$

$\Delta t \quad \Delta y$

$\Delta t \longrightarrow \infty$ higher value

Teoría:

CFD. Generación de malla – Calidad de la malla

En las capas límite, dd las variables de flujo cambian con rapidez en la dirección normal a la pared y se requieren mallas de alta resolución en la cercanía a esta, las mallas estructuradas permiten una resolución mucho + fina que las no estructuradas.

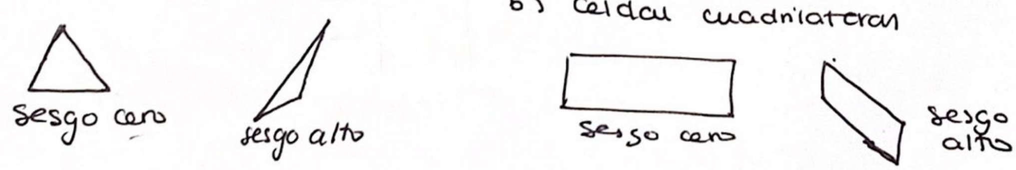
La calidad de la malla es imprescindible para soluciones confiables de la DFC.

El tipo de sesgo + apropiado para celdas bidimensionales es el sesgo equiángulo, definido como:

$$\left[\ Q_{EAS} = MAX\left(\frac{\theta_{max} - \theta_{igual}}{180° - \theta_{igual}}, \frac{\theta_{igual} - \theta_{min}}{\theta_{igual}}\right)\ \right]$$

a) Celdas triangulares

b) Celdas cuadriláteras



donde:

θ_{min} = angulo mínimo
θ_{max} = angulo max } (en grados) entre dos lados cualesquiera de la celda

θ igual = angulo entre dos lados de una celda equilátera ideal con el mismo n° de lados.

↳ C. triangulares: $\theta_{igual} = 60°$

↳ C. cuadriláteras: $\theta_{igual} = 90°$

con la ecuación Q_{EAS}: $0 < Q_{EAS} < 1$ para cualquier celda de 2D.

- otros factores afectan tb a la calidad de la malla, por ejemplo: los cambios abruptos en el tamaño de celda los cuales conducen a las dificultades numericas o de convergencia.