1. Consideramos en el espacio de las funciones continuas en el intervalo $[0,1]$, $C[0,1]$, el producto escalar $<f,g> = \int_0^1 f(x)g(x)dx$ para $f,g \in C[0,1]$. El producto escalar entre $f(x) = x^2-1$ y $g(x) = x$ es:

$$<f,g> = \int_0^1 (x^2-1)\cdot x\, dx = \int_0^1 (x^3-x)\, dx = \left.\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

2. Sea $x: (-2,2) \to \mathbb{R}^3$ la curva de ecuacion

$$\bar{x}(t) = (2\operatorname{sen} t, \cos \pi t, t+1)$$

Señálese la ecuación del plano normal a la curva en $(0,1,1)$

El punto $(0,1,1)$ es la imagen de $t=0$

$$x'(t) = (2\cos t, -\pi \operatorname{sen} \pi t, 1)$$

$$x'(0) = (2,0,1)$$

El vector tangent es:

$$t(0) = \frac{x'(0)}{\|x'(0)\|} = \frac{(2,0,1)}{\sqrt{5}} = \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$$

La ecuación del plano normal es:

$$(x - x(0))\cdot t(0) = 0$$

$$(x - (0,1,1))\left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right) = 0$$

$$(x, y-1, z-1)\left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right) = 0$$

$$\frac{2}{\sqrt{5}}x + \frac{z}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0 \;, \quad \frac{1}{\sqrt{5}}(2x+z-1) = 0$$

$$\boxed{2x+z-1=0}$$

3. Sea C la curva definida por las ecuaciones $x(t) = (t^2+t, 2t, t^3)$.

Elija la respuesta correcta:

$$\kappa(t) = \frac{\|x'(t) \times x''(t)\|}{\|x'(t)\|^3}$$

$$x'(t) = (2t+1, 2, 3t^2) \quad ; \quad x'(0) = (1, 2, 0)$$

$$x''(t) = (2, 0, 6t) \quad ; \quad x''(0) = (2, 0, 0)$$

$$x'(t) \times x''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -4\vec{k}$$

$$\kappa(t) = \frac{4}{(\sqrt{5})^3} \rightarrow \text{Radio curvatura} = \frac{1}{\kappa(t)} = \frac{(\sqrt{5})^3}{4}$$

4. Sea S la superficie dada por $\bar{r}(u,v) = (u\cos\theta, u^2\cos\theta, u)$. Elíjase el valor en $(-1,1)$ de la primera forma cuadrática fundamental en el punto $\bar{r}(-1,0)$.

$$x_u(u,\theta) = (\cos\theta, 2u\cos\theta, 1) \quad ; \quad x_u(-1,0) = (1,-2,1)$$

$$x_\theta(u,\theta) = (-u\,\text{sen}\,\theta, -u^2\text{sen}\,\theta, 0) \quad ; \quad x_\theta(-1,0) = (0,0,0)$$

$$E = x_u x_u = (1,-2,1)(1,-2,1) = 1+4+1 = 6$$

$$F = x_u x_v = 0$$

$$G = x_v x_v = 0$$

1ª forma fundamental para $(-1, 1)$, $h = -1$, $k = 1$

$$I_p = Eh^2 + 2Fhk + Gk^2$$

$$I_p = 6(-1)^2 + 2\cdot 0\,(-1)(1) + 0\,(+1)^2 = 6$$

5. Sea S la superficie dada por $z = x^2 - y^2 + y$. Entonces, las curvaturas principales en el punto $(1,0,1)$ son:

$$X(u,v) = (u, v, u^2 - v^2 + v)$$

$$X_u(u,v) = (1, 0, 2u)$$
$$X_v(u,v) = (0, 1, -2v+1)$$
$$X_{uu}(u,v) = (0,0,2)$$
$$X_{uv}(u,v) = (0,0,0)$$
$$X_{vv}(u,v) = (0,0,-2)$$

$$X_u(1,0) = (1,0,2) \qquad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \vec{k} - 2\vec{i} - \vec{j} = (-2,-1,1)$$
$$X_v(1,0) = (0,1,1)$$

$$N = \frac{X_u \cdot X_v}{\|X_u \times X_v\|} = \frac{(-2,-1,1)}{\sqrt{6}}$$

$$E = X_u \cdot X_u = (1,0,2)(1,0,2) = 5$$
$$F = X_u \cdot X_v = (1,0,2)(0,1,1) = 2$$
$$G = X_v \cdot X_v = (0,1,1)(0,1,1) = 2$$
$$e = N \cdot X_{uu} = \left(\frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)(0,0,2) = \frac{2}{\sqrt{6}}$$
$$f = N \cdot X_{uv} = \left(-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)(0,0,0) = 0$$
$$g = N \cdot X_{vv} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\cdot(0,0,-2) = -\frac{2}{\sqrt{6}}$$

Ecuación de curvaturas principales:

$$k^2(EG - F^2) - (Eg - 2F\cancel{f}^{\,0} + Ge)k - \cancel{f^2}^{\,0} + eg = 0$$
$$k^2(5\cdot 2 - 2^2) - \left(5\cdot\left(\frac{-2}{\sqrt{6}}\right) + 2\left(\frac{-2}{\sqrt{6}}\right)\right)k - \frac{4}{6} = 0$$
$$6k^2 + \frac{6}{\sqrt{6}}k - \frac{2}{3} = 0$$
$$k = \frac{-\frac{6}{\sqrt{6}} \pm \sqrt{6+16}}{12} = \begin{cases} \dfrac{-\sqrt{6}+\sqrt{22}}{12} \\[2ex] \dfrac{-\sqrt{6}-\sqrt{22}}{12} \end{cases}$$