UNIT 2

Exercises of UNIT 2- Part II: Spectral Characteristics

2.10

Given that $X(t) = \sum_{i=1}^{N} \alpha_i X_i(t)$ where α_i are real constants, show that

$$\mathcal{S}_{XX}(\omega) = \sum_{i=1}^{N} \alpha_i^2 \mathcal{S}_{X_iX_i}(\omega)$$

if

(a) the processes $X_i(t)$ are orthogonal

(b) the processes are independent with zero mean

a) $S_{XX}(\omega)$?

$\boxed{X_i, X_j \quad i \neq j}$ are orthogonal $\Rightarrow E[X_i(t_1)X_j(t_2)] = 0$

$R_{X_iX_j}(t_1,t_2) = 0$

$S_{XX}(\omega) = FT\langle R_{XX}(\tau)\rangle$

$\sum\sum \alpha_i X_i \alpha_j X_j$

$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = E\left[\sum_{i=1}^{N}\alpha_i X_i(t_1) \sum_{j=1}^{N}\alpha_j X_j(t_2)\right]$

$X(t) = \sum_{i=1}^{N} \alpha_i X_i(t)$

$= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j X_i(t_1)X_j(t_2)\right]$

$= \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j \underbrace{E[X_i(t_1)X_j(t_2)]}_{R_{X_iX_j}\ ?}$

$\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j E[X_i(t_1)X_j(t_2)]$

$= \sum_i \alpha_i^2 \underbrace{E[X_i(t_1)X_i(t_2)]}_{R_{X_iX_i}(t_1,t_2)} + \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}}_{i\neq j}\alpha_i\alpha_j \cancel{\underbrace{E[X_i(t_1)X_j(t_2)]}_{R_{X_iX_j}(t_1,t_2)=0}} \to 0$

$= \sum_i \alpha_i^2 R_{X_iX_i}(t_1,t_2) = R_{XX}(t_1,t_2)$ $\quad$ X_i, X_j are orthogonal

$$S_{XX}(w) = FT\langle R_{XX}(\tau)\rangle = FT\langle \sum_{i} \alpha_i^2 R_{X_iX_i}(\tau)\rangle$$

$$= \sum_{i=1}^{N} \alpha_i^2 \underbrace{FT\langle R_{X_iX_i}(\tau)\rangle}_{S_{X_iX_i}(w)} = \sum_{i=1}^{N} \alpha_i^2 S_{X_iX_i}(w)$$

b) $S_{XX}(w)$?

X_i, X_j INDEPENDENT $i \neq j$

$\forall i,\ E[X_i(t)] = 0$

ORTHOGONAL $\Rightarrow$ b) $\rightsquigarrow$ a)

IND $\longrightarrow$ $E[X_i(t)X_j(t_2)] = E[\cancel{X_i(t_1)}]E[\cancel{X_j(t_1)}] = 0$

ZERO MEAN

$\Downarrow$

ORTHOGONAL

2.11

If $X(t)$ is a stationary process, find the power spectrum of

$Y(t) = A_0 + B_0X(t)$ in terms of the power spectrum of $X(t)$ if A_0 and B_0 are real constants.

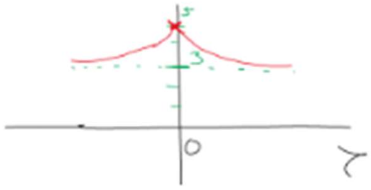
2.12 The autocorrelation function of a random process $X(t)$ is

$R_{XX}(\tau) = 3 + 2e^{-4\tau^2}$

Find

(c) The fraction of power that lies in the frequency band

$$-\sqrt{\frac{1}{2}} \leq \omega \leq \sqrt{1/2}$$



2	$\alpha/2\pi$	$\alpha\delta(\omega)$
20	$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$

$$R_{xx}(\tau)=3+2e^{-4\tau^2}$$

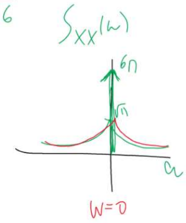
a) $S_{xx}(w)$?

$$S_{xx}(w) = FT\{R_{xx}(\tau)\}$$

$$= FT\{3\} + FT\{2e^{-4\tau^2}\}$$

$$= 2\pi \cdot 3\, \delta(w) + \frac{2}{\sqrt{8}}\sqrt{2\pi}\, e^{-\frac{w^2}{16}}$$

$$= 6\pi\, \delta(w) + \sqrt{\pi}\, e^{-\frac{w^2}{16}}$$



b) P_{xx}?

$$P_{xx} = R_{xx}(0) = 3+2 = 5 \text{ W}$$

$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(w)\, dw = \frac{1}{2\pi}\left[\underbrace{\int_{-\infty}^{\infty} 6\pi\, \delta(w)\, dw}_{6\pi} + \int_{-\infty}^{\infty} \sqrt{\pi}\, e^{-\frac{w^2}{16}}\, dw\right]$$

$$= 3 + \frac{\sqrt{\pi}}{2\pi}\underbrace{\int_{-\infty}^{\infty} e^{-\frac{w^2}{16}} dw}$$

$$\frac{1}{2\sqrt{\pi}} e^{-\frac{w^2}{16}}$$

$\bar{x}=0$

$$\underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{?} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \quad 2\sigma^2=16 \quad = \underbrace{\frac{1}{\sqrt{\pi 16}}} e^{-\frac{x^2}{16}}$$

$$= 3 + \frac{1}{2\sqrt{\pi}}\sqrt{16\pi}\,\underbrace{\frac{1}{\sqrt{16\pi}}\int_{-\infty}^{\infty} e^{-\frac{w^2}{16}} dw}_{N(0,\sigma^2=8)} = 3 + \frac{\overbrace{\sqrt{16\pi}}^{\sqrt{4}=2}}{\sqrt{4\pi}} = 3 + 2 = 5\,W$$

c) $P_{XX}\big|_{|w|\le \frac{1}{\sqrt{2}}} = \frac{1}{2\pi}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} S_{XX}(w)\,dw = \frac{1}{2\pi}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 6\pi\,\delta(w)\,dw + \frac{1}{2\pi}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{2\sqrt{\pi}} e^{-\frac{w^2}{16}} dw$

$R_{XX}(0)$?

$$= 3 + 2 \times \underbrace{\frac{1}{\sqrt{16\pi}}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} e^{-\frac{w^2}{16}} dw}_{N(0,\sigma^2=8)}$$

$\sigma = \sqrt{8}$

$$\int_{x_1}^{x_2} f_X(u)\,du = F_X(x_2) - F_X(x_1)$$

$$= 3 + 2\left[\underbrace{F_W\left(\frac{1}{\sqrt{2}}\right) - F_W\left(-\frac{1}{\sqrt{2}}\right)}\right]$$

- NORMALISE
- USE TABLES

ANALOGY WITH RANDOM

2.13 Given a random process with autocorrelation $R_{XX}(\tau) = P\cos^4(\omega_0\tau)$.

(a) $S_{XX}(\omega)$

(b) P_{XX} from $S_{XX}(\omega)$

(c) P_{XX} from $R_{XX}(\tau)$

TIP: The clue for this exercise is to express $\cos^4$

as a summation of cosine functions:

$$\cos^4(\omega\tau) = \cos^2(\omega\tau) \cdot \cos^2(\omega\tau)$$

$$= \frac{1}{2}(1 + \cos(2\omega\tau))(1 + \cos(2\omega\tau))$$

$$= \frac{1}{2}(1 + \cos(2\omega\tau) + \cos(2\omega\tau) + \cos^2(2\omega\tau))$$

$$= \frac{1}{2}(1 + 2\cos(2\omega\tau) + 1 + \cos(4\omega\tau))$$

$$= 1 + 1\cos(2\omega\tau) + \frac{1}{2}\cos(4\omega\tau)$$

2.14 Given a random process with autocorrelation $R_{XX}(\tau) = Ae^{-\alpha|\tau|}\cos(\omega_0\tau)$ where $A > 0$, $\alpha > 0$, and ω_0 are real constants, find the power spectrum.

$$R_{XX}(\tau) = A e^{-\alpha|\tau|} \cos(w_0 \tau)$$

$A, w_0 \rightarrow$ constant

$$S_{XX}(w) = FT\{R_{XX}(\tau)\} = FT\{A e^{-\alpha|\tau|} \cos w_0 \tau\}$$

$$= A\left[\underbrace{FT\{e^{-\alpha|\tau|}\}}_{\frac{2\alpha}{\alpha^2+w^2}} * \underbrace{FT\{\cos w_0 \tau\}}_{\pi(\delta(w-w_0)+\delta(w+w_0))}\right]$$

$$= A \cdot \pi \cdot \left(\frac{2\alpha}{\alpha^2+w^2}\right) * \left[\delta(w-w_0) + \delta(w+w_0)\right]$$

$$= 2\pi\alpha A\left[\frac{1}{\alpha^2+(w-w_0)^2} + \frac{1}{\alpha^2+(w+w_0)^2}\right]$$

2.15 A random process is given by

$W(t) = AX(t) + BY(t)$ where A and B are real constants and $X(t)$ and $Y(t)$ are jointly widesense stationary processes. Find

(a) The power spectrum of $W(t)$ as a function of $S_{XX}(\omega)$, $S_{YY}(\omega)$, $S_{XY}(\omega)$ and $S_{YX}(\omega)$

(b) The power spectrum of $W(t)$ as a function of $S_{XX}(\omega)$, $S_{YY}(\omega)$, $\bar{X}$ and $\bar{Y}$, if $X(t)$ and $Y(t)$ are uncorrelated

(c) $S_{XW}(\omega)$ and S_{WX} as functions of $S_{XX}(\omega)$, $S_{YY}(\omega)$, $S_{XY}(\omega)$ and $S_{YX}(\omega)$

a) $S_{WW}(\omega)$?

$$S_{WW}(\omega) = FT\langle R_{WW}(\tau)\rangle$$

$$R_{WW}(t,t+\tau) = E[W(t)\cdot W(t+\tau)] = E[(AX(t)+BY(t))(AX(t+\tau)+BY(t+\tau))]$$

$$= E[A^2 X(t)X(t+\tau)] + E[AB\,X(t)Y(t+\tau)] + E[AB\,Y(t)X(t+\tau)] + E[B^2 Y(t)Y(t+\tau)]$$

$$= A^2 R_{XX}(\tau) + AB\,R_{XY}(\tau) + AB\,R_{YX}(\tau) + B^2 R_{YY}(\tau) = R_{WW}(\tau)$$

$$S_{WW}(\omega) = FT\langle R_{WW}(\tau)\rangle = A^2 S_{XX}(\omega) + AB\cdot S_{XY}(\omega) + AB\,S_{YX}(\omega) + B^2 S_{YY}(\omega)$$

b) $X(t)$ & $Y(t)$ are uncorrelated $\Rightarrow$ $R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] \overset{\text{uncorrelation}}{=} E[X(t)]\,E[Y(t+\tau)] \overset{\text{w.c.r.}}{=} \bar{X}\cdot\bar{Y}$

TIME	ω
$\underbrace{\dfrac{\alpha}{2\pi}}_{\alpha'}$	$\alpha\cdot\delta(\omega)$
	$2\pi\cdot\alpha'\,\delta(\omega)$

$$R_{WW}(\tau) = A^2 R_{XX}(\tau) + AB\underbrace{R_{XY}(\tau)}_{\bar{X}\cdot\bar{Y}} + AB\underbrace{R_{YX}(\tau)}_{\bar{X}\cdot\bar{Y}} + B^2 R_{YY}(\tau)$$

$$= A^2 R_{XX}(\tau) + 2AB\bar{X}\cdot\bar{Y} + B^2 R_{YY}(\tau)$$

$$S_{WW}(\omega) = FT\langle R_{WW}(\tau)\rangle = A^2 S_{XX}(\omega) + 2\pi\, 2AB\,\bar{X}\,\bar{Y}\,\delta(\omega) + B^2 S_{YY}(\omega)$$

$$= A^2 S_{XX}(\omega) + B^2 S_{YY}(\omega) + 4\pi\cdot AB\,\bar{X}\,\bar{Y}\,\delta(\omega)$$

c) 1. $S_{WX}(\omega)$?
2. $S_{XW}(\omega)$?

1. $S_{WX}(\omega) = FT\langle R_{WX}(\tau) \rangle$

$R_{WX}(\tau) = E\left[W(t) \cdot X(t+\tau)\right] = E\left[(AX(t)+BY(t))\,X(t+\tau)\right]$

$= A R_{XX}(\tau) + B R_{YX}(\tau)$

$S_{WX}(\omega) = A S_{XX}(\omega) + B S_{YX}(\omega)$

2. $S_{WX}(\omega) = f(S_{XW}(\omega))$? $\rightarrow$ ...

Look for the relationship between $S_{WX}(\omega)$ and $S_{XW}(\omega)$.

2.16 A wide-sense stationary $X(t)$ is applied to an ideal differentiator, so that $Y(t) = dX(t)/dt$. The cross-correlation of the input-output processes is known to be

$R_{XY}(\tau) = dR_{XX}(\tau)/dt$

(a) Determine $S_{XY}(\omega)$ in terms of $S_{XX}(\omega)$

(b) Determine $S_{YX}(\omega)$ in terms of $S_{XX}(\omega)$

TIP: The key here is that the $FT\left\{\frac{df(\tau)}{d\tau}\right\} = j\omega \cdot FT\{f(\tau)\}$

2.17 The cross-correlation of jointly wide-sense stationary processes $X(t)$ and $Y(t)$ is assumed to be

$R_{XY}(\tau) = Be^{-W\tau}u(\tau)$ where $B > 0$ and $W > 0$

are constants. Find

(a) $R_{YX}(\tau)$

(b) $S_{XY}(\omega)$ (use appendix C from Peebles' book)

(c) $S_{YX}(\omega)$ (use cross-power density properties)

W/(rad/s). Given an LTI system with impulse response $h(t) = e^{-\alpha t}u(t)$, with $\alpha > 0$. Find

(a) The mean value of the response of the LTI system if the input is $X_1(t)$

(b) The average power (second-order moment) of the response of the system if the input is $X_2(t)$.

a) $E[X_1(t)] = A > 0$

W.S.S.

$E[Y(t)]\ ? \rightarrow \bar{Y}\ ?$

$X_1(t) \rightarrow \boxed{LTI} \rightarrow Y(t)$

$h(t) = e^{-\alpha t} u(t) = e^{-\alpha t},\ t \geq 0$

$$E[Y(t)] = \bar{X} \cdot \int_{-\infty}^{\infty} h(t)dt = \bar{Y} \int_{0}^{\infty} e^{-\alpha t} dt$$

$$= \bar{X} \cdot \left(\frac{-1}{\alpha}\right) e^{-\alpha t} \Big|_{0}^{\infty} = \frac{\bar{X}}{\alpha}$$

b) $X_2(t) \to \boxed{LTI} \to Y(t)$ $\quad P_{YY}?$

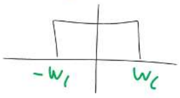
WHITE NOISE

$$P_{YY} = \int_{u_2=-\infty}^{\infty}\int_{u_1=-\infty}^{\infty} \underbrace{R_{XX}(u_1-u_2)}_{R_{XX}(\tau)=\frac{N_0}{2}\delta(\tau)}\, h(u_2)\,h(u_1)\,du_1\,du_2$$

WHITE NOISE

$$P_{YY} = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)\,dt = \frac{N_0}{2}\int_0^{\infty} e^{-2\alpha t}\,dt = \frac{N_0}{4\alpha}$$

$$H(w) = FT\{h(t)\} = \frac{1}{\alpha + jw} = \frac{1/\alpha}{1 + j\frac{w}{\alpha}}$$



$w_c = \alpha$

2.19 A random process $X(t)$ with known mean $\bar{X}$ is the input of an LTI system with impulse response

$h(t) = te^{-Wt}u(t)$.

Find

(a) The mean value of the response of the LTI system

(b) The average power (second-order moment) of the response of the system if $X(t)$ is a white noise with power density 5 W/(rad/s).

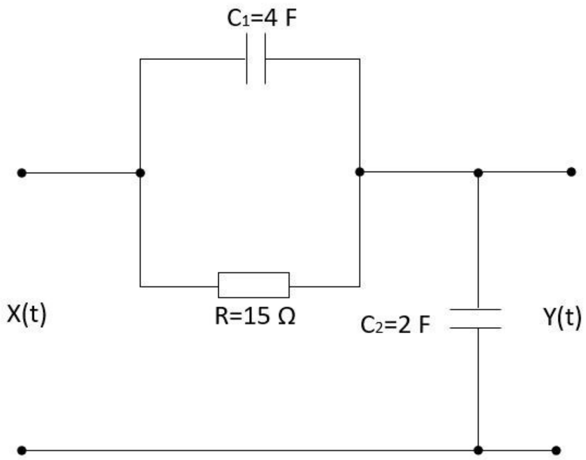
Quite similar to 2.18.

2.20 A white noise with power density $N_0/2$ is applied to a network with impulse response of a system with impulse response

$h(t) = Wte^{-Wt}u(t)$

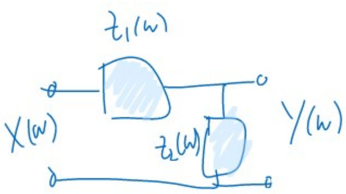
where W is a real positive constant. Find the cross-correlation of the response of input and the output of the system.

2.21 A stationary random process $X(t)$, having an autocorrelation function $R_{XX} = 2e^{-4|\tau|}$ is applied to the network of the figure below. Find the power spectrum of the output of the system.



$S_{YY}(\omega)$?

$S_{YY}(\omega) = S_{XX}(\omega)\,|H(\omega)|^2$



$$\boxed{\begin{aligned} Z_{C_1}(w) &= \frac{1}{jwC_1} \\ Z_R(w) &= R \\ Z_{C_2}(w) &= \frac{1}{jwC_2} \end{aligned}}$$

$$H(w) = \frac{Z_2(w)}{Z_1(w) + Z_2(w)}$$

$$Z_1(w) = Z_{C_1}(w) \parallel Z_R(w) = \frac{Z_{C_1}(w)\, Z_R(w)}{Z_{C_1}(w) + Z_R(w)}$$

$$Z_2(w) = Z_{C_2}(w)$$

$$Z_1(\omega) = \frac{\frac{1}{j\omega C_1} \times R}{\frac{1}{j\omega C_1} + R} = \frac{R}{1 + j\omega C_1 R}$$

$$Z_2(\omega) = \frac{1}{j\omega C_2}$$

$$H(\omega) = \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)} = \frac{\frac{1}{j\omega C_2}}{\frac{R}{1 + j\omega C_1 R} + \frac{1}{j\omega C_2}} = \frac{1}{\frac{j\omega R C_2}{1 + j\omega R C_1} + 1}$$

$$= \frac{1 + j\omega R C_1}{j\omega R C_2 + 1 + j\omega R C_1} = \frac{1 + j\omega R C_1}{1 + j\omega R (C_1 + C_2)}$$

ZERO

POLE

$$S_{yy}(\omega) = S_{xx}(\omega) \cdot |H(\omega)|^2 = S_{xx}(\omega) \frac{1 + (\omega R C_1)^2}{1 + (\omega R [C_1 + C_2])^2}$$

$$= FT\{2e^{-4|\tau|}\} \frac{1 + (\omega R C_1)^2}{1 + (\omega R [C_1 + C_2])^2}$$

$$= \overbrace{2 \frac{8}{\underbrace{16 + \omega^2}}}^{16} \frac{1 + (\omega R C_1)^2}{1 + (\omega R [C_1 + C_2])^2}$$

2.22 A white noise $X(t)$ with $R_{XX} = 4{\cdot}10^{-3}{\cdot}\delta(\tau)$ is filtered with the network of the figure below. Find the average power of the input and output of the system.