$$\int_{-\infty}^{+\infty} dx\; e^{-\alpha^2 x^2} = \frac{\sqrt{\pi}}{\alpha} \qquad \forall \alpha \in \mathbb{C} \;/\; -\frac{\pi}{4} < \operatorname{Arg}\alpha < \frac{\pi}{4}$$

¿Por qué $-\frac{\pi}{4} < \operatorname{Arg}\alpha < \frac{\pi}{4}$?

$\alpha = |\alpha| e^{i\theta} \quad \alpha^2 = |\alpha|^2 e^{2i\theta} = |\alpha|^2(\cos 2\theta + i \operatorname{sen} 2\theta)$

Si $\cos 2\theta \leq 0$, entonces $\left|e^{-\alpha^2 x^2}\right| = e^{|\alpha|^2 |\cos 2\theta| x^2} \not\to \infty$ y la integral diverge.

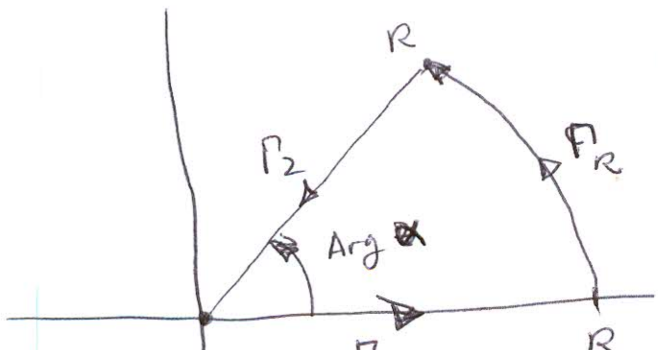
$\cos 2\theta > 0 \iff |2\theta| < \frac{\pi}{2}$, si $\theta \in [0, 2\pi]$.

$$\left|\int_{-\infty}^{+\infty} dx\; e^{-\alpha^2 x^2}\right| < \int_{-\infty}^{+\infty} dx\; e^{-|\alpha|^2 \cos 2\theta\, x^2} < +\infty \quad \text{ya que } |\alpha|^2 \cos 2\theta > 0$$

cuando $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

$$\int_{-\infty}^{+\infty} dx\; e^{-\alpha^2 x^2} = 2\int_0^{\infty} dx\; e^{-\alpha^2 x^2}.$$ Supongamos $\underline{\underline{\operatorname{Arg}\alpha \geq 0}}$

Tomemos el contorno



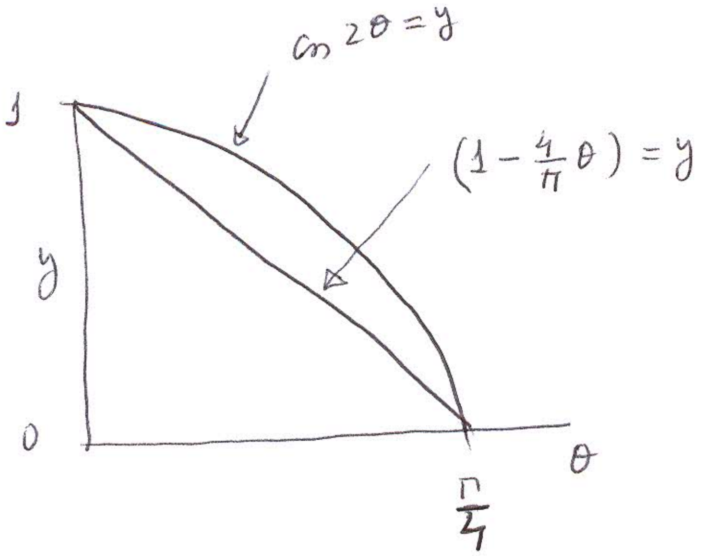
$C = \Gamma_1 \cup \Gamma_2 \cup \Gamma_R$

Entonces

$$\left|\int_0^{\operatorname{Arg}\alpha} iRe^{i\theta}\, e^{-R^2(\cos 2\theta + i\sin 2\theta)}\, d\theta\right| \le$$

$$R\int_0^{\operatorname{Arg}\alpha} e^{-R^2\cos 2\theta}\, d\theta \qquad (3.1)$$

$$\cos 2\theta \geq \left(1-\frac{4}{\pi}\theta\right) \quad \forall\theta\in\left[0,\frac{\pi}{4}\right] \qquad (3.2)$$



$$e^{-R^2\cos 2\theta} \le e^{-R^2\left(1-\frac{4}{\pi}\theta\right)} \quad \forall\theta\in\left[0,\frac{\pi}{4}\right] \qquad (3.3)$$

De acuerdo con (3.1) y (3.3)

$$\left|\int_0^{\operatorname{Arg}\alpha} iRe^{i\theta}\, e^{-R^2(\cos 2\theta + i\sin 2\theta)}\, d\theta\right| \le R\int_0^{\operatorname{Arg}\alpha} e^{-R^2\left(1-\frac{4}{\pi}\theta\right)}\, d\theta$$

$$= Re^{-R^2}\int_0^{\operatorname{Arg}\alpha} e^{R^2\frac{4}{\pi}\theta}\, d\theta = \frac{\pi R}{4R^2}\, e^{-R^2}\left(e^{R^2\frac{4}{\pi}\operatorname{Arg}\alpha}-1\right)$$

$$= \frac{\pi}{4}\,\frac{1}{R}\left(e^{-R^2\left(1-\frac{4}{\pi}\operatorname{Arg}\alpha\right)} - e^{-R^2}\right) \to 0 \text{ cuando } R\to+\infty \qquad (3.4)$$

ya que $1-\frac{4}{\pi}\operatorname{Arg}\alpha > 0 \quad \forall\alpha \,/\, 0\le \operatorname{Arg}\alpha < \frac{\pi}{4}$

(3.4) implica que

$$\lim_{R\to\infty}\left|\int_0^{\operatorname{Arg}\alpha} iRe^{i\theta}\, e^{-R^2e^{2i\theta}}\, d\theta\right| = 0 \iff$$

$$\iff \lim_{R\to\infty}\int_0^{\operatorname{Arg}\alpha} iRe^{i\theta}\, e^{-R^2e^{2i\theta}}\, d\theta = 0 \qquad \text{q.e.d.}$$

$\Gamma_1 = \{ z \in \mathbb{C} \mid z = x \ ; \ x \in [0, R] \} \to dz = dx$

$\Gamma_2 = \{ z \in \mathbb{C} \mid z = \alpha x \ ; \ x \in [R, 0] \} \to dz = \alpha\, dx$

$\Gamma_R = \{ z \in \mathbb{C} \mid z = R(\cos\theta + i \operatorname{sen}\theta) \ ; \ \theta \in [0, \operatorname{Arg}\alpha] \} \to dz = iRe^{i\theta} d\theta$

$$\int_{\Gamma_1} dz\, e^{-z^2} = \int_0^R dx\, e^{-x^2}$$

$$\int_{\Gamma_2} dz\, e^{-z^2} = \int_R^0 \alpha\, dx\, e^{-\alpha^2 x^2} = -\alpha \int_0^R dx\, e^{-\alpha^2 x^2}$$

$$\int_{\Gamma_R} dz\, e^{-z^2} = \int_0^{\operatorname{Arg}\alpha} iRe^{i\theta} d\theta\, e^{-R^2\cos 2\theta - iR^2 \operatorname{sen} 2\theta)}$$

$$C = \Gamma_1 \cup \Gamma_2 \cup \Gamma_R$$

Por el teorema de Cauchy $0 = \oint_C dz\, e^{-z^2} \Rightarrow$

$$\Rightarrow 0 = \int_{\Gamma_1} dz\, e^{-z^2} + \int_{\Gamma_2} dz\, e^{-z^2} + \int_{\Gamma_R} dz\, e^{-z^2} =$$

$$= \int_0^R dx\, e^{-x^2} + (-\alpha)\int_0^R dx\, e^{-\alpha^2 x^2} + \int_0^{\operatorname{Arg}\alpha} iRe^{i\theta} d\theta\, e^{-R^2(\cos 2\theta + i \operatorname{sen} 2\theta)}$$

Tomando el límite $R \to +\infty$

$$0 = \int_0^\infty dx\, e^{-x^2} - (\alpha) \int_0^\infty dx\, e^{-\alpha^2 x^2} + \lim_{R\to\infty} \int_0^{\operatorname{Arg}\alpha} iRe^{i\theta} d\theta\, e^{-R^2(\cos 2\theta + i \operatorname{sen} 2\theta)} \qquad (2.1)$$

Probemos que $\lim_{R\to+\infty} \int_0^{\operatorname{Arg}\alpha} iR\, e^{i\theta} d\theta\, e^{-R^2(\cos 2\theta + i \operatorname{sen} 2\theta)} = 0 \qquad (2.2)$

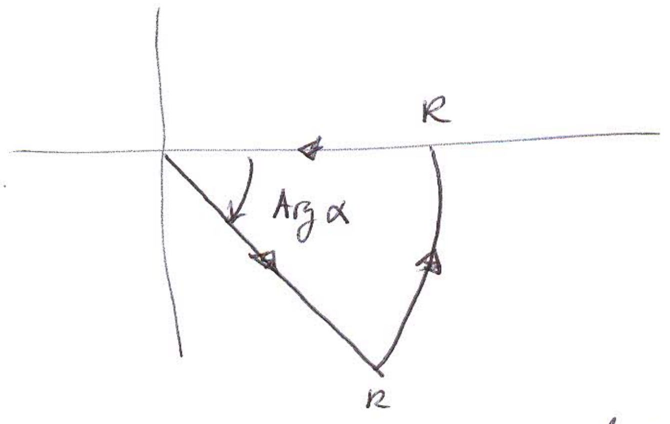
(2.1) y (2.2) conducen a que

$$\int_0^\infty dx\, e^{-x^2} = \alpha \int_0^\infty dx\, e^{-\alpha^2 x^2} \Leftrightarrow$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} dx\, e^{-x^2} = \alpha \int_{-\infty}^{+\infty} dx\, e^{-\alpha^2 x^2} \Leftrightarrow$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} dx\, e^{-\alpha^2 x^2} = \frac{\sqrt{\pi}}{\alpha} \quad \text{ya que} \int_{-\infty}^{+\infty} dx\, e^{-x^2} = \sqrt{\pi}$$

Si $\operatorname{Arg}\alpha \,/\, -\frac{\pi}{4} < \operatorname{Arg}\alpha \leq 0$, el contorno es



pero la prueba es análoga.

$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha^2(x+\beta)^2} = \frac{\sqrt{\pi}}{\alpha} \quad \forall \alpha \in \mathbb{C} \;/\; -\frac{\pi}{4} < \operatorname{Arg}\alpha < \frac{\pi}{4}$$

$\beta = \operatorname{Re}\beta + i\,\operatorname{Im}\beta$

$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha^2(x+\beta)^2} = \int_{-\infty}^{+\infty} dx\, e^{-\alpha^2(x+\operatorname{Re}\beta+i\operatorname{Im}\beta)^2} =$$

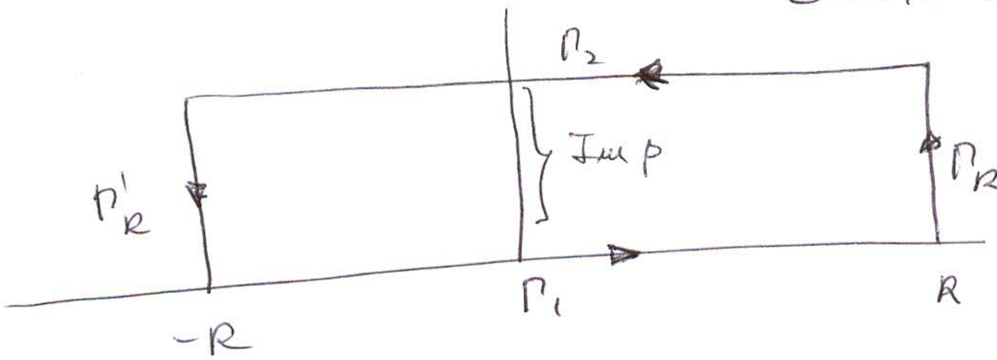
$x + \operatorname{Re}\beta \to x \Rightarrow dx \to dx$

$$= \int_{-\infty}^{+\infty} dx\, e^{-\alpha^2(x+i\operatorname{Im}\beta)^2}$$

así que sin pérdida de generalidad supondremos que $\operatorname{Re}\beta = 0$

Tomemos $\operatorname{Im}\beta > 0$ y consideremos el contorno

$C = \Gamma_1 \cup \Gamma_2 \cup \Gamma_R \cup \Gamma'_R$



$\Gamma_1 = \{z \in \mathbb{C} \;/\; z = x,\; x \in [-R, R]\} \Rightarrow dz = dx$

$\Gamma_2 = \{z \in \mathbb{C} \;/\; z = x + i\operatorname{Im}\beta \;;\; x \in [R, -R]\} \Rightarrow dz = dx$

$\Gamma_R = \{z \in \mathbb{C} \;/\; z = R + iy \;;\; y \in [0, \operatorname{Im}\beta]\} \Rightarrow dz = i\,dy$

$\Gamma'_R = \{z \in \mathbb{C} \;/\; z = -R + iy \;;\; y \in [\operatorname{Im}\beta, 0]\} \Rightarrow dz = i\,dy$

Entonces,

$$\int_{\Gamma_1} dz\, e^{-\alpha^2 z^2} = \int_{-R}^{R} dx\, e^{-\alpha^2 x^2} \qquad (6.1)$$

$$\int_{\Gamma_2} dz\, e^{-\alpha^2 z^2} = \int_{+R}^{-R} dx\, e^{-\alpha^2 (x + i\,\mathrm{Im}\,\beta)} = -\int_{-R}^{R} dx\, e^{-\alpha^2 (x + i\,\mathrm{Im}\,\beta)} \qquad (6.2)$$

$$= \int_{-\Gamma_2} dz\, e^{-\alpha^2 z^2}$$

$$\int_{\Gamma_R} dz\, e^{-\alpha^2 z^2} = i \int_{0}^{\mathrm{Im}\,\beta} dy\, e^{-\alpha^2 (R + iy)^2} \qquad (6.3)$$

$$\int_{\Gamma'_R} dz\, e^{-\alpha^2 z^2} = i \int_{\mathrm{Im}\,\beta}^{0} dy\, e^{-\alpha^2 (-R + iy)^2} = -i \int_{0}^{\mathrm{Im}\,\beta} dy\, e^{-\alpha^2 (-R + iy)^2}$$

$$= -\int_{-\Gamma'_R} dz\, e^{-\alpha^2 z^2} \qquad (6.4)$$

Probemos que

$$\lim_{R \to +\infty} \int_{\Gamma_R} dz\, e^{-\alpha^2 z^2} = 0 \qquad (6.4)$$

$$\left| \int_{\Gamma_R} dz\, e^{-\alpha^2 z^2} \right| = \left| \int_{0}^{\mathrm{Im}\,\beta} dy\, e^{-\alpha^2 (R+iy)^2} \right| \leq$$

$$\leq \int_{0}^{\mathrm{Im}\,\beta} dy \left| e^{-\alpha^2 [(R^2 - y^2) + 2Ryi]} \right| =$$

$$= \int_{0}^{\mathrm{Im}\,\beta} dy\, e^{-\alpha^2 R^2} e^{\alpha^2 y^2} = e^{-\alpha^2 R^2} \int_{0}^{\mathrm{Im}\,\beta} dy\, e^{\alpha^2 y^2}$$

$$\lim_{R \to \infty} \left| \int_{\Gamma_R} dz\, e^{-\alpha^2 z^2} \right| = \lim_{R \to \infty} e^{-\alpha^2 R^2} \underbrace{\int_{0}^{\mathrm{Im}\,\beta} dy\, e^{\alpha^2 y^2}}_{\text{cte}} = 0$$

$$\lim_{R\to\infty}\left|\int_{\Gamma_R} dz\, e^{-\alpha^2 z^2}\right| = 0 \Leftrightarrow \lim_{R\to\infty}\int_{\Gamma_R} dz\, e^{-\alpha^2 z^2} = 0$$

Analogamente, se demuestra que

$$\lim_{R\to+\infty}\int_{\Gamma'_R} dz\, e^{-\alpha^2 z^2} = 0 \qquad (7.1)$$

Sea $C = \Gamma_1 \cup \Gamma_2 \cup \Gamma_R \cup \Gamma'_R$, como $e^{-\alpha^2 z^2}$ es analítica en todo el plano complejo, el Th de Cauchy garantiza que

$$0 = \oint_C dz\, e^{-\alpha^2 z^2} = \int_{\Gamma_1} dz\, e^{-\alpha^2 z^2} + \int_{\Gamma_2} dz\, e^{-\alpha^2 z^2} + \int_{\Gamma_R} dz\, e^{-\alpha^2 z^2} + \int_{\Gamma'_R} dz\, e^{-\alpha^2 z^2}$$

así que en el limite $R\to+\infty$

$$0 = \int_{-\infty}^{+\infty} dx\, e^{-\alpha^2 x^2} - \int_{-\infty}^{+\infty} dx\, e^{-\alpha^2(x+i\,\mathrm{Im}\,p)} + 0$$

— se ha tenido en cuenta (6.4) y (7.1), y (6.1), (6.2), (6.3) y (6.4) —

Por tanto

$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha^2(x+i\,\mathrm{Im}\,p)} = \int_{-\infty}^{+\infty} dx\, e^{-\alpha^2 x^2} = \frac{\sqrt{\pi}}{\alpha} \quad \text{q.e.d}$$

Si $\mathrm{Im}\,p < 0$, el contorno es