La función de onda de una partícula libre de masa m es, en el instante $t=0$, la siguiente

$$\phi(x) = \frac{A}{x^2+a^2} \quad , \quad a>0$$

1) Calcular A de modo que $\phi(x)$ esté normalizada a la unidad.
2) Calcular la función de onda, $\tilde{\phi}(p)$, en la representación de momentos en $t=0$.
3) Calcular la probabilidad de que en $t=0$ la partícula esté en el segmento $[0, na]$ $n \in \mathbb{N}$.
4) Calcular la probabilidad de que en $t=0$, el momento de la partícula sea mayor que $\frac{\hbar}{a}$.
5) Calcúlese $\Delta_\phi X$ en $t=0$
6) Calcúlese $\Delta_\phi P$ en $t=0$
7) Compruébese que $\Delta_\phi X \cdot \Delta_\phi P > \frac{\hbar}{2}$. ¿Por qué no se cumple $\Delta_\phi X \, \Delta_\phi P = \frac{\hbar}{2}$?
8) Suponiendo que $\forall t>0$ la partícula sigue siendo libre, obténgase $\phi(x,t)$.

Ayuda:

$$\int_0^\infty dq \; e^{-iq^2 - qz} = \frac{1+i}{\sqrt{2}} F\left(\frac{1+i}{\sqrt{2}} \frac{z}{2}\right) \qquad z=\alpha+i\beta \; ; \; \alpha>0$$

donde $F(u)$ $u \in \mathbb{C}$ es la llamada función de Dawson.

$$1 = \|\phi\| = \sqrt{(\phi,\phi)} = \left(\int_{-\infty}^{+\infty} dx\, \phi^*(x)\phi(x)\right)^{1/2} = |A|\left(\int_{-\infty}^{+\infty} dx\, \frac{1}{(a^2+x^2)^2}\right)^{1/2}$$

$$\Rightarrow \quad A = e^{i\theta}\left(\int_{-\infty}^{+\infty} dx\, \frac{1}{(x^2+a^2)^2}\right)^{1/2} = \frac{\sqrt{2}\, a^{3/2}}{\sqrt{\pi}}\, e^{i\theta}.$$

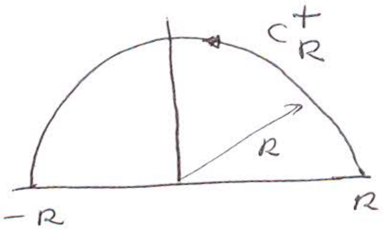
Tomamos, sin pérdida de generalidad, $\theta = 0$; ya que $e^{i\theta}$ es global.

$$\phi(x) = \sqrt{\frac{2a^3}{\pi}}\; \frac{1}{a^2+x^2}$$

2)

$$\tilde{\phi}(p) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, \phi(x)\, e^{-\frac{ipx}{\hbar}} = \sqrt{\frac{2a^3}{\pi}}\, \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\, \frac{e^{-\frac{ipx}{\hbar}}}{x^2+a^2} \qquad (2,1)$$

Para calcular la integral anterior utilizaremos el Th de Cauchy y el lema de Jordan siguiente



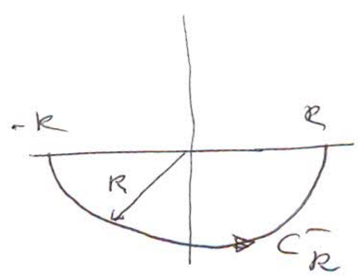
<u>Lema de Jordan</u> (Weinberger, Page 302)

1) Sea $C_R^+ = \{ z \in \mathbb{C} \;/\; z = Re^{i\theta};\ \theta \in [0,\pi] \}$

Sea $f(z)$ tal que $\max_{z \in C_R^+} |f(z)| \to 0$ cuando $R \to +\infty$.

Entonces, si <u>$\omega \geq 0$</u>.

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\, e^{i\omega z} = 0$$



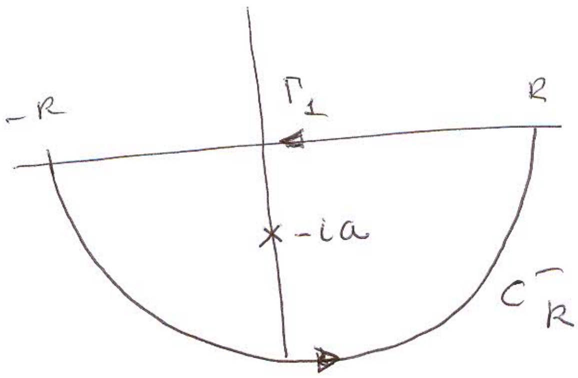
2) Sea $C_R^- = \{ z \in \mathbb{C} \;/\; z = Re^{i\theta},\ \theta \in [-\pi, 0] \}$

Sea $f(z)$ tal que $\max_{z\in C_R^-} |f(z)| \to 0$ cuando $R \to \infty$

Entonces, si $\omega \geq 0$

$$\lim_{R\to\infty} \int_{C_R^-} f(z)\, e^{-i\omega z} = 0$$

Sea $p \geq 0$, tomemos $\Gamma = \Gamma_1 \cup C_R^-$



El th de Cauchy dice que

$$\oint_\Gamma dz \frac{e^{-i\frac{p}{\hbar}z}}{z^2+a^2} = 2\pi i \operatorname{Res}\Big|_{z=-ia} \left(\frac{e^{-\frac{ipz}{\hbar}}}{z^2+a^2} \right) = \cancel{2}\pi \cancel{i} \frac{e^{-\frac{ip}{\hbar}(-ia)}}{-\cancel{2}\cancel{i}a}$$

$$\oint_\Gamma dz \frac{e^{-i\frac{p}{\hbar}z}}{z^2+a^2} = -\frac{\pi}{a} e^{-\frac{pa}{\hbar}} \qquad (3.1)$$

$$\oint_\Gamma dz \frac{e^{-i\frac{p}{\hbar}z}}{z^2+a^2} = \int_{\Gamma_1} dz \frac{e^{-i\frac{p}{\hbar}z}}{z^2+a^2} + \int_{C_R^-} \frac{e^{-i\frac{p}{\hbar}z}}{z^2+a^2} dz$$

Sea $z \in C_R^-$, entonces

$$\frac{1}{|z^2+a^2|} = \frac{1}{|R^2e^{2i\theta}+a^2|} \leq \frac{1}{\left||R^2e^{2i\theta}| - |a^2|\right|} \leq \frac{1}{|R^2-a^2|} \rightarrow$$

$\Rightarrow \frac{1}{|z^2+a^2|} \rightarrow 0$ cuando $R \rightarrow \infty \Rightarrow$ Podemos aplicar el lema de Jordan: $\lim_{R\to\infty} \int_{C_R^-} \frac{e^{-i\frac{p}{\hbar}z}}{z^2+a^2} dz = 0$

$$\int_{\Gamma_1} dz \frac{e^{-i\frac{p}{\hbar}z}}{z^2+a^2} = -\int_{-R}^{+R} dx \frac{e^{-i\frac{px}{\hbar}}}{x^2+a^2}$$ ya que en Γ_1; $z = x$, $x \in [+R, -R]$

Ahora bien,

$$\lim_{R\to\infty} \oint_\Gamma dz \frac{e^{-ip/\hbar z}}{z^2+a^2} = -\int_{-\infty}^{+\infty} dx \frac{e^{-\frac{ipx}{\hbar}}}{x^2+a^2} + \lim_{R\to\infty} \int_{C_R^+} dz \frac{e^{-i/\hbar pz}}{z^2+a^2}$$

$$= -\int_{-\infty}^{+\infty} dx \frac{e^{-ipx}}{x^2+a^2} \qquad (3.2)$$

(3.1) y (3.2) conducen a que

$$\int_{-\infty}^{+\infty} dx \, \frac{e^{-ipx}}{x^2+a^2} = \frac{\pi}{a} e^{-\frac{pa}{\hbar}} \qquad (4.1)$$

Substituyendo (4.1) en (2.1) se obtiene que si

$p \geq 0$

$$\tilde{\phi}(p) = \sqrt{\frac{2a^3}{\pi}} \frac{1}{\sqrt{2\pi\hbar}} \frac{\pi}{a} e^{-\frac{pa}{\hbar}} = \sqrt{\frac{a}{\hbar}} e^{-\frac{pa}{\hbar}}$$

Si $p \leq 0$, entonces

valor absoluto de p

$$\int_{-\infty}^{+\infty} dx \, \frac{e^{-ipx}}{x^2+a^2} = \int_{-\infty}^{+\infty} dx \, \frac{e^{i|p|x}}{x^2+a^2} = \left(\int_{-\infty}^{+\infty} dx \, \frac{e^{-i|p|x}}{x^2+a^2} \right)^*$$

$$\overset{(4.1)}{=} \left(\frac{\pi}{a} e^{-\frac{|p|a}{\hbar}} \right)^* = \frac{\pi}{a} e^{-\frac{|p|a}{\hbar}}$$

Por tanto,

$$\boxed{\tilde{\phi}(p) = \sqrt{\frac{a}{\hbar}} e^{-\frac{|p|a}{\hbar}} \quad \forall p \in [-\infty, \infty]}$$

Nótese que $\int_{-\infty}^{+\infty} |\phi(p)|^2 dp = \frac{a}{\hbar} \int_{-\infty}^{+\infty} e^{-\frac{2|p|a}{\hbar}} dp = \frac{2a}{\hbar} \int_0^{\infty} dp \, e^{-\frac{2pa}{\hbar}} = 1$

3) $$\mathcal{P}([0,na]) = \int_0^{na} dx \, |\phi(x)|^2 = \int_0^{na} dx \, \frac{A^2}{(1+x^2)^2} =$$

$$= A^2 \int_0^{na} dx \, \frac{1}{(1+x^2)^2} = \left(\sqrt{\frac{2a^3}{\pi}}\right)^2 \left(\frac{ax}{a^2+x^2} + \mathrm{Arctg}\left(\frac{x}{a}\right)\right) \frac{1}{\pi} \Bigg|_0^{na} =$$

$$\int dx \, \frac{1}{(1+x^2)^2} = \left(\frac{ax}{a^2+x^2} + \mathrm{Arctg}\left(\frac{x}{a}\right)\right) \frac{1}{\pi}$$

$$= \frac{n + \mathrm{Arctg}(n) + n^2 \mathrm{Arctg}(n)}{\pi + n^2 \pi}$$

4) $$\mathcal{P}\left(p > \frac{\hbar}{a}\right) = \int_{\hbar/a}^{\infty} dp \, |\tilde{\phi}(p)|^2 = \frac{a}{\hbar} \int_{\hbar/a}^{\infty} e^{-\frac{2pa}{\hbar}} dp =$$

$$= \frac{\cancel{a}}{\cancel{\hbar}}\left(\frac{-\cancel{\hbar}}{2\cancel{a}}\right) e^{-\frac{2pa}{\hbar}} \Bigg|_{\hbar/a}^{\infty} = \frac{1}{2} e^{-2}$$

5) $$\Delta_\phi X = \sqrt{\langle x^2 \rangle_\phi - (\langle x \rangle_\phi)^2}$$

$$\langle x \rangle_\phi = \int_{-\infty}^{+\infty} dx \, x \, |\phi(x)|^2 = 0 \quad \text{ya que} \quad \phi(x) = \frac{A}{(x^2+a^2)} \text{ es par.}$$

$$\langle x^2 \rangle_\phi = \int_{-\infty}^{+\infty} dx \, x^2 |\phi(x)|^2 = A^2 \int_{-\infty}^{+\infty} dx \, \frac{x^2}{(x^2+a^2)^2} = a^2$$

$$\Delta_\phi X = \sqrt{\langle x^2 \rangle_\phi - (\langle x \rangle_\phi)^2} = \sqrt{a^2} = a$$

(6)

$$\Delta_\phi P = \sqrt{\langle p^2\rangle_{\tilde\phi} - (\langle p\rangle_{\tilde\phi})^2}$$

$$\langle p^2\rangle_{\tilde\phi} = \int_{-\infty}^{+\infty} dp\ p^2 |\tilde\phi(p)|^2 = \frac{a}{\hbar}\int_{-\infty}^{+\infty} dp\ p^2 e^{-\frac{2|p|a}{\hbar}} =$$

$$= 2\frac{a}{\hbar}\int_0^\infty dp\ p^2 e^{-\frac{2pa}{\hbar}} = \frac{\hbar^2}{2a^2}$$

$$(\langle p\rangle_{\tilde\phi})^2 = \int_{-\infty}^{+\infty} dp\ p|\tilde\phi(p)|^2 = \frac{a}{\hbar}\int_{-\infty}^{\infty} dp\ p\ e^{-\frac{2|p|a}{\hbar}} = 0$$ ya que

$e^{-\frac{2|p|a}{\hbar}}$ es par.

$$\Delta_\phi P = \sqrt{\langle p^2\rangle_{\tilde\phi} - (\langle p\rangle_{\tilde\phi})^2} = \sqrt{\frac{\hbar^2}{2a^2}} = \frac{\hbar}{\sqrt2\, a}$$

(7) $$\Delta_\phi X\, \Delta_\phi P = a\cdot\frac{\hbar}{\sqrt2\cdot a} = \frac{\hbar}{\sqrt2} > \frac{\hbar}{2}$$

$\Delta_\phi X \Delta_\phi P \neq \frac{\hbar}{2}$ ya que ϕ no es un paquete de ondas <u>mínimo</u>.

(8) La solución general de la E^a de Schrödinger para una partícula libre →

$$\phi(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi\hbar}} \tilde{\phi}(p) e^{-\frac{i}{\hbar}(E(p)t - px)} \quad ; \quad E(p) = \frac{p^2}{2m}$$

~~[illegible]~~ $\tilde{\phi}(p)$ se ha de fijar de modo que

$$\phi(x, t=0) = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi\hbar}} \tilde{\phi}(p) e^{i/\hbar\, px}$$

En nuestro caso $\tilde{\phi}(p) = \sqrt{\frac{a}{\hbar}}\, e^{-\frac{|p|a}{\hbar}}$

Por tanto.

$$\phi(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}}\, e^{-\frac{|p|a}{\hbar}} e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)}$$

$$= \int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}}\, e^{-\frac{pa}{\hbar}} e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)} +$$

$$+ \int_{-\infty}^{0} \frac{dp}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}}\, e^{\frac{pa}{\hbar}} e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)} = \quad \longrightarrow p \to -p$$

$$= \int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}}\, e^{-\frac{pa}{\hbar}} e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)} +$$

$$+ \int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}}\, e^{-\frac{pa}{\hbar}} e^{-i/\hbar\left(\frac{p^2 t}{2m} + px\right)} \qquad (7.1)$$

~~[illegible]~~ Expresemos, en términos de la función de Dawson,

$$\int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}}\, e^{-\frac{pa}{\hbar}} e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)} =$$

$$q = p\sqrt{\frac{t}{2m\hbar}}$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{\cancel{\hbar}}}\sqrt{\frac{2m\cancel{\hbar}}{t}}\int_0^\infty dq\, e^{-iq^2 - qa\sqrt{\frac{2m}{\hbar t}} + iqx\sqrt{\frac{2m}{\hbar t}}}$$

$$= \sqrt{\frac{ma}{\pi\hbar t}}\int_0^\infty dq\, e^{-iq^2 - qz} \qquad \text{con } z = \sqrt{\frac{2m}{\hbar t}}\,(a - ix)$$

$$= \sqrt{\frac{ma}{\pi\hbar t}}\,\frac{1+i}{\sqrt{2}}\,F\left[\frac{1+i}{\sqrt{2}}\sqrt{\frac{2m}{\hbar t}}\,\frac{(a-ix)}{2}\right] =$$

$$= \sqrt{\frac{ma}{2\pi\hbar t}}\,(1+i)\,F\left[(1+i)\left(\frac{m}{\hbar t}\right)^{1/2}\frac{(a-ix)}{2}\right]$$

$$\int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{\hbar}}\,e^{-\frac{pa}{\hbar}}\,e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)} =$$

$$= \sqrt{\frac{ma}{2\pi\hbar t}}\,(1+i)\,F\left[(1+i)\left(\frac{m}{\hbar t}\right)^{1/2}\frac{(a-ix)}{2}\right] \qquad (8.1)$$

$F(u)$, con $u \in \mathbb{C}$, es la función de Dawson que es la solución al siguiente problema:

$$\frac{dF(u)}{du} = -2uF(u) + 1$$

$$F(0) = 0$$

$F(u)$ es una función entera en $\mathbb{C}$, cuyo desarrollo en serie es

$$F(u) = u\sum_{k=0}^{\infty}\frac{(-1)^k k!}{(2k+1)!}(2u)^{2k} \qquad u \in \mathbb{C}$$

La serie converge $\forall u$.

(8.1) conduce a

$$\int_0^\infty \frac{dp}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}}\, e^{-\frac{pa}{\hbar}}\, e^{-i/\hbar\left(\frac{p^2 t}{2m} + px\right)} =$$

$$= \sqrt{\frac{ma}{2\pi\hbar t}}\,(1+i)\,F\left((1+i)\left(\frac{m}{\hbar t}\right)^{1/2}\frac{(a+ix)}{2}\right) \qquad (9.1)$$

(8.1), (9.1) y (7.1) dan lugar al siguiente resultado

$$\boxed{\psi(x,t) = \sqrt{\frac{ma}{2\pi\hbar t}}\,(1+i)\left\{F\left[(1+i)\left(\frac{m}{\hbar t}\right)^{1/2}\left(\frac{a-ix}{2}\right)\right] + F\left[(1+i)\left(\frac{m}{\hbar t}\right)^{1/2}\left(\frac{a+ix}{2}\right)\right]\right\}}$$