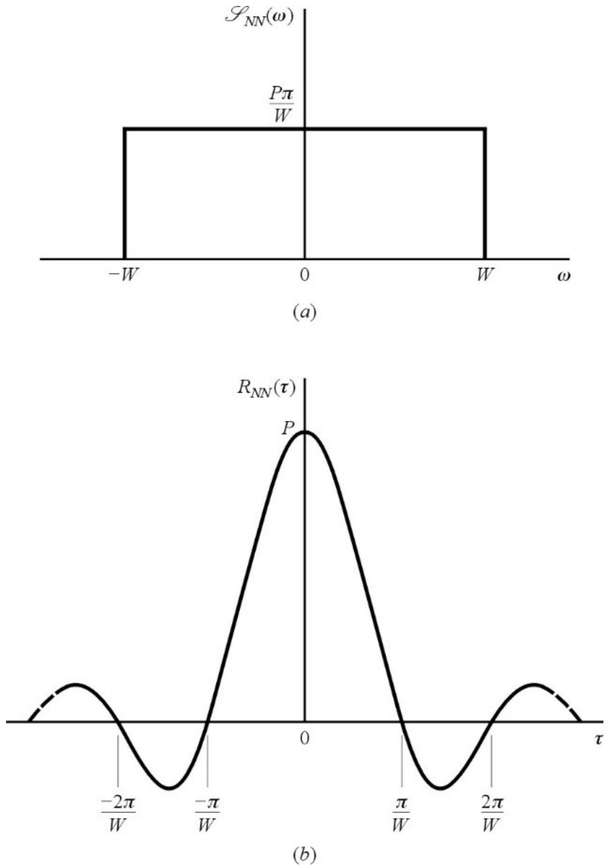
CEU
Universidad
San Pablo

UNIT 2 – Part II: Random Processes
Spectral Characteristics

Gabriel Caffarena Fernández
3rd Year Biomedical Engineering Degree
EPS – Univ. San Pablo – CEU

Spectral characteristics of Random Processes

- A random process can be studied in the frequency domain by means of the **power density spectrum**
- The power density spectrum is also related to the autocorrelation
- It can be used to analyze the frequency components of an r.p.
- Also, it can be used to analyze the effect that LTI systems have on r.p.



Spectral characteristics of Random Processes

- The spectral properties of a deterministic signal *x(t)* are contained in its *Fourier Transform* ***X(ω)***

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega}\ dt$$

- However the FT cannot be directly applied to a r.p., due to the fact that not all sample functions have a valid FT
- However, using a ***power density function*** solves this problem, leading to the

Power Density Spectrum

Power Density Spectrum

- First, let us define the **portion sample function** $\mathrm{x_T}(t)$

$$\mathrm{x_T}(t) = \begin{cases} x(t) & -T < t < T \\ 0 & elsewhere \end{cases}$$

- The FT of $\mathrm{x_T}(t)$ is

$$X_{\mathrm{T}}(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt = \int_{-T}^{T} x(t) e^{-j\omega} \ dt$$

- The energy of $\mathrm{x_T}(t)$ is called $\mathrm{E(T)}$ and it is related to both $\mathrm{x(t)}$ and $X_{\mathrm{T}}(\omega)$ (by means of Parserval's theorem)

$$E(T) = \int_{-T}^{T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

Power Density Spectrum

- We can obtain the power of $x_T(t)$ by averaging its energy

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

- We can infer that $\frac{|X_T(\omega)|^2}{2T}$ is a power density spectrum

---

- To obtain the power of a r.p. *X(t)*, P_{XX}, we have to account for the following:
  - *T* must tend to ∞
  - We must use *X(t)* instead of *x(t), which is random*
- So, we have to
  - Apply the limit when T→∞
  - Use $E[X^2(t)]$ instead of $X^2(t)$

Power Density Spectrum

- So now we can define the power of a r.p. *X(t)* as

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega) d\omega \ [\mathrm{W}]$$

- We call $\mathcal{S}_{XX}(\omega)[\frac{\mathrm{W}}{\mathrm{Hz}}]$ the **power density function**
- Note that

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]dt = \mathrm{A}[E[X^2(t)]] \xrightarrow{X(t) \text{ w.s.s.}} P_{XX} = E[X^2(t)] = R_{XX}(0)$$

Power Density Spectrum

- Example: Find the average power of the r.p.

$$\mathrm{X(t)} = \mathrm{A}_0 \cos(\omega_0 t + \Theta)$$

where A_0 and ω_0 are constants and $\Theta \sim U(0, \frac{\pi}{2})$

$$P_{XX} = \mathrm{A}[E[X^2(t)]] = \mathrm{A}[\frac{A_0^2}{2} - \frac{A_0^2}{\pi} sin(2\omega_0 t) \quad] = \frac{A_0^2}{2}$$

Properties of the Power Density Spectrum

- $\mathcal{S}_{XX}(\omega)$ is real and ≥ 0
- $\mathcal{S}_{XX}(-\omega) = \mathcal{S}_{XX}(\omega)$
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)d\omega = \mathrm{A}[E[X^2(t)]] = P_{XX}$
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)e^{j\omega\tau}d\omega = \mathrm{A}[\mathrm{R_{XX}}(\mathrm{t}, \mathrm{t}+\tau)]$ $\xrightarrow{X(t) \text{ w.s.s.}}$ $\mathrm{R_{XX}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)e^{j\omega\tau}d\omega$
- $\mathcal{S}_{XX}(\omega) = \int_{-\infty}^{\infty} \mathrm{R_{XX}}(\mathrm{t}, \mathrm{t}+\tau)e^{-j\omega\tau}d\tau$ $\xrightarrow{X(t) \text{ w.s.s.}}$ $\mathcal{S}_{XX}(\omega) = \int_{-\infty}^{\infty} \mathrm{R_{XX}}(\tau)e^{-j\omega\tau}d\tau$

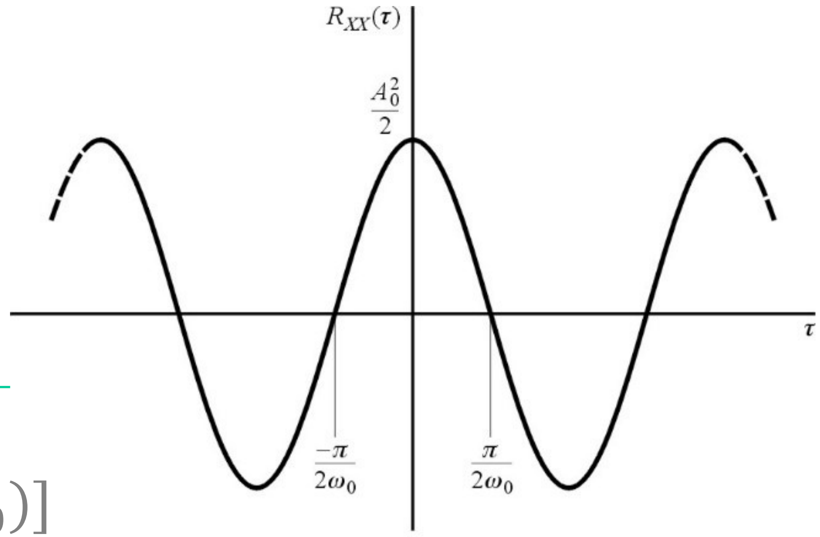
Properties of Power Density Spectrum

- Example: Find the p.d.s. of r.p. with autocorrelation

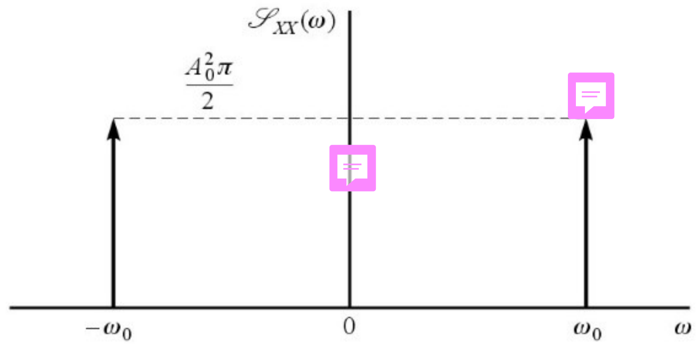
$$R_{XX}(\tau) = (A_0^2/2)\cos(\omega_0 \tau)$$

where A_0 and ω_0 are constants

$$\mathcal{S}_{XX}(\omega) = TF\{R_{XX}(\tau)\} = (A_0^2\pi/2))[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



(a)



(b)

The Cross-Power Density Spectrum

- We define the cross-power of r.p. X(t) and Y(t) as

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X(t)Y(t)]dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_{XY}(\omega) d\omega$$

- We call $\mathcal{S}_{XY}(\omega)$ the cross-**power density function**
- Note that

$$P_{XY} = \mathrm{A}[\mathrm{E}[X(t)Y(t)]] \xrightarrow{\text{j.}w\text{.s.s.}} P_{XY} = E[X(t)Y(t)] = R_{XY}(0)$$

Properties of the Cross-Power Density Spectrum

- $\mathcal{S}_{XY}(\omega) = \mathcal{S}_{YX}(-\omega) = \mathcal{S}_{YX}^{*}(\omega)$
- $Re[\mathcal{S}_{XY}(\omega)]$ and $Re[\mathcal{S}_{YX}(\omega)]$ are **even** functions of ω
- $Im[\mathcal{S}_{XY}(\omega)]$ and $Im[\mathcal{S}_{YX}(\omega)]$ are **odd** functions of ω
- If X(t) and Y(t) are **orthogonal** then $\mathcal{S}_{XY}(\omega) = \mathcal{S}_{YX}(\omega) = 0$
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XY}(\omega)d\omega = A[E[X(t)Y(t)]] = P_{XY}$ $\xrightarrow{\text{j.w.s.s.}}$ $P_{XY} = E[X(t)Y(t)] = R_{XY}(0)$

Properties of the Cross-Power Density Spectrum

- $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XY}(\omega)d\omega = \mathrm{A}[\mathrm{E}[X(t)Y(t)]] = P_{XY}$ $\xrightarrow{\text{j.w.s.s.}}$ $P_{XY} = E[X(t)Y(t)] = R_{XY}(0)$
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XY}(\omega)e^{j\omega\tau}d\omega = \mathrm{A}[\mathrm{R}_{XY}(\mathrm{t},\mathrm{t}+\tau)]$ $\xrightarrow{\text{j.w.s.s.}}$ $\mathrm{R}_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XY}(\omega)e^{j\omega\tau}d\omega$
- $\mathcal{S}_{XY}(\omega) = \int_{-\infty}^{\infty} \mathrm{R}_{XY}(\mathrm{t},\mathrm{t}+\tau)e^{-j\omega\tau}d\tau$ $\xrightarrow{\text{j.w.s.s.}}$ $\mathcal{S}_{XY}(\omega) = \int_{-\infty}^{\infty} \mathrm{R}_{XY}(\tau)e^{-j\omega\tau}d\tau$
- If X(t) and Y(t) are **uncorrelated** with **constant means** $\bar{X}$ and $\bar{Y}$ then

$$\mathcal{S}_{XY}(\omega) = 2\pi\overline{XY}\delta(\omega)$$

Noise

- Noise in present in bioinstrumentation systems
- Noise degrades the quality of the signal under study
- It is interesting to characterize noise through the power density spectrum
- Knowing the spectrum of noise helps to design better bioinstrumentation systems
  - For instance, we can filter noise that is outside the bandwidth of the signal

White Noise

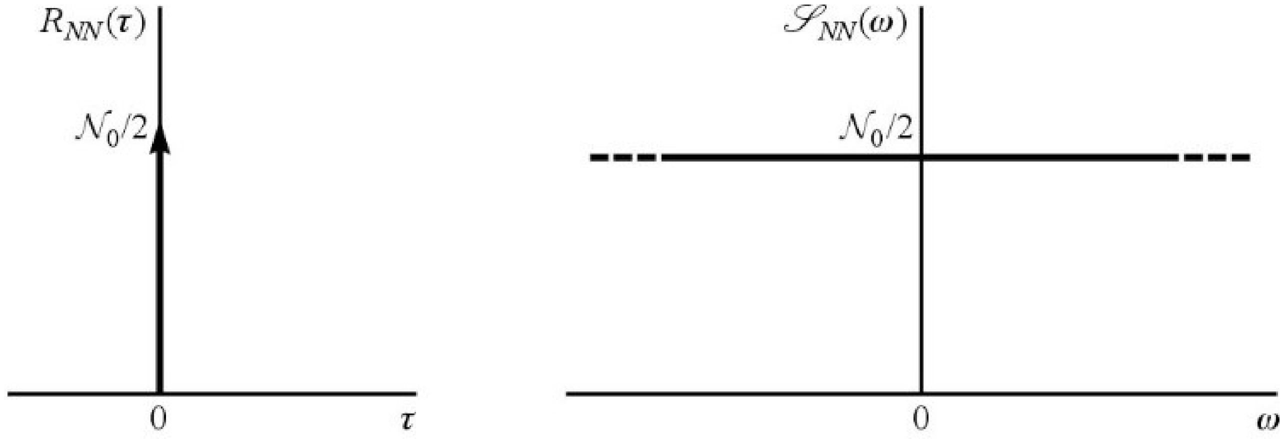
- A white noise has a constant p.d.s.

$$\mathcal{S}_{NN}(\omega) = N_0/2$$

where N_0 is a real positive constant

- The autocorrelation is

$$\mathrm{R_{NN}}(\tau) = \left(\frac{N_0}{2}\right)\delta(\tau)$$



- It is unrealizable since its power is infinite
- However, there are real cases where noise is almost constant for a wide bandwidth, so it can be approximated as a white noise

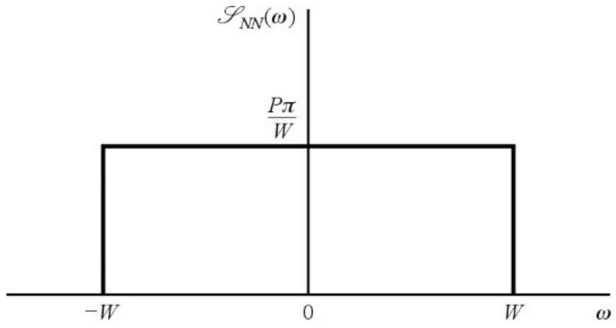
Band-limited White Noise

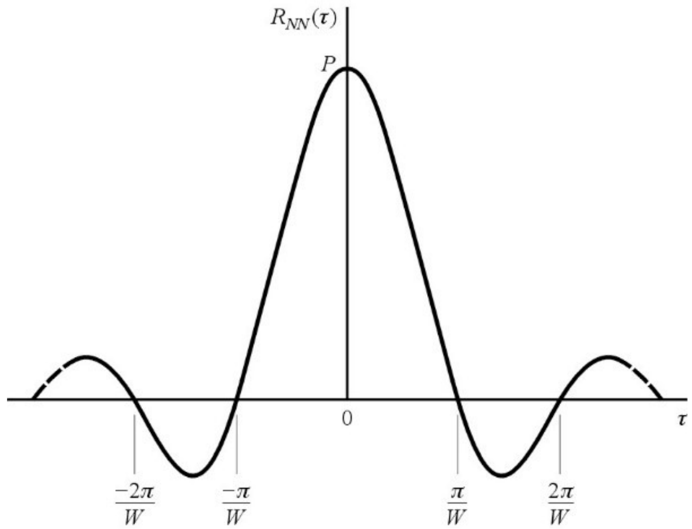
- White noise is filtered to reduce its effect on the quality of a processing algorithm
- If we assume an ideal filter, the resulting p.d.s. is constant in a limited interval of frequencies
- If a lowpass filter is applied to the noise (because the signal is lowpass), the resulting p.d.s and autocorrelation are

$$\mathcal{S}_{NN}(\omega) = \begin{cases} P\pi/W & -W < \omega < W \\ 0 & elsewhere \end{cases}$$

- The autocorrelation is

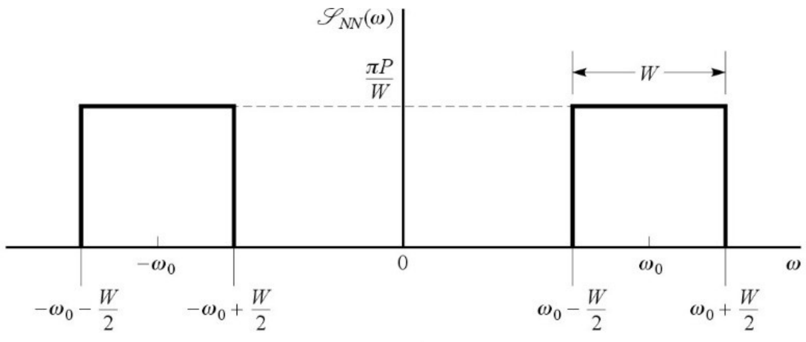
$$\mathrm{R}_{\mathrm{NN}}(\tau) = P\,\frac{\sin(W\tau)}{W\tau}$$





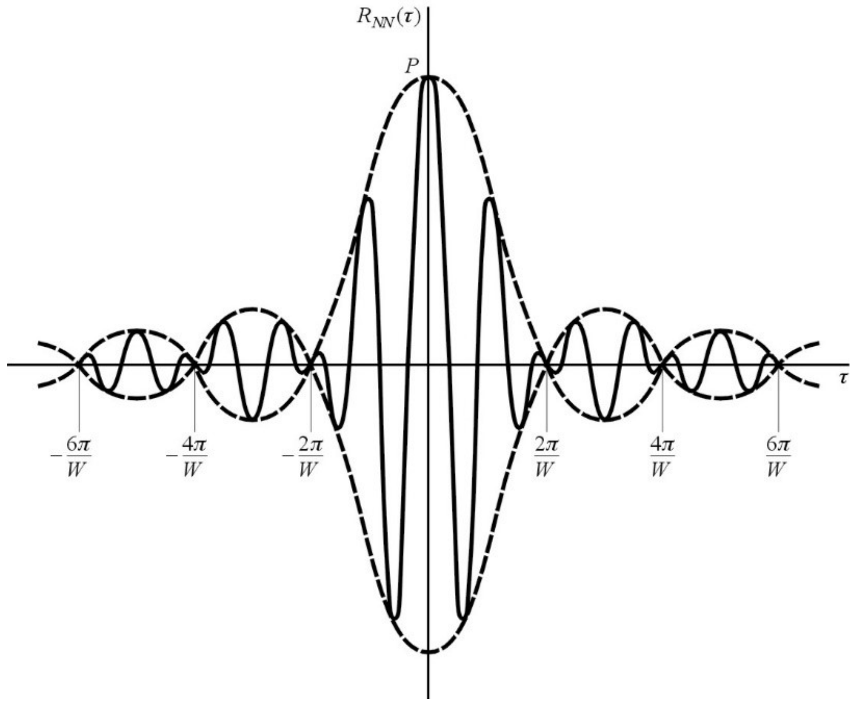
Band-limited White Noise

- If a bandpass filter is applied to the noise (because the signal is bandpass), the resulting p.d.s and autocorrelation are



$$\mathcal{S}_{NN}(\omega) = \begin{cases} P\pi/W & \omega_0 - \left(\frac{W}{2}\right) < |\omega| < \omega_0 + \left(\frac{W}{2}\right) \\ 0 & elsewhere \end{cases}$$

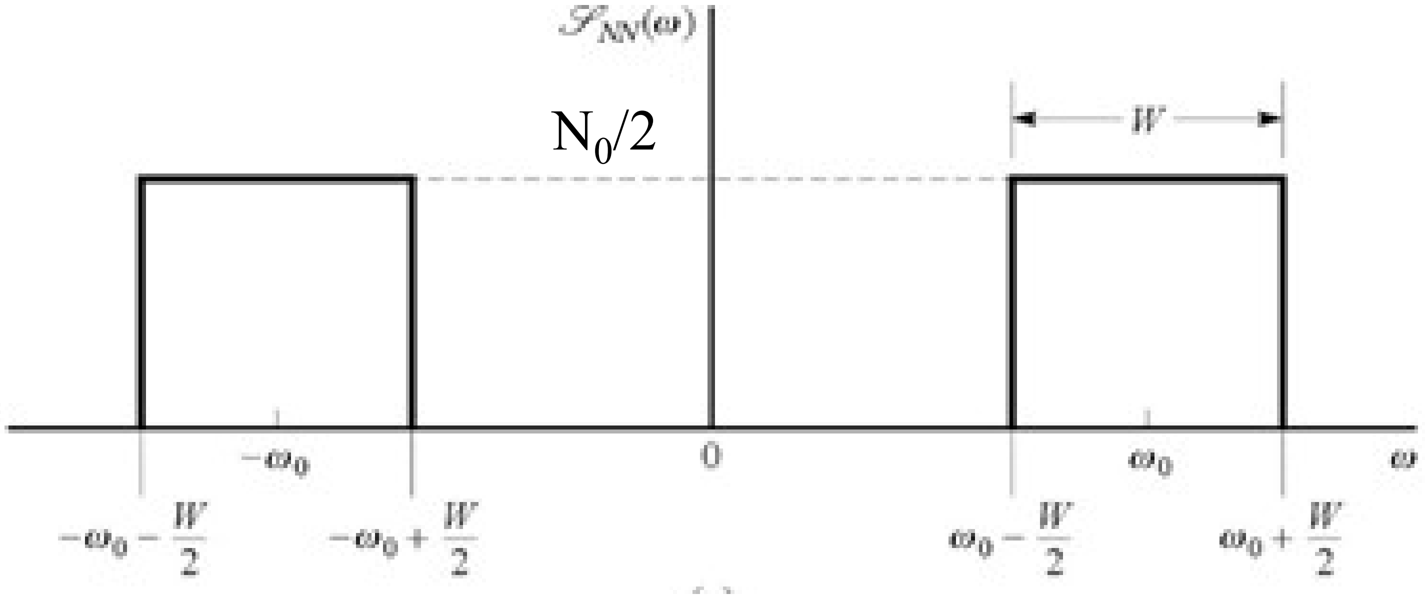
- The autocorrelation is



$$\mathrm{R}_{\mathrm{NN}}(\tau) = P\,\frac{\sin\left(\frac{W\tau}{2}\right)}{\frac{W\tau}{2}}\cos(\omega_0\tau)$$

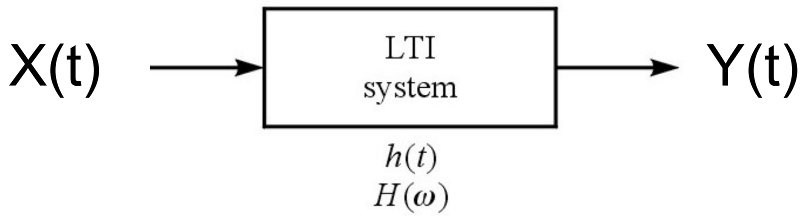
Band-limited White Noise

CHALLENGE: Compute the power of a band-limited white noise as a function of W ($W<2\omega_0$)



Linear systems with random inputs

- Consider an LTI system fed with a random process X(t) with known $R_{XX}(t)$

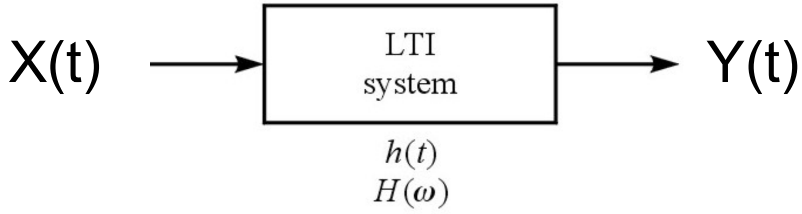


- It is possible to find
  - The mean, variance and autocorrelation of the output: $\bar{Y},\ \sigma_Y^2, R_{YY}(\tau)$
  - The p.d.s. of Y(t) → $\mathcal{S}_{YY}(\omega)$
  - The cross-correlation between the input and the output of the system

    $R_{XY}(\tau)$ $R_{YX}(\tau)$
  - The cross-p.d.s. between the input and the output of the system

    $\mathcal{S}_{XY}(\omega)$ $\mathcal{S}_{YX}(\omega)$

Mean and second moment of Y(t)

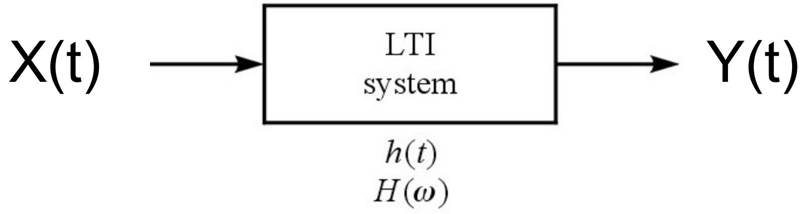


- We base our calculations on the fact that Y(t)=X(t)*h(t)
- We regard X(t) and Y(t) as **j.w.s.s.**
- For the mean:

$$E[Y(t)]=E\left[\int_{-\infty}^{\infty} h(u)X(t-u)du\right]=\int_{-\infty}^{\infty} h(u)E[X(t-u)]du$$

$$=\bar{X}\int_{-\infty}^{\infty} h(u)du=\bar{Y}$$

Mean and second order of Y(t)



- As for the second order moment:

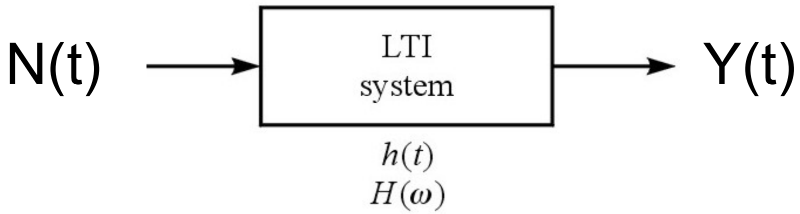
$$E[Y^2(t)] = E\left[\int_{-\infty}^{\infty} h(u_1)X(t-u_1)du_1 \quad \int_{-\infty}^{\infty} h(u_2)X(t-u_2)du_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-u_1)X(t-u_2)]\, h(u_1)h(u_2)du_1du_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(u_1-u_2)\, h(u_1)h(u_2)du_1du_2$$

Mean and second order of Y(t)

Example: LTI system with white noise as input:



$$\bar{N} = 0 \quad \text{and} \quad R_{NN}(\tau) = \left(\frac{N_0}{2}\right)\delta(\tau)$$

$$\bar{Y} = 0$$

$$E[Y^2(t)] = \left(\frac{N_0}{2}\right)\int_{-\infty}^{\infty} h^2(u)du$$

Autocorrelation of Y(t)

- For the autocorrelation we have:

$$R_{YY}(t,t+\tau) = \mathrm{E}[Y(t)Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u_1)X(t-u_1)du_1 \quad \int_{-\infty}^{\infty} h(u_2)X(t+\tau-u_2)du_2 \right]$$

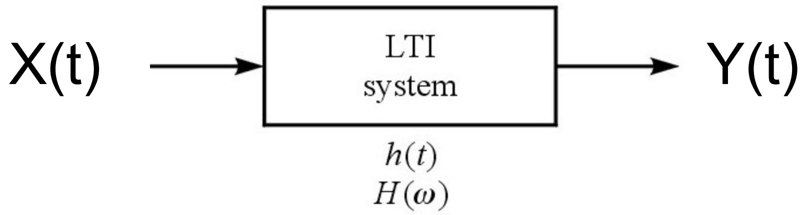
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-u_1)X(t+\tau-u_2)]\, h(u_1)h(u_2)du_1du_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+u_1-u_2)\, h(u_!)h(u_2)du_1du_2$$

$$= R_{XX}(\tau) * h(-\tau) * h(\tau)$$

$$\boldsymbol{R_{YY}(t,t+\tau) = R_{XX(\tau)} * h(-\tau) * h(\tau)}$$

Cross-Correlation of Y(t)



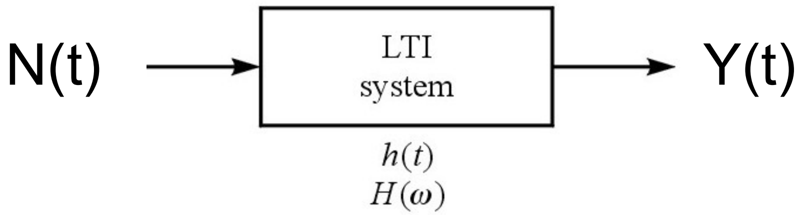
- Following an analysis similar to the one we applied to the autocorrelation:

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$
$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$$
$$R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$$

Cross-correlation of Y(t)

Example: Compute R_{YN} and R_{NY} given the following LTI system with a white noise as input:

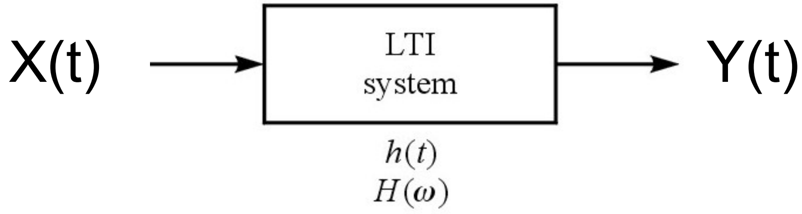


$$\bar{N} = 0 \quad \text{and} \quad R_{NN}(\tau) = \left(\frac{N_0}{2}\right)\delta(\tau)$$

$$R_{NY}(\tau) = \left(\frac{N_0}{2}\right)h(\tau)$$

$$R_{YN}(\tau) = \left(\frac{N_0}{2}\right)h(-\tau) = R_{NY}(-\tau)$$

Power Density Spectrum of Y(t)



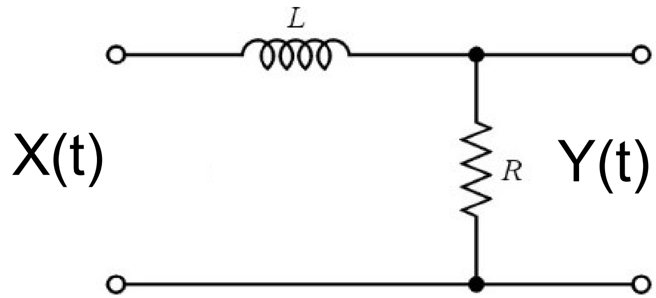
- Given that the auto- and cross-p.d.s. are the Fourier Transform of the auto- and cross-correlations:
  - $R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$ → $S_{XY}(\omega) = S_{XX}(\omega)H(\omega)$
  - $R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$ → $S_{YX}(\omega) = S_{XX}(\omega)H^*(\omega)$
  - $R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$ → $S_{YY}(\omega) = S_{XY}(\omega)H^*(\omega)$
  - $R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$ → $S_{YY}(\omega) = S_{YX}(\omega)H(\omega)$

  - $R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$ → $S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$

Cross-correlation of Y(t)

Example: Find the p.d.s. and power of of Y(t) :

$$\mathcal{S}_{NN}(\omega) = \left(\frac{N_0}{2}\right)$$



$$|H(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega L}{R}\right)^2}$$

$$\mathcal{S}_{YY}(\omega) = \frac{N_0/2}{1+\left(\frac{\omega L}{R}\right)^2}$$

$$P_{YY} = \frac{N_0 R}{4L}$$

SUMMARY

- Power density spectrum
- Noise
- Random processes and LTI systems