CEU
Universidad
San Pablo

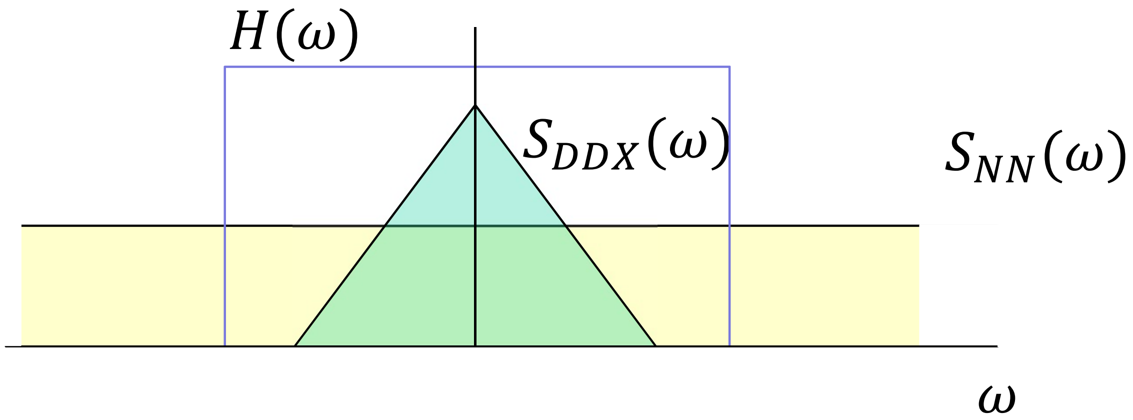
UNIT 4: Optimal Filtering

Gabriel Caffarena Fernández
3rd Year Biomedical Engineering Degree
EPS – Univ. San Pablo – CEU
(based on “Biomedical Signal Analysis”, 2nd edition, © Willey 2015)

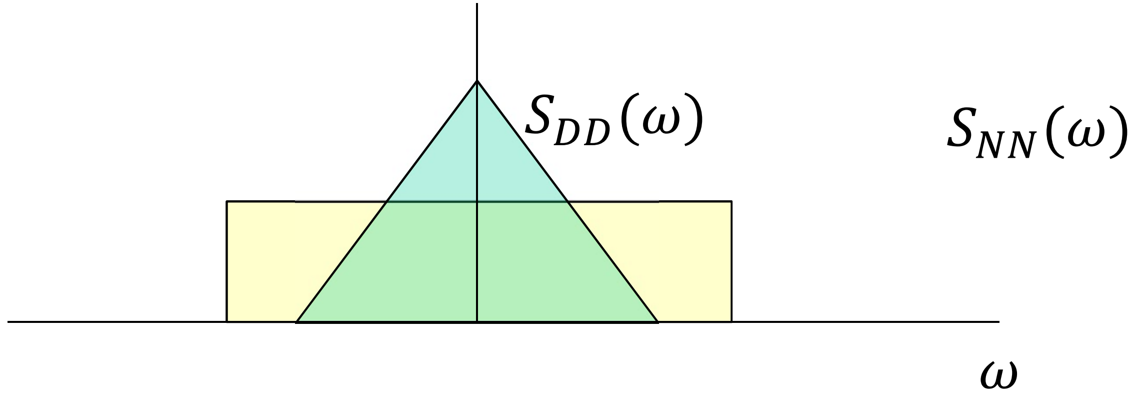
Noise filtering

- A common technique to reduce the effect of noise is filtering with a band pass equal to the frequency band of the signal of interest. The effect is a reduction in the signal to noise ratio (SNR)

$$SNR = 10\log\left(\frac{P_D}{P_N}\right)$$



SNR is low



SNR is increased

Noise filtering

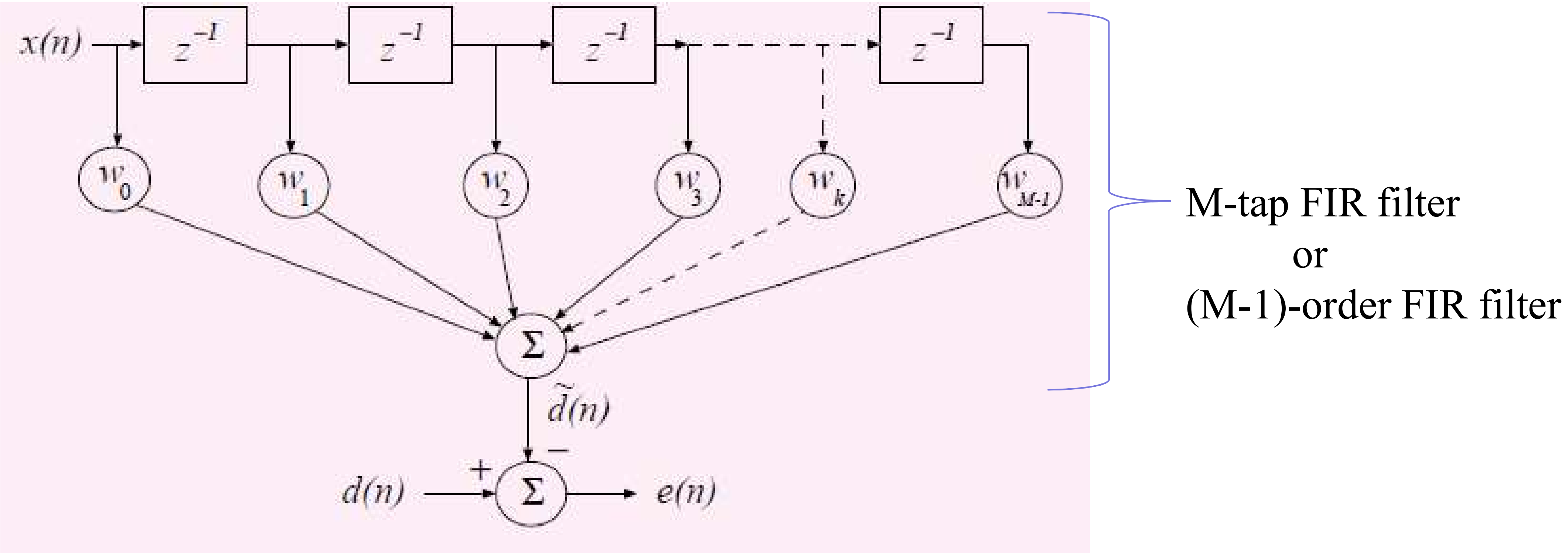
- In order to do so, we have to select the following:
  - Type of filter: **FIR** or **IIR** (digital domain)
  - Order of filter
  - Location of poles and zeros
- The previous decisions lead to many different filter properties that can be desired or not:
  - Flat or rippled band pass
  - Phase distortion or linear phase
  - Smooth or abrupt transition band
- However, it is possible to design a digital filter considering the **statistical properties** of signal and noise and **maximizing SNR**

Optimal filtering

- **Problem**:
  - Design an optimal filter to remove noise from a signal, given that the signal and noise processes are random, independent and stationary.
  - We assume the "desired" or ideal characteristics of the uncorrupted signal as well as characteristics of the noise **to be known**.
- **Solution**:
  - Wiener filter theory provides a means to optimize filter parameters with reference to a performance criterion.
  - The output is guaranteed to be the best achievable result

The Wiener filter

- **d(n)** is the <u>desired</u> signal
- **x(n)** is d(n) plus noise
- **x(n)** is filtered producing an estimate $\hat{d}(n)$ of the desired signal
- Our goal is to minimized the error $\mathbf{e(n) = d(n)} - \hat{d}(n)$



The Wiener filter

- The filter has M tap weigths $\mathrm{w_i}, \mathrm{i} \in \{0,1,2,\dots,M-1\}$, so for an FIR filter the impulse reponse is $\mathrm{h(n)} = \begin{cases} w_n, n \in [0, M-1] \\ 0, otherwise \end{cases}$
- The output of the filter is the convolution of the impulse response with **x(n)**

$$\hat{d}(n) = h(n) * x(n) = \sum_{k=0}^{M-1} h(n)x(n-k) = \sum_{k=0}^{M-1} w_k x(n-k)$$

- Let's introduce vectors to use more compact expressions

$\vec{w} = [w_0, w_1, w_2, \dots, w_{M-1}]^T$ **coefficient vector**

$\vec{x}(n) = [x(n), x(n-1), \dots, x(n-(M-1))]^T$ **delayed input vector**

NOTE that they are column vectors

- So now we can rewrite $\widehat{\boldsymbol{d}}(\boldsymbol{n})$ and $\boldsymbol{e}(\boldsymbol{n})$

$$\hat{d}(n) = \sum_{k=0}^{M-1} w_k x(n-k) = \vec{w}^T \vec{x}(n)$$

$$e(n) = d(n) - \hat{d}(n) = d(n) - \vec{w}^T \vec{x}(n)$$

The Wiener filter

- The criterion used to optimize the error is to minimize the **Mean Squared Error**

$$\mathrm{MSE} = \mathrm{E}[\mathrm{e}^2(\mathrm{n})]$$

$$= \mathrm{E}\left[\left(d(n) - \hat{d}(n)\right)^2\right]$$

$$= \mathrm{E}[(d(n) - \vec{w}^T\vec{x}(n))(d(n) - \vec{x}(n)^T\vec{w})]$$

$$= \mathrm{E}[d^2(n)\ ] - \vec{w}^T E[\vec{x}(n)d(n)] - E[d(n)\vec{x}(n)^T]\vec{w} + \vec{w}^T E[\vec{x}(n)\vec{x}(n)^T]\ \vec{w}$$

The Wiener filter

- Let's now add vector $\overrightarrow{R_{XD}}$

$$\mathrm{MSE} = \mathrm{E}[\mathrm{e}^2(\mathrm{n})]$$

$$= \mathrm{E}[d^2(n)\,] - \vec{w}^T E[\vec{x}(n)d(n)] - E[d(n)\vec{x}(n)^T]\vec{w} + \vec{w}^T E[\vec{x}(n)\vec{x}(n)^T]\,\vec{w}$$

$$E[\vec{x}(n)d(n)] = \left[E[x(n)d(n)], E[x(n-1)d(n)], \ldots, E[x(n-(M-1))d(n)]\right]^T$$

$$= [R_{XD}(0), R_{XD}(1), \ldots, R_{XD}(M-1)]^T \;\; = \overrightarrow{R_{XD}}$$

$$E[d(n)\vec{x}(n)^T] = \left[E[d(n)x(n)], E[d(n)x(n-1)], \ldots, E[d(n)x(n-(M-1))d(n)]\right]$$

$$= [R_{DX}(0), R_{DX}(-1), \ldots, R_{DX}(-(M-1))] \;\; =$$

$$= [R_{XD}(0), R_{XD}(1), \ldots, R_{XD}(M-1)] \;\; = \overrightarrow{R_{XD}}^T$$

The Wiener filter

- Let's now add vector $\overrightarrow{R_{XD}}$

$$\text{MSE} = \text{E}[\text{e}^2(\text{n})]$$

$$= \text{E}[d^2(n)] - \vec{w}^T E[\vec{x}(n)d(n)] - E[d(n)\vec{x}(n)^T]\vec{w} + \vec{w}^T E[\vec{x}(n)\vec{x}(n)^T]\,\vec{w}$$

$$= \text{E}[d^2(n)] - \vec{w}^T\,\overrightarrow{R_{XD}} - \overrightarrow{R_{XD}}^T\,\vec{w} + \vec{w}^T E[\vec{x}(n)\vec{x}(n)^T]\,\vec{w}$$

$$E[\vec{x}(n)d(n)] = [R_{XD}(0), R_{XD}(1), \dots, R_{XD}(M-1)]^T = \overrightarrow{R_{XD}}$$

$$E[d(n)\vec{x}(n)^T] = [R_{DX}(0), R_{DX}(-1), \dots, R_{DX}(-(M-1))] = \overrightarrow{R_{XD}}^T$$

- It is trivial to check that $\vec{w}^T\,\overrightarrow{R_{XD}} = \overrightarrow{R_{XD}}^T\,\vec{w}$, so

$R_{XX}?$

$$\text{MSE} = \text{E}[\text{e}^2(\text{n})] = \text{E}[d^2(n)] - 2\vec{w}^T\,\overrightarrow{R_{XD}} + \vec{w}^T E[\vec{x}(n)\vec{x}(n)^T]\,\vec{w}$$

The Wiener filter

- And finally, matrix $\mathbf{R_{XX}}$

$$\mathrm{MSE} = \mathrm{E}[\mathrm{e}^2(\mathrm{n})]$$

$$= \mathrm{E}[d^2(n)\,] - 2\vec{w}^T\,\overrightarrow{R_{XD}} + \vec{w}^T\boldsymbol{R_{XX}}\,\vec{w}$$

$$\mathbf{R_{XX}} = E[\vec{x}(n)\vec{x}(n)^T]$$

$$= \begin{pmatrix} E[x(n)x(n)] & E[x(n)x(n-1)] & \dots & E[x(n)x(n-M+1)] \\ E[x(n-1)x(n)] & E[x(n-1)x(n-1)] & \cdots & E[x(n-1)x(n-M+1)] \\ \vdots & \vdots & \ddots & \vdots \\ E[x(n-M+1)x(n)] & E[x(n-M+1)x(n-1)] & \cdots & E[x(n-M+1)x(n-M+1)] \end{pmatrix}$$

$$= \begin{pmatrix} R_{XX}(0) & R_{XX}(-1)] & \dots & R_{XX}(1-M) \\ R_{XX}(1) & R_{XX}(0)] & \cdots & R_{XX}(2-M) \\ \vdots & \vdots & \ddots & \vdots \\ R_{XX}(M-1) & R_{XX}(M-2)] & \cdots & R_{XX}(0)] \end{pmatrix}$$

The Wiener filter

- Assuming that **d(n)** has zero mean, $\mathrm{E}[d^2(n)] = \sigma_d^2$, therefore

$$\mathrm{MSE} = \sigma_d^2 - 2\vec{w}^T\,\overrightarrow{R_{XD}} + \vec{w}^T \boldsymbol{R_{XX}}\,\vec{w}$$

$$\overrightarrow{R_{XD}} = E[\vec{x}(n)d(n)] = \big[E[x(n)d(n)], E[x(n-1)d(n)], \dots, E[x(n-(M-1))d(n)]\big]^T$$

$$= [R_{XD}(0), R_{XD}(1), \dots, R_{XD}(M-1)]^T$$

$$\mathbf{R_{XX}} = \begin{pmatrix} R_{XX}(0) & R_{XX}(-1)] & \dots & R_{XX}(-(M-1)) \\ R_{XX}(1) & R_{XX}(0)] & \cdots & R_{XX}(-(M-2)) \\ \vdots & \vdots & \ddots & \vdots \\ R_{XX}(M-1) & R_{XX}(M-2)] & \cdots & R_{XX}(0)] \end{pmatrix}$$

The Wiener filter: optimal coefficients

- Let us now obtain the derivative of the MSE to find the minimum

$$\text{MSE} = \sigma_d^2 - 2\vec{w}^T \overrightarrow{R_{XD}} + \vec{w}^T \boldsymbol{R_{XX}} \vec{w}$$

$$\frac{dMSE}{d\vec{w}} = -\mathbf{2}\frac{d\left(\vec{w}^T \overrightarrow{R_{XD}}\right)}{d\vec{w}} + \frac{d\left(\vec{w}^T \boldsymbol{R_{XX}} \vec{w}\right)}{d\vec{w}}$$

$$\frac{d\left(\vec{w}^T \overrightarrow{R_{XD}}\right)}{d\vec{w}} = \left[\frac{\partial\left(\vec{w}^T \overrightarrow{R_{XD}}\right)}{\partial w_0}, \frac{\partial\left(\vec{w}^T \overrightarrow{R_{XD}}\right)}{\partial w_1}, \dots, \frac{\partial\left(\vec{w}^T \overrightarrow{R_{XD}}\right)}{\partial w_{M-1}}\right]^{\boldsymbol{T}}$$

$$= \left[\frac{\partial\left(\sum_{i=0}^{M-1} \vec{w}(i)\overrightarrow{R_{XD}}(i)\right)}{\partial w_0}, \frac{\partial\left(\sum_{i=0}^{M-1} \overbrace{\vec{w}(i)}^{w_i}\overrightarrow{R_{XD}}(i)\right)}{\partial w_1}, \dots, \frac{\partial\left(\sum_{i=0}^{M-1} \vec{w}(i)\overrightarrow{R_{XD}}(i)\right)}{\partial w_{M-1}}\right]^{\boldsymbol{T}}$$

$$= \left[\overrightarrow{R_{XD}}(0), \overrightarrow{R_{XD}}(1), \dots, \overrightarrow{R_{XD}}(M-1)\right]^{\boldsymbol{T}}$$

$$= \left[R_{XD}(0), R_{XD}\,(1), \dots, R_{XD}\,(M-1)\right]^{\boldsymbol{T}} = \overrightarrow{R_{XD}}$$

The Wiener filter: optimal coefficients

$$\frac{dMSE}{d\vec{w}} = -2\frac{d\left(\vec{w}^T\,\overrightarrow{R_{XD}}\right)}{d\vec{w}} + \frac{d\left(\vec{w}^T\boldsymbol{R_{XX}}\,\vec{w}\right)}{d\vec{w}} = -2\overrightarrow{R_{XD}} + \frac{d\left(\vec{w}^T\boldsymbol{R_{XX}}\,\vec{w}\right)}{d\vec{w}}$$

$$\frac{d(\vec{w}^T\boldsymbol{R_{XX}}\,\vec{w})}{d\vec{w}} = \frac{d\left(\sum_{i=0}^{M-1} w_i \sum_{j=0}^{M-1} w_j R_{XX}(j-i)\right)}{d\vec{w}}$$

$$= \Bigg[\sum_{i=0}^{M-1} w_i\,R_{XX}(i) + \sum_{j=0}^{M-1} w_j\,R_{XX}(-j), \quad \sum_{i=0}^{M-1} w_i\,R_{XX}(i-1) + \sum_{j=0}^{M-1} w_j\,R_{XX}(1-j),$$

$$\dots\quad , \sum_{i=0}^{M-1} w_i\,R_{XX}\big(i-(M-1)\big) + \sum_{j=0}^{M-1} w_j\,R_{XX}\big((M-1)-j\big)\Bigg]^T$$

$$= \left[\sum_{i=0}^{M-1} w_i\big(R_{XX}(i) + R_{XX}(-j)\big), \sum_{i=0}^{M-1} w_i\big(R_{XX}(i-1) + R_{XX}(1-j)\big),\right.$$

$$\left.\dots, \sum_{i=0}^{M-1} w_i\big(R_{XX}(i-(M-1)) + R_{XX}((M-1)-j)\big)\right]^T$$

X is stationary

$$= \left[2\sum_{i=0}^{M-1} w_i R_{XX}(-i), 2\sum_{i=0}^{M-1} w_i R_{XX}(1-i), \dots, 2\sum_{i=0}^{M-1} w_i R_{XX}((M-1)-i)\right]^T$$

$$= 2\boldsymbol{R_{XX}}\vec{w}$$

The Wiener filter: optimal coefficients

- To minimize the MSE we set its derivative to zero

$$\frac{dMSE}{d\vec{w}} = -2\overrightarrow{R_{XD}} + 2\boldsymbol{R_{XX}}\,\vec{w} = 0$$

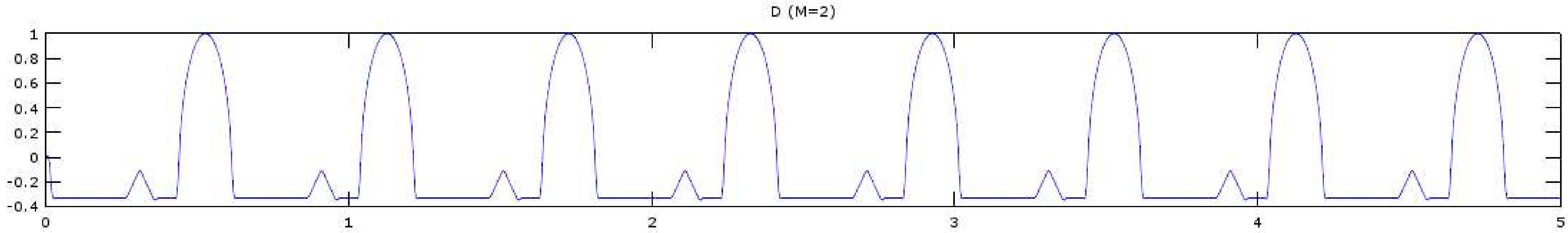
- Therefore

$$\boldsymbol{R_{XX}}\,\vec{w} = \overrightarrow{R_{XD}}$$

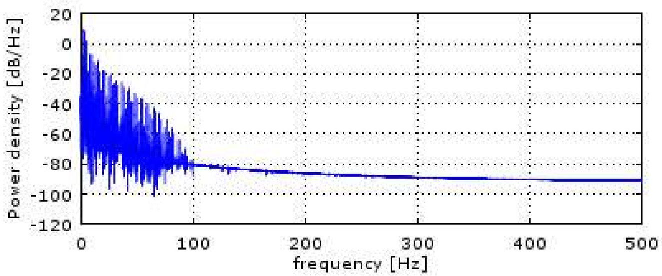
$$\vec{w} = \boldsymbol{R_{XX}^{-1}}\overrightarrow{R_{XD}}$$

The Wiener filter: Case study

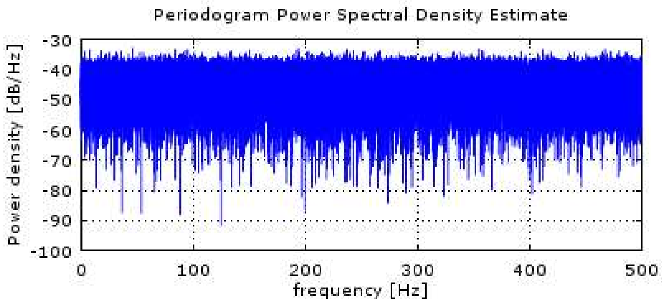
- A synthetic ECG is corrupted with white noise
- The original SNR is 10 dB
- The corrupted signal is filtered using Wiener filters with M taps (order M-1)



Sdd(f)



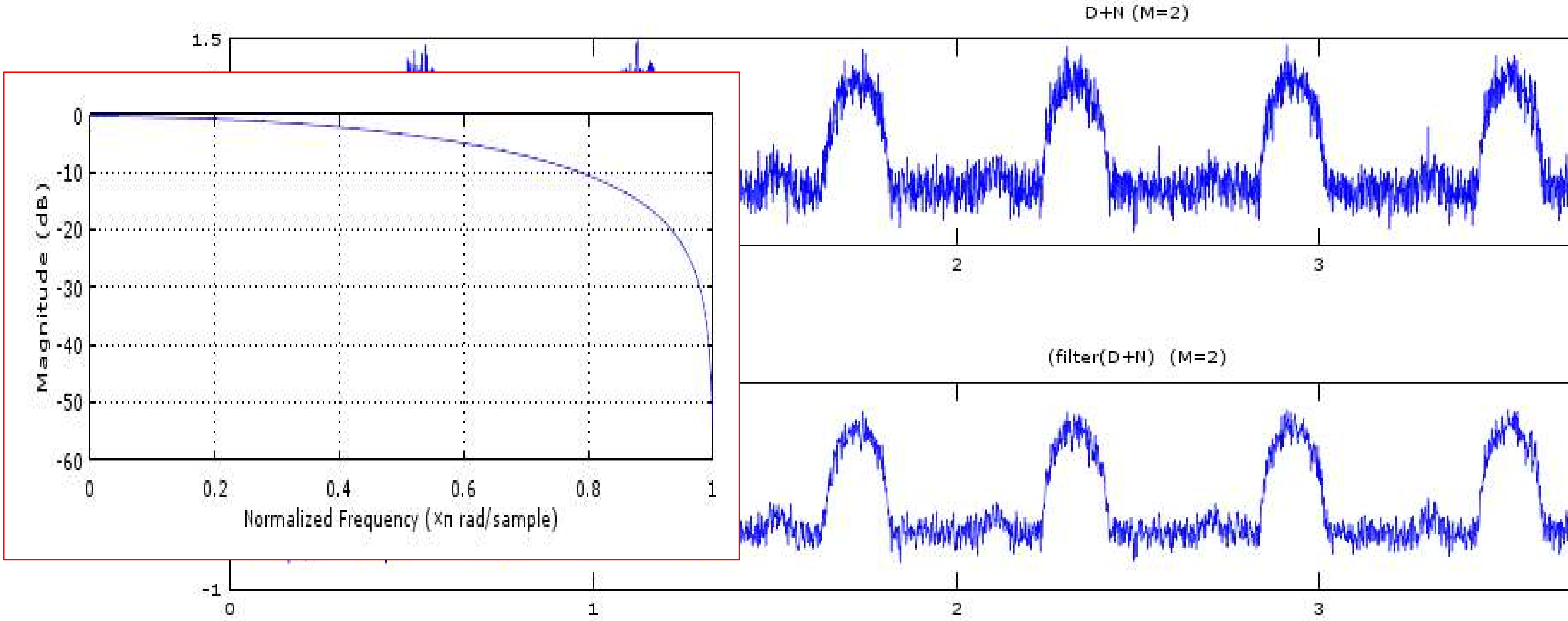
Snn(f)



The Wiener filter: Case study

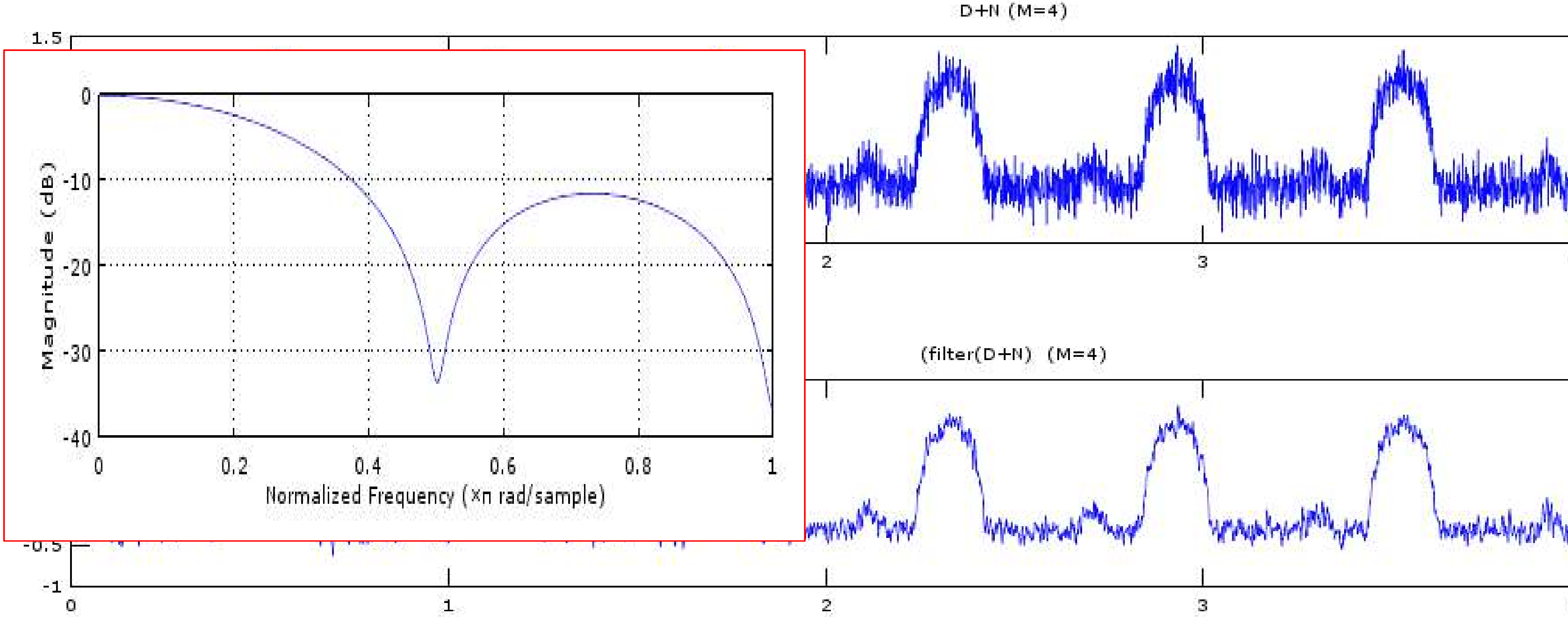
- M=2 (order 1)

SNR=10 dB → 13 dB



The Wiener filter: Case study

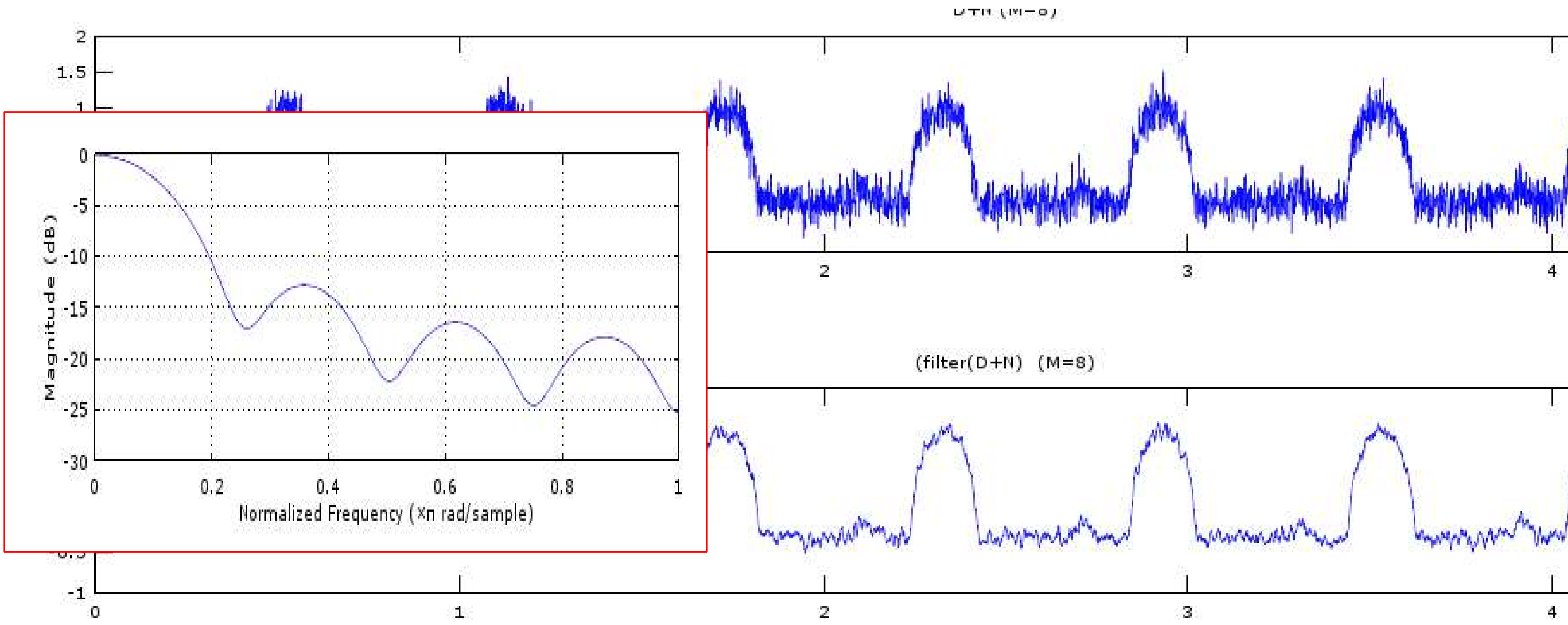
- M=4 (order 3) SNR=10 dB → 16 dB



The Wiener filter: Case study

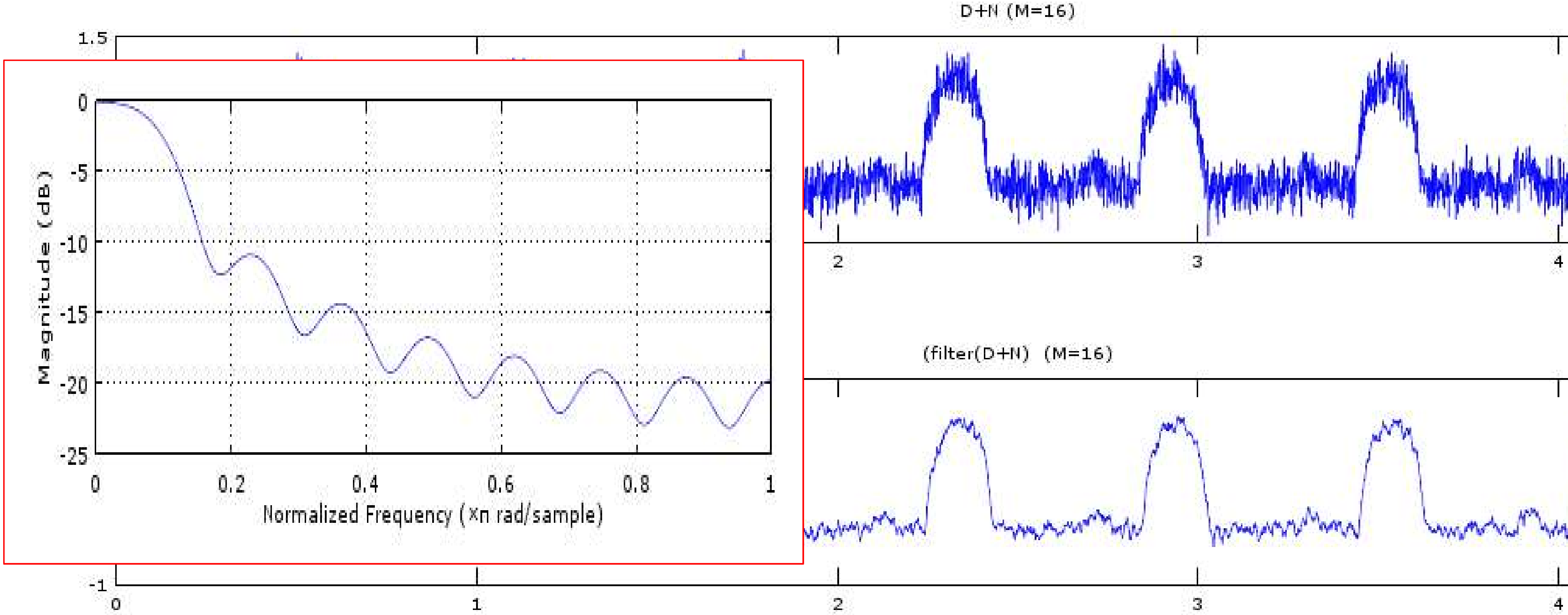
- M=8 (order 7)

SNR=10 dB → 18.7 dB



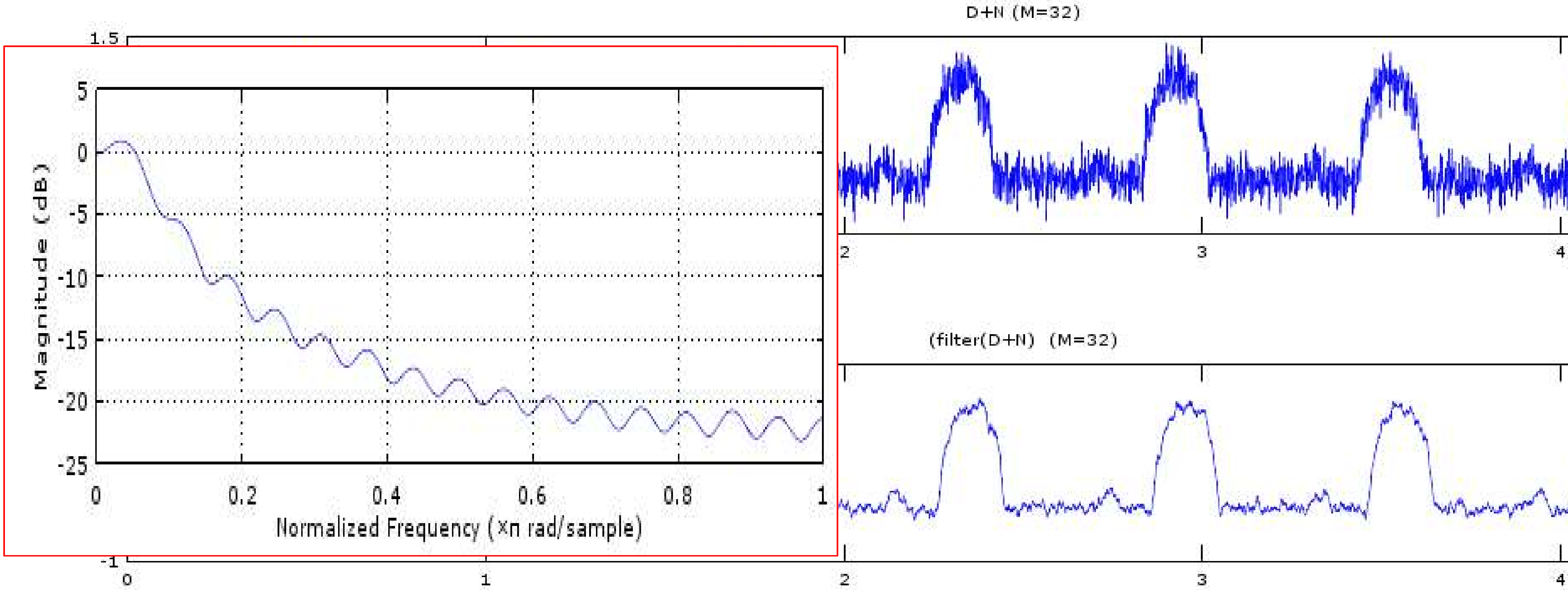
The Wiener filter: Case study

- M=16 (order 15) SNR=10 dB → 19.2 dB



The Wiener filter: Case study

- M=32 (order 31) SNR=10 dB → 19.5 dB



SUMMARY

- Noise filtering
- Wiener filtering
- Wiener-Hopf equations
- ECG example