pic 1: Review of Stochastic Processes

Telecommunication Systems Fundamentals

Profs. Javier Ramos & Eduardo Morgado

Academic year 2.013-2.014

...epts in this Chapter

- ...w of Signals models and classification
  - ...amples of actual signals
  - ...nal modeling
  - ...nals classification
- ...w of Statistical Basics: Modeling of Stochastic Processes
  - ...plitude distribution (probability density function, pdf) and averages
  - ...ocorrelation
  - ...ependence
  - ...tionarity
  - ...odicity
  - ...ss-correlation
  - ...wer and Energy Spectral Density

Theory classes: 3 sessions (6 hours)
Problems resolution: 1 session (2 hours)
Lab (Matlab): 2 hours

graphy

mmunication Systems Engineering. John. G. Proakis. ntice Hall

emas de Comunicación. S. Haykin. Wiley

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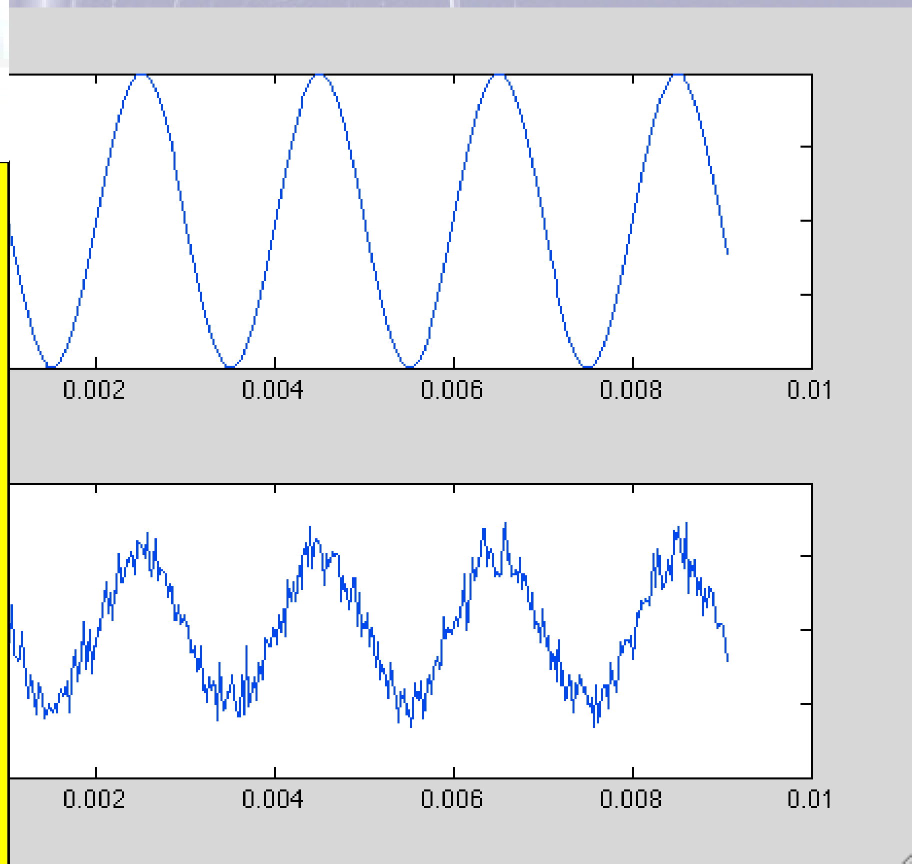
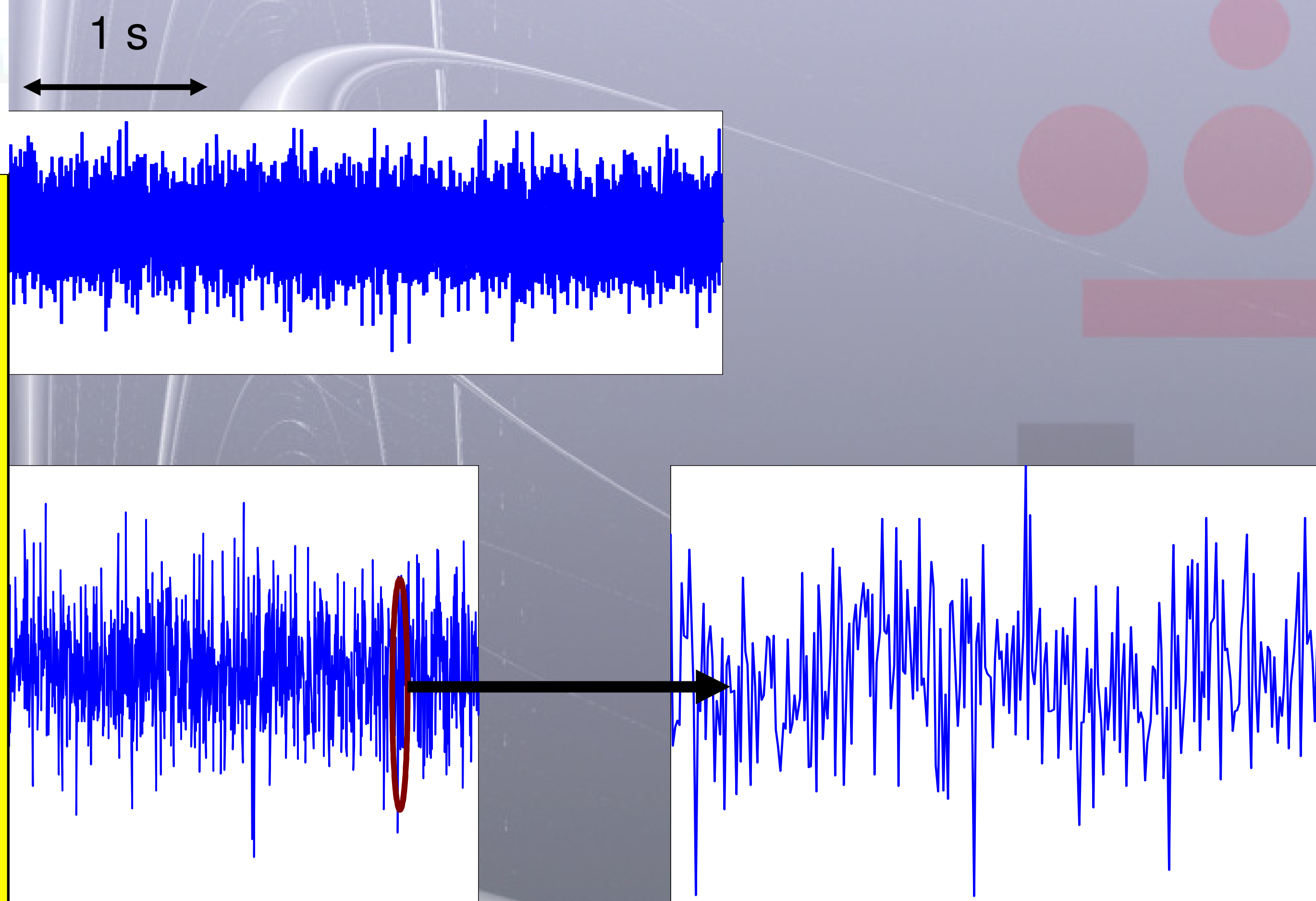
Tone



Noise Free

Aditive Noise

quencies



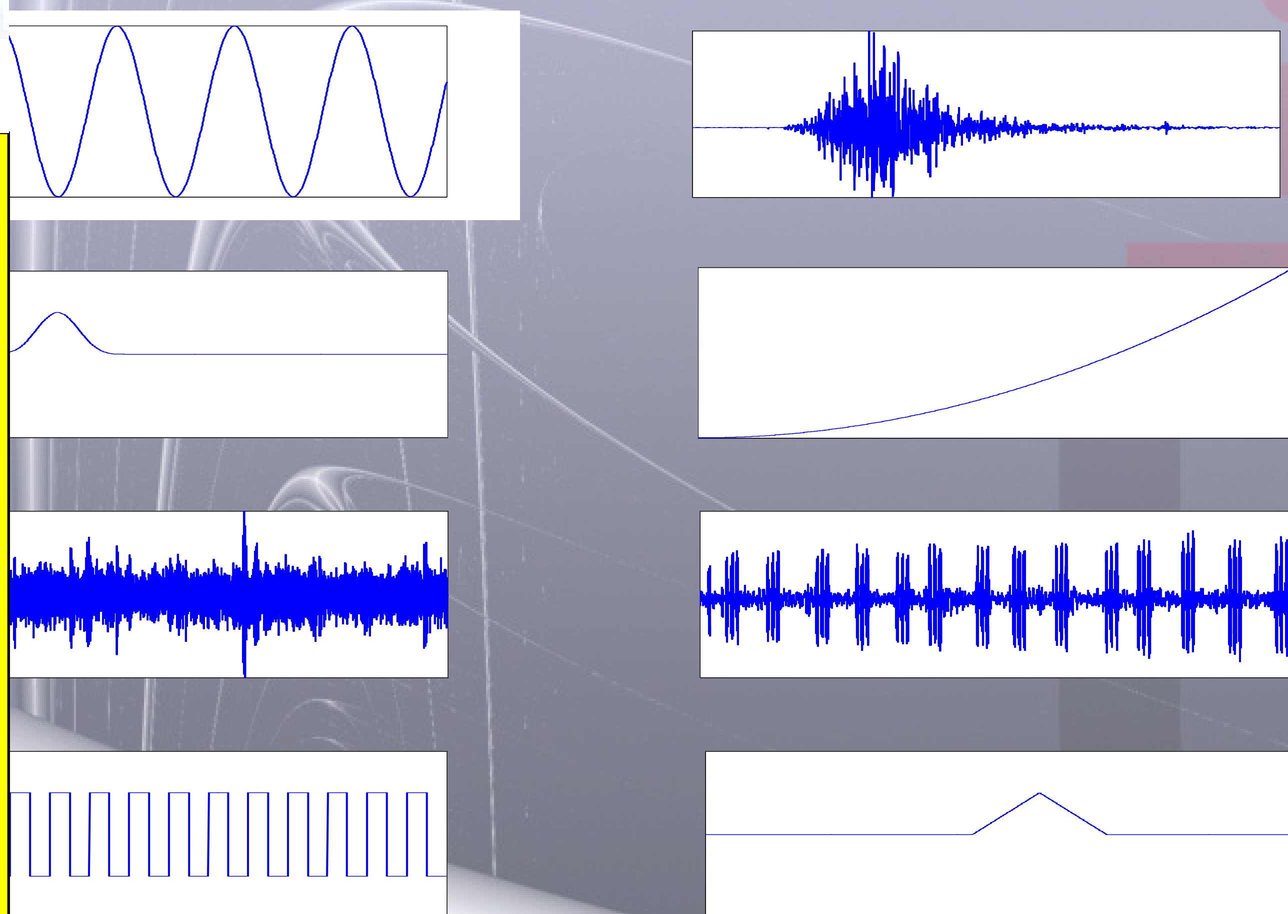
…odeling Signals?

…nswer the following questions:

- …hat information does the signal contain? How is the info …ded into the signal? How much info does the signal contain?
- …w does the channel affect the transmitted signal?
- …w is the telecommunication system designed?

…escribe signals by their mathematical model – …urable characteristics of the signal

...would you describe them?



Modeling

oint-by-point description, the value of the signal at each time t is stored in a look-up-table

t	x(t)
…	…
0	7
1	2
…	…

oint-by-point description is valid for any signal (assuming the ing rate is fast enough) and contains all the information the signal, but “seeing” the information is not evident

Modeling

signals can be modeled by a mathematical expression that es its amplitude as function of time

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega_0 t)$$

ype of signals are named “Deterministic” because their lack domness

y few signals in telecommunications systems can be modeled as simple as

Modeling

n briefly describe a signal by some of its characteristics

an value

an squared value (power)

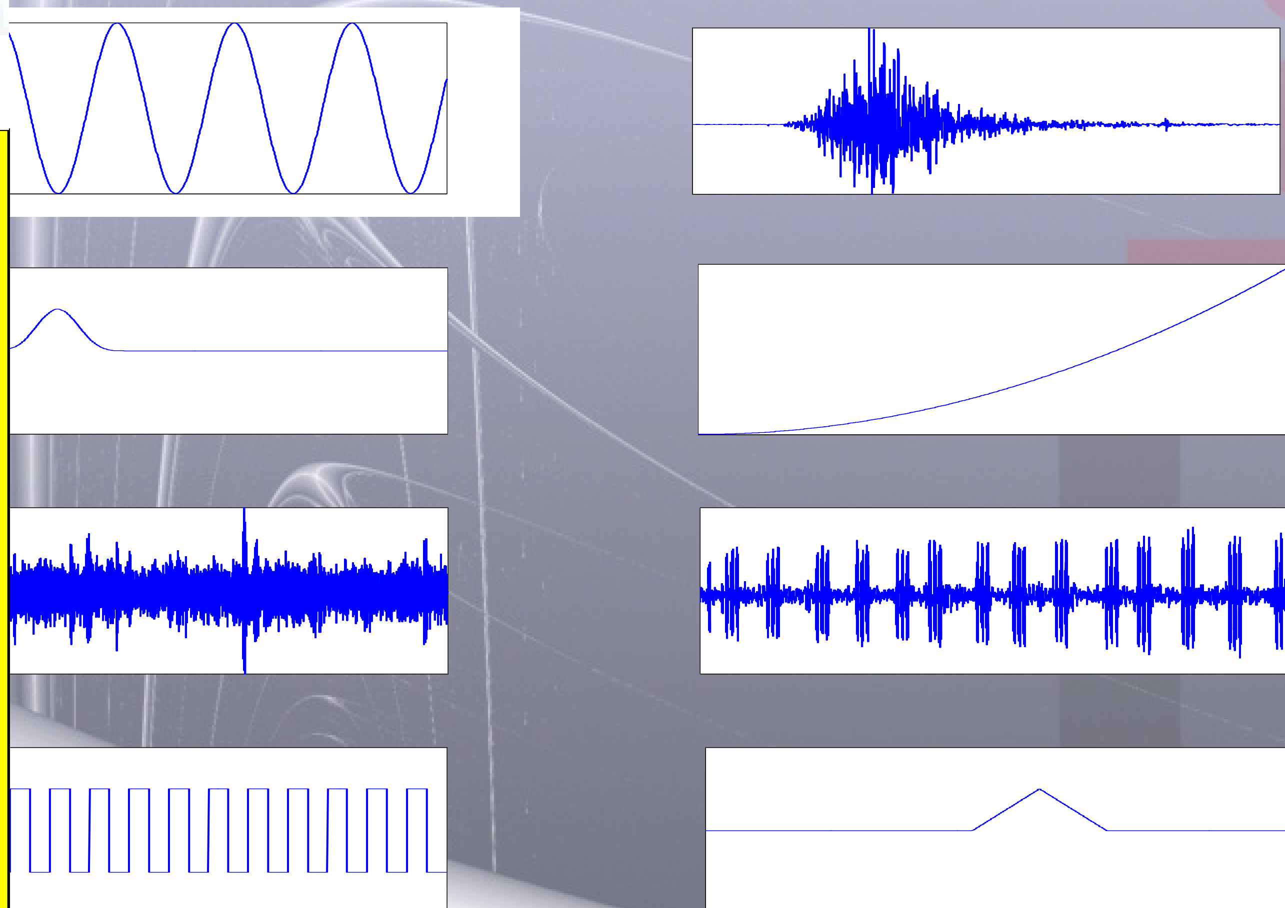
ergy

ndard deviation

tocorrelation

universal procedure (usable for any kind of signal), and it some criteria to classify signals. However, it does not be the signal completely (univocally).

vould you describe them?



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Value

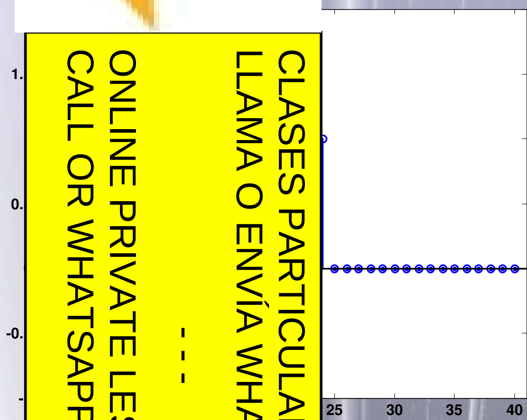
ne-discrete signals, mean value is defined as:

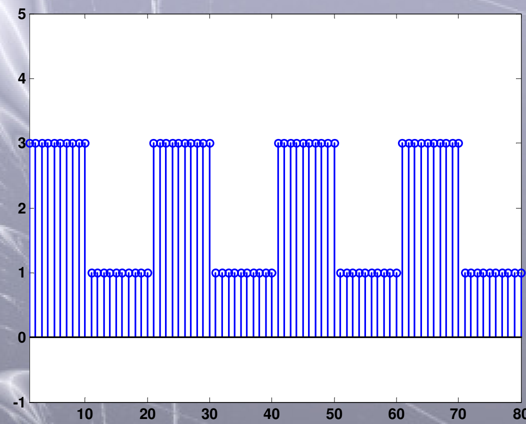
$$\langle x[n]\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}x[n]$$

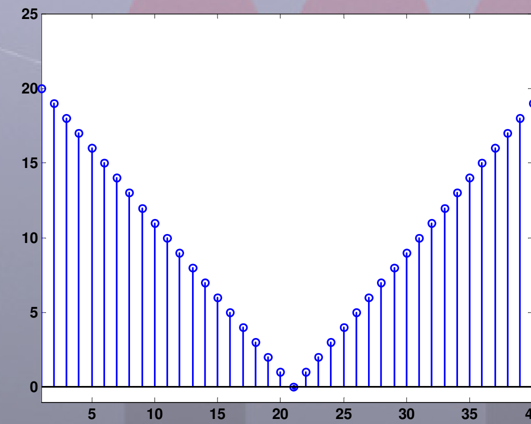
ne-continuous signals, mean value is defined as:

$$\langle x(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)\,dt$$

Value







$= 0$

$\langle x_2(t) \rangle = 2$

$\langle x_3(t) \rangle = \infty$

y

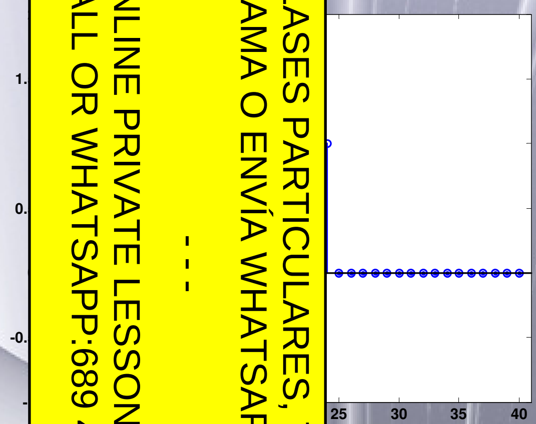
cisive classification of signals is related to its gy and power: finite energy, or power defined. For energy signals, it is defined

r discrete signals:

$$\mathbf{E_x} = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

r continuous signals:

$$\mathbf{E_x} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$



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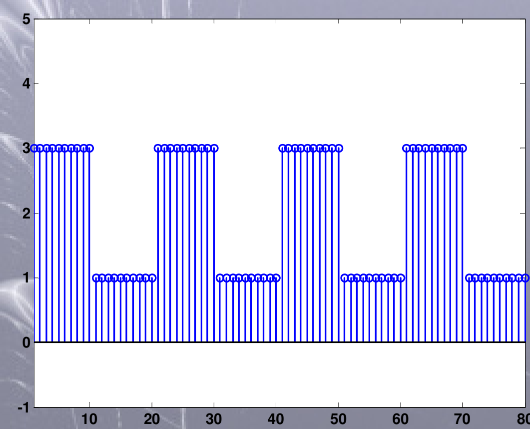
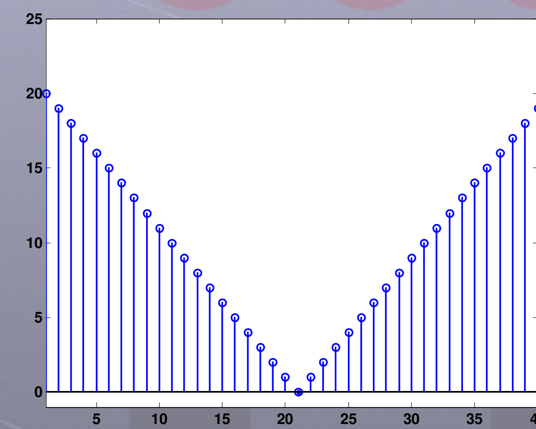


$$E_{x_2} = \infty$$



$$E_{x_3} = \infty$$

...ge Power

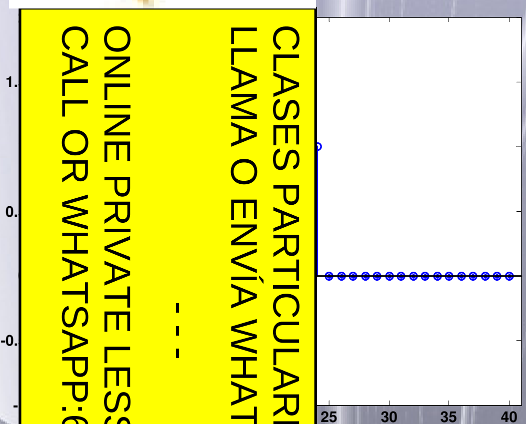
...average power of discrete signals is defined as:

$$\mathbf{P_x} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

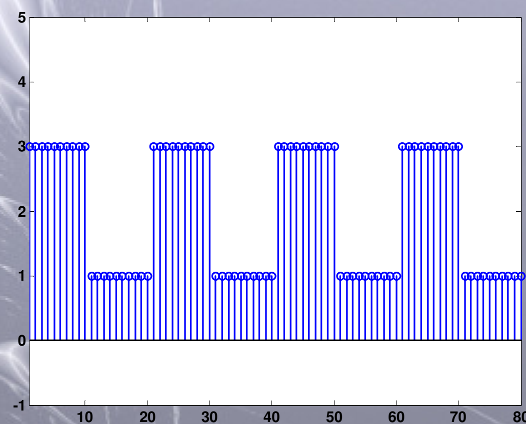
...e the average power of continuous signal is ...ed as:

$$\mathbf{P_x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

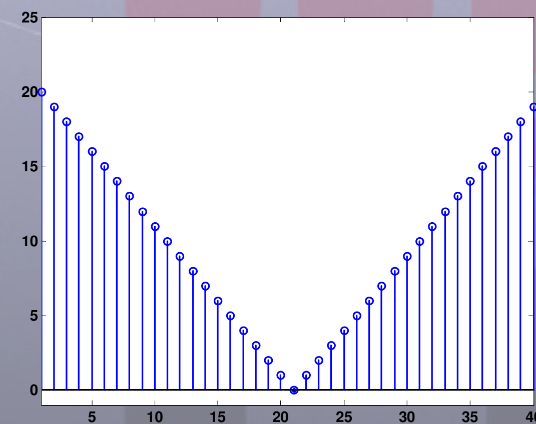
...ge Power



...0



$P_{x_2} = 5$



$P_{x_3} = \infty$

...ro when
...of the
...creases

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...al classification

Signal

...e
...gy
...efined)

$$\ldots_{x} < \infty$$

Finite
Average
Power
(Power Defined)

$$E_x = \infty$$

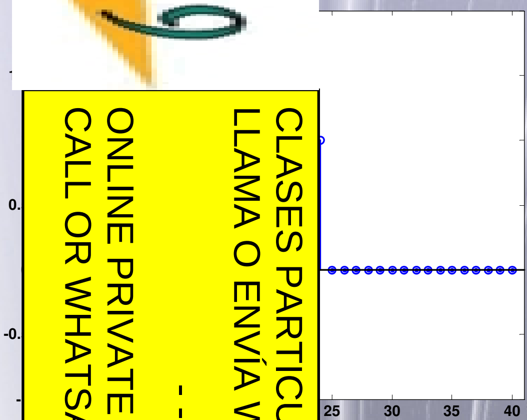
$$0 < P_x < \infty$$

Unbounded
Average
Power

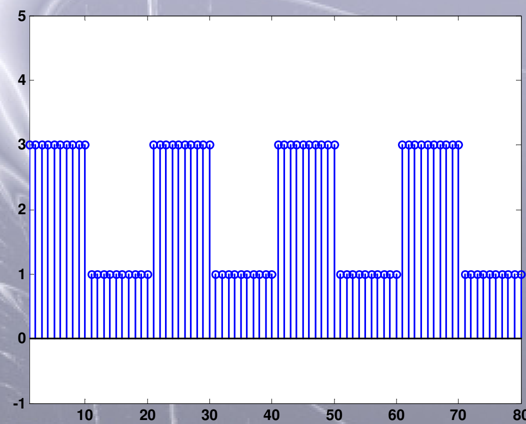
$$E_x = \infty$$

$$P_x = \infty$$

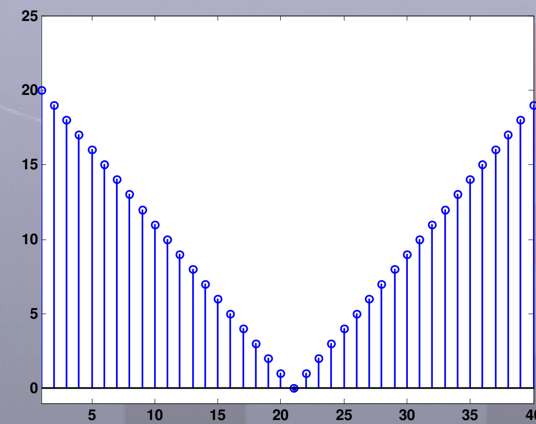
y/Power signal classification



e
gy
efined)

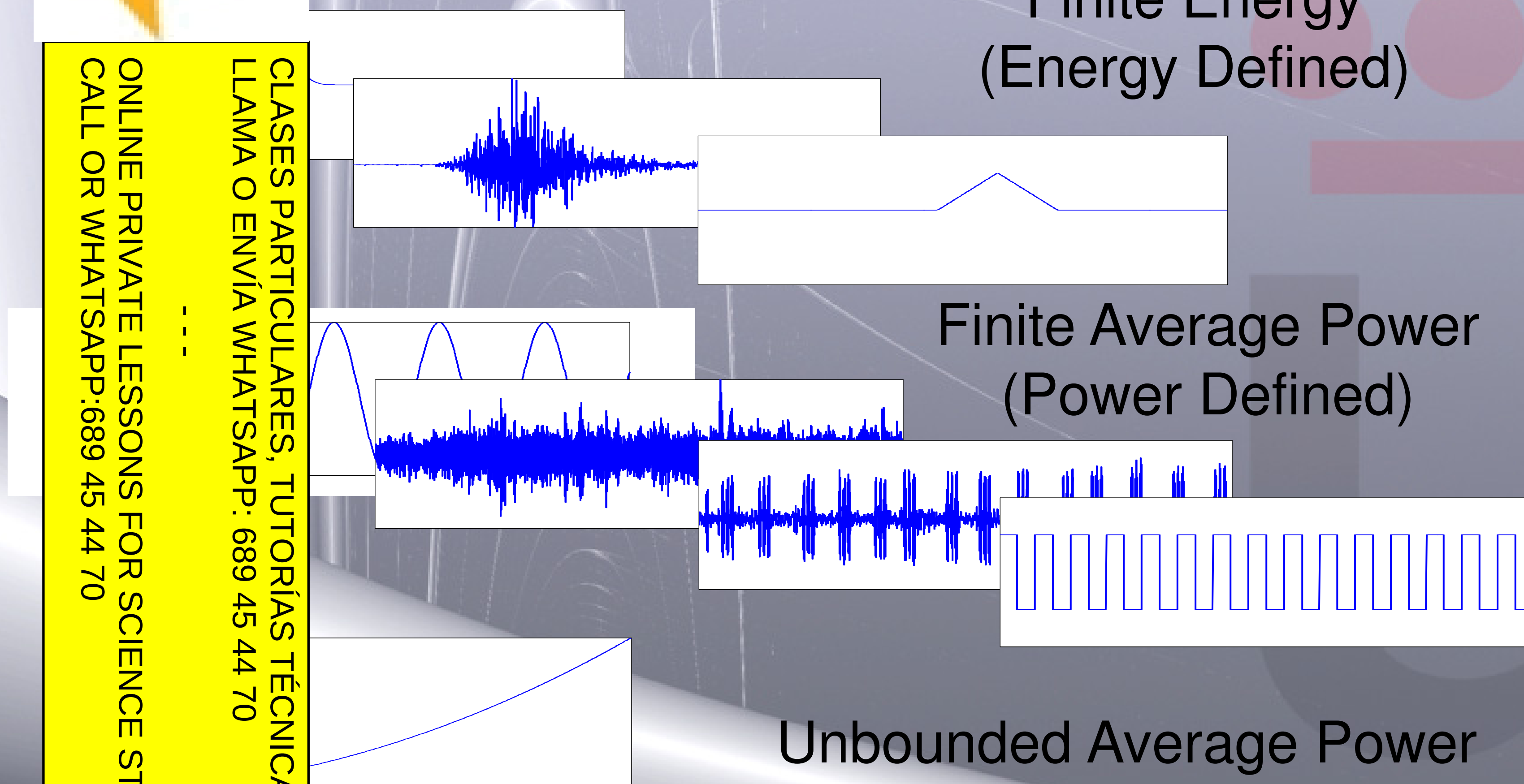


Finite
Average
Power
(Power Defined)



Unbounded
Average
Power
(assuming it
increases
indefinitely)

...y/Power signal classification



Finite Energy
(Energy Defined)

Finite Average Power
(Power Defined)

Unbounded Average Power

work

ute: i) Average Value; ii) Energy and ii) Average Power of
o following signals:

$$x_1(t) = A\cos(2\pi ft)$$

$$x_2(t) = Ae^{j2\pi ft}$$

fying Signals: A Taxonomy

- nuous / Discrete
- alog / Digital
- ministic / Stochastic (random signals)
  - terministic:
    - Energy Defined (time limited)
    - Power Defined
      - Periodic / Non periodic
  - chastic
    - Stationary
      - Ergodic / Non-Ergodic
    - Non-Stationary
- classifications
  - al valued / Complex
  - en / Odd
  - rmitical / Non-Hermitical

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  - …ver and Energy Spectral Density

: Time Averaging and Expected Value

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

$$\updownarrow$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

work: Time Averaging and Expected Value

erate a Random Variable uniformly distributed
een 0 and 1

~U(0,1)

f - f(x) = 1, for 0 < x < 1, and 0 otherwise.

a simulation (Matlab) of 10.000 samples of U(0,1)

pute the average value of the 10.000 samples

ytically calculate expected value of U(0,1)

pare values obtained in 2 and 3.

: Time Averaging and Expected Value

$$\left\langle x^2(t) \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt$$

$$\Updownarrow$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

work: Time Averaging and Expected Value

erate a Random Variable uniformly distributed
een 0 and 1

~U(0,1)

f - f(x) = 1, for 0 < x < 1, and 0 otherwise.

a simulation (Matlab) of 10.000 samples of U(0,1)

pute the average power of the 10.000 samples

ytically calculate second moment of U(0,1)

pare values obtained in 2 and 3.

tatistical modeling is useful?

- aracterizing a stochastic process would require the cification of the signal at every single instant

- t cases we do not know the signal ***a priori***

- get the whole signal in very rare occasions

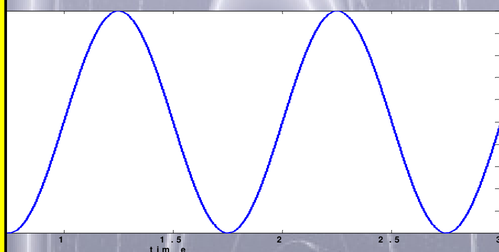
Statistical model to:

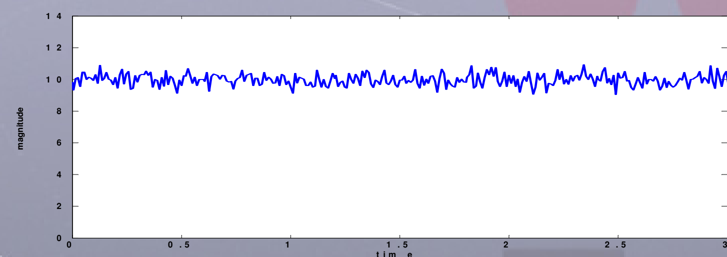
rize the on of a signal r

… describe **sets of** signals

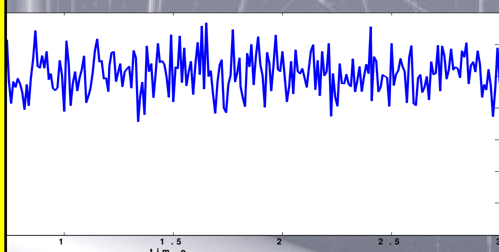
… describe the whole signal from a **finite time interval**

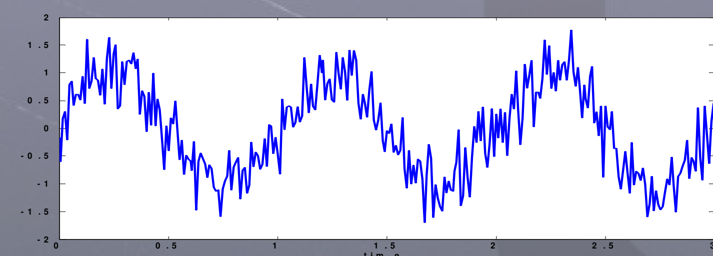
o you describe them?



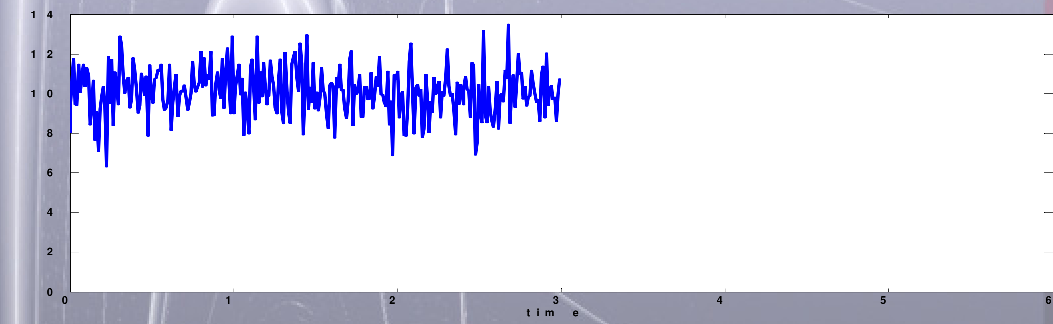


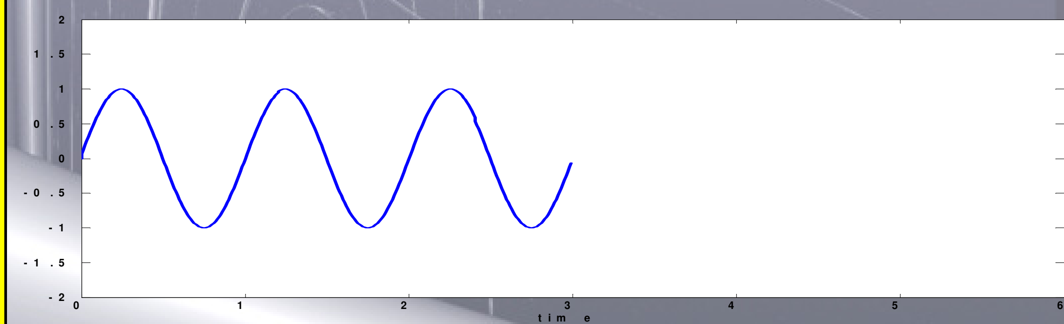
Why Statistical Modeling is Useful?



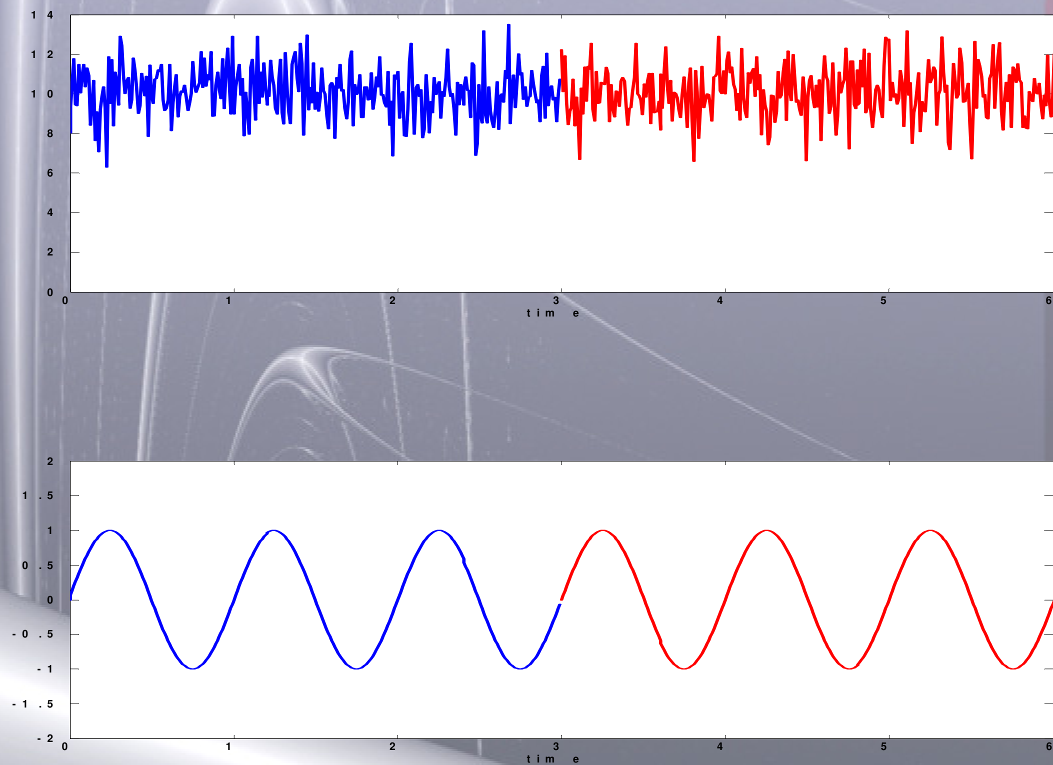


...s next?





s next?



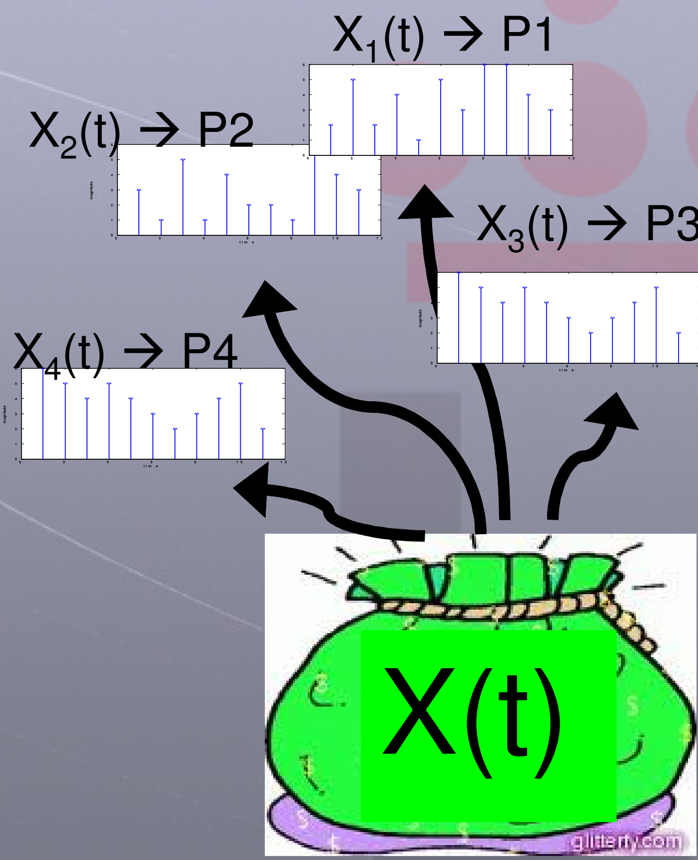
w of the Concept of Stochastic Process

n 1: a SP can be seen as series of
n Variables; or it can be also seen
 that is time-variant

n 2: a SP can be seen as a set of
riant signals, each one with its
lity of happening (imagine a bag
the possible signals and you get
hem).

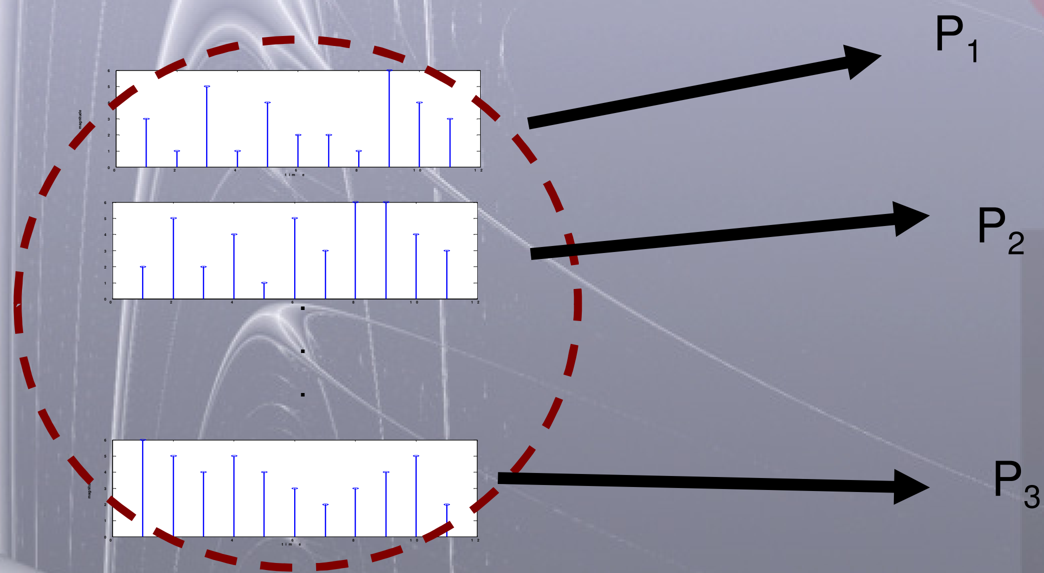
e talk about "time" when referring to
pendent variable, but it could be
t:

mple: thickness of bar as function of its length



astic Process Model

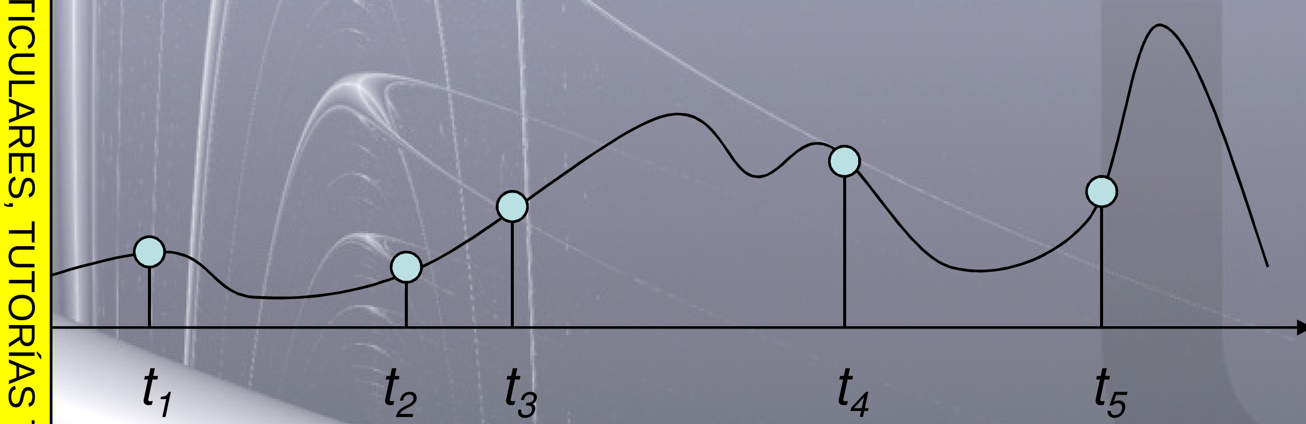
characterize an SP a probability measurement of each possible ion has to be provided.



words, we should be able to tell how likely is any given observation tion) to happen

...astic Process Model

...plete description of a SP, $X(t)$, requires the definition of the ...nce ($X(t_1)$, …, $X(t_k)$) for any value of k and any value of the ...e (t_1, …, t_n).



astic Process Model

neral practice, we will not look for a complete ription of the SP, but we will define by two main cts:

- nplitude distribution
- tocorrelation, which contains the time variation description atistical relationship betwen two instants of the signal)

correlation can be expressed also as Power tral Density – the Fourier Transform of the correlation

, we can analyse the impact of a linear channel on P, i.e. the impact of the channel on the amplitude bution and on the autocorrelation

...arizing main concepts of SP

...s a mechanism that generates time-variant amplitudes – a ... Each of the signal produced by a SP is called “realization”

...P model also applies to each realization. In other words, a ... for a SP models also every possible realization of it.

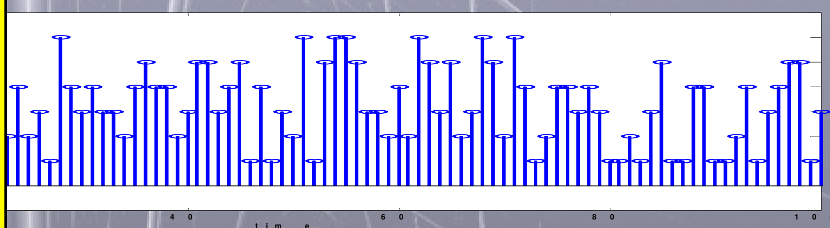
...l model SP by their amplitude distribution and autocorrelation. ...ude distribution models the realization values at a given time, ...tocorrelation models the time variation of the SP.

...nputing Fourier Transform of autocorrelation we get the Power ...al Density – the information of the amount of energy contained ...n frequency – the spectrum

...in telecommunications is modeled as a SP

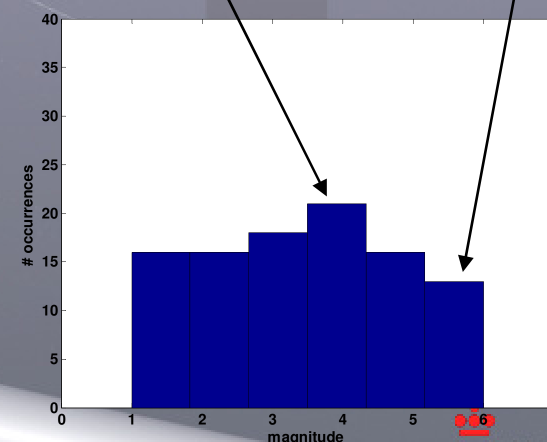
...tude Distribution

- ...ch instant of time of the SP (sample), its amplitude is a ...om Variable following a Probability Density Function (pdf)
- ...ude Distribution is modeled by its pdf. It can be modeled ...y its amplitude (two values for complex signals) or by its ... Thus, a pdf of the signal amplitude or a pdf of the signal ... should be provided



Most frequent value

Less frequent value



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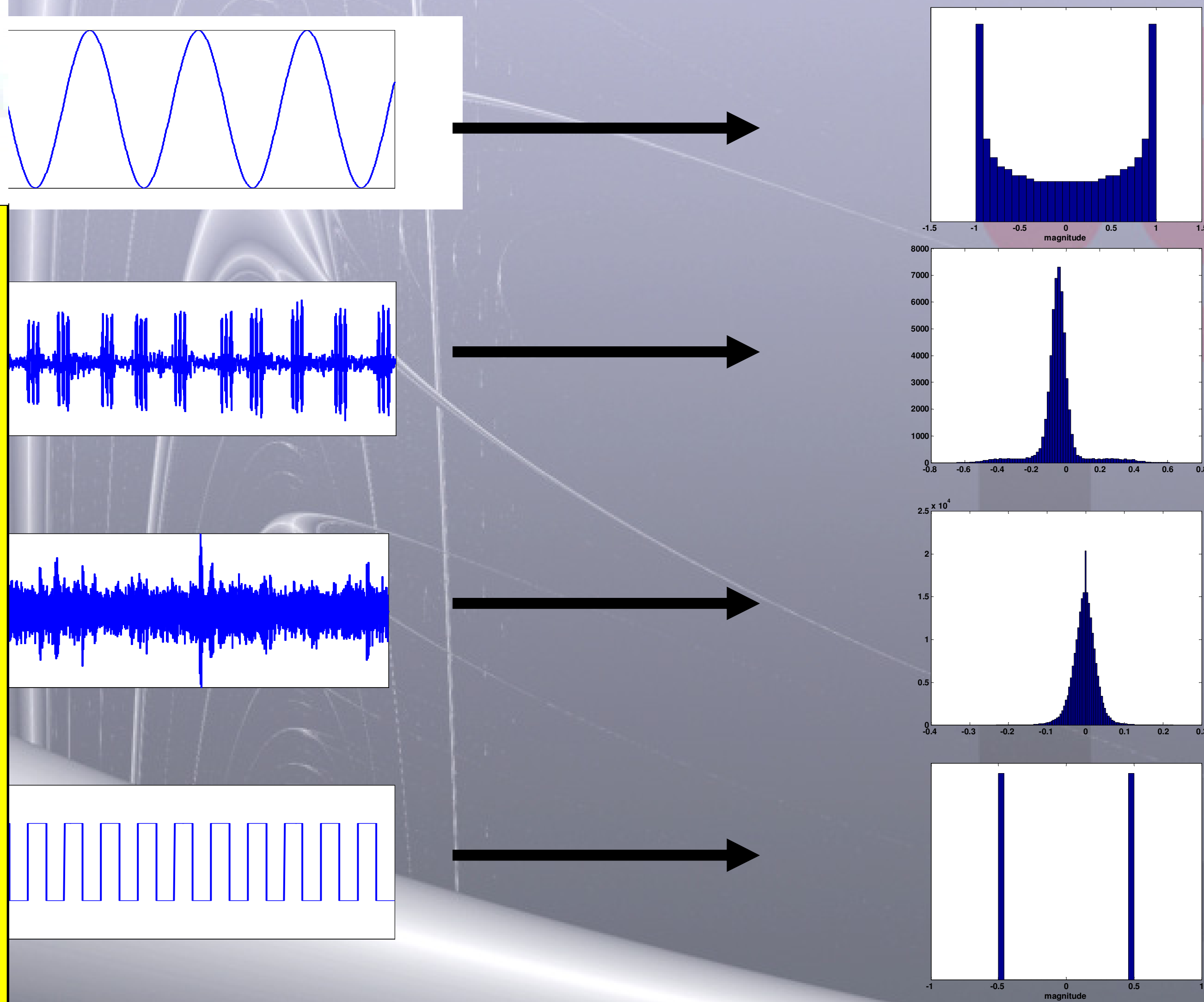
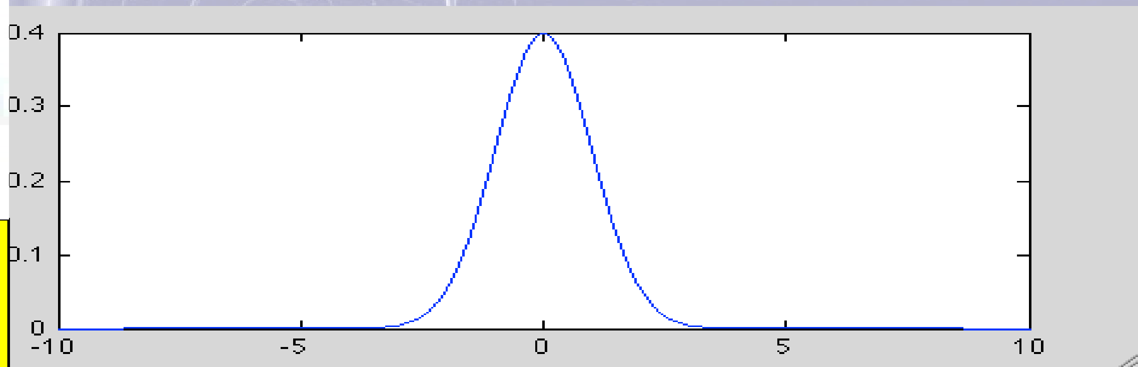
…tude Distribution



work



y the magnitudes on each axes of the above graph

If we interpret the plot as a Gaussian-shaped signal

If we interpret the plot as the pdf of the voltage of a noisy signal

s the mean value for each case?

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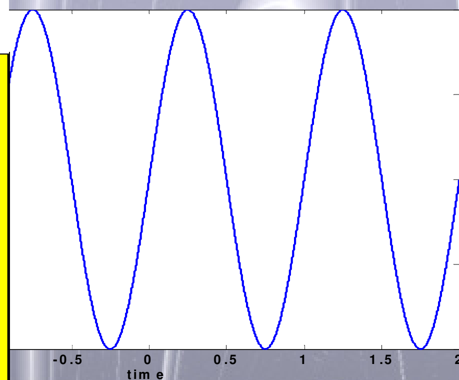
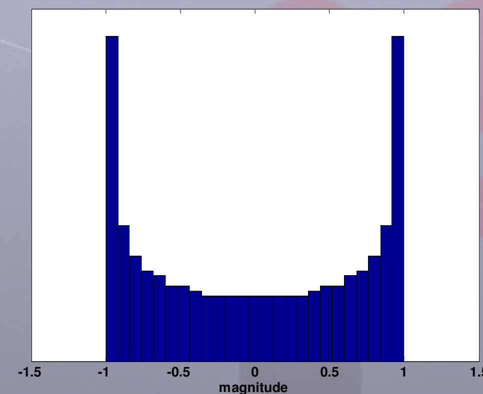
...alue: time domain and statistical approach

- ... average
- Statistical mean value



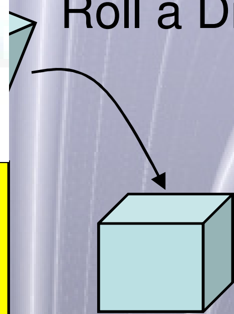


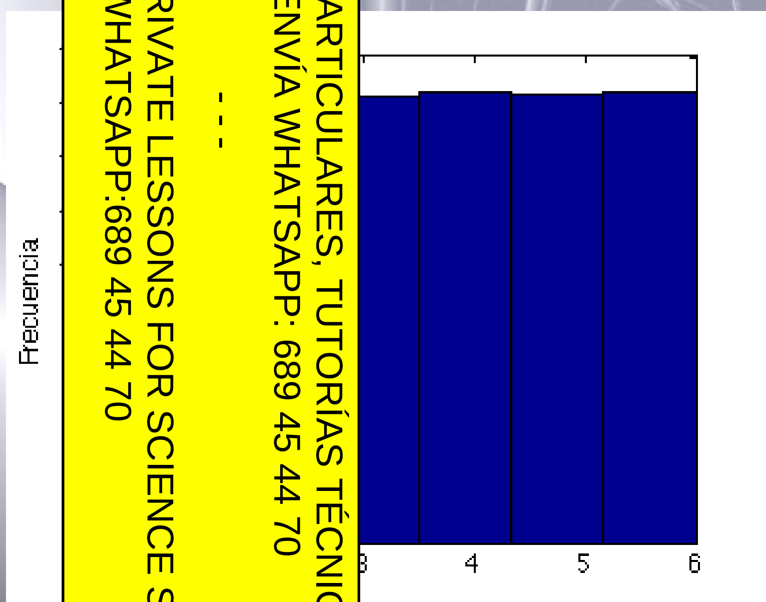
$$\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \quad \longleftrightarrow \quad E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$$

$$\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt \quad \longleftrightarrow \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

rical Example

Roll a Dice





- Time average

x[n] = 5 6 2 1 2 3 2 6 1 4 1 6 2 4 6
3 1 2 3 3 2 6 5 6 3 6 1 4 5 2
6 5 5 4 5 2 2 2 4 5 2 3 6 4 6
2 5 5 2 3 3 4 5 1 5 6 3 3 4 5
5 5 3 3 6 4 6 2 4 4 6 1 4 4 1
6 ...

$$\rangle x[n] \langle = \frac{1}{N} \sum_{n=0}^{N-1} x[n] = 3{,}7$$

3,7 for 100 samples, for 100.000 ➔ 3,4997

- Statiscal average

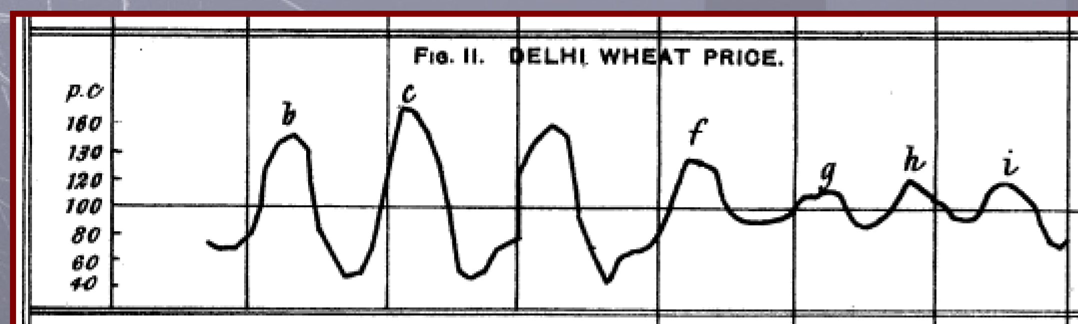
$$E(X) = \sum_{n=1}^{N} n f_x[n] =$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

orrelogram

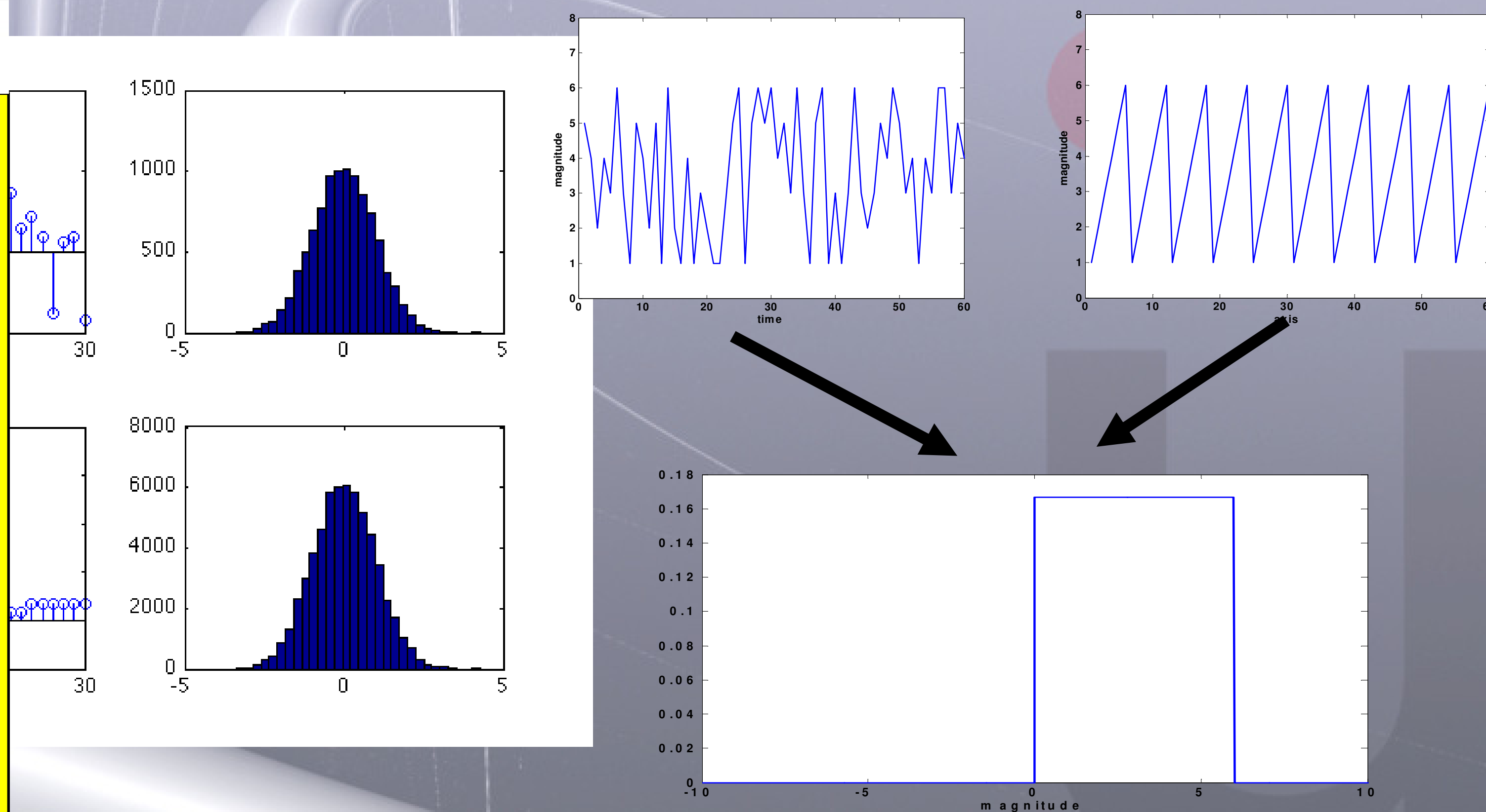
A COMPARISON of the FLUCTUATIONS in the PRICE of WHEAT and in the COTTON and SILK IMPORTS into GREAT BRITAIN. By J. H. POYNTING, M.A., *late Fellow of Trinity College, Cambridge; Professor of Physics, Mason College, Birmingham.*

[Read before the Statistical Society, 15th January, 1884. Sir RAWSON W. RAWSON, K.C.M.G., C.B., a Vice-President, in the Chair.]



ry Poynting
- 1914)

...tude distribution is not enough to ...be a time-variant SP



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orrelation

power-defined signals (time-unbounded) exhibit
tion patterns. Although such signals are not periodic,
nave some periodicity on their amplitude distribution.
are quasi-periodic

can we study such signals?

proach to analyze quasi-periodic patterns is to check
eness between the signal and a delayed version of

utocorrelation function describes the likeness of a
with a delayed version of itself. Therefore, we can
fy quasi-periodic patterns by computing the signal
orrelation

orrelation

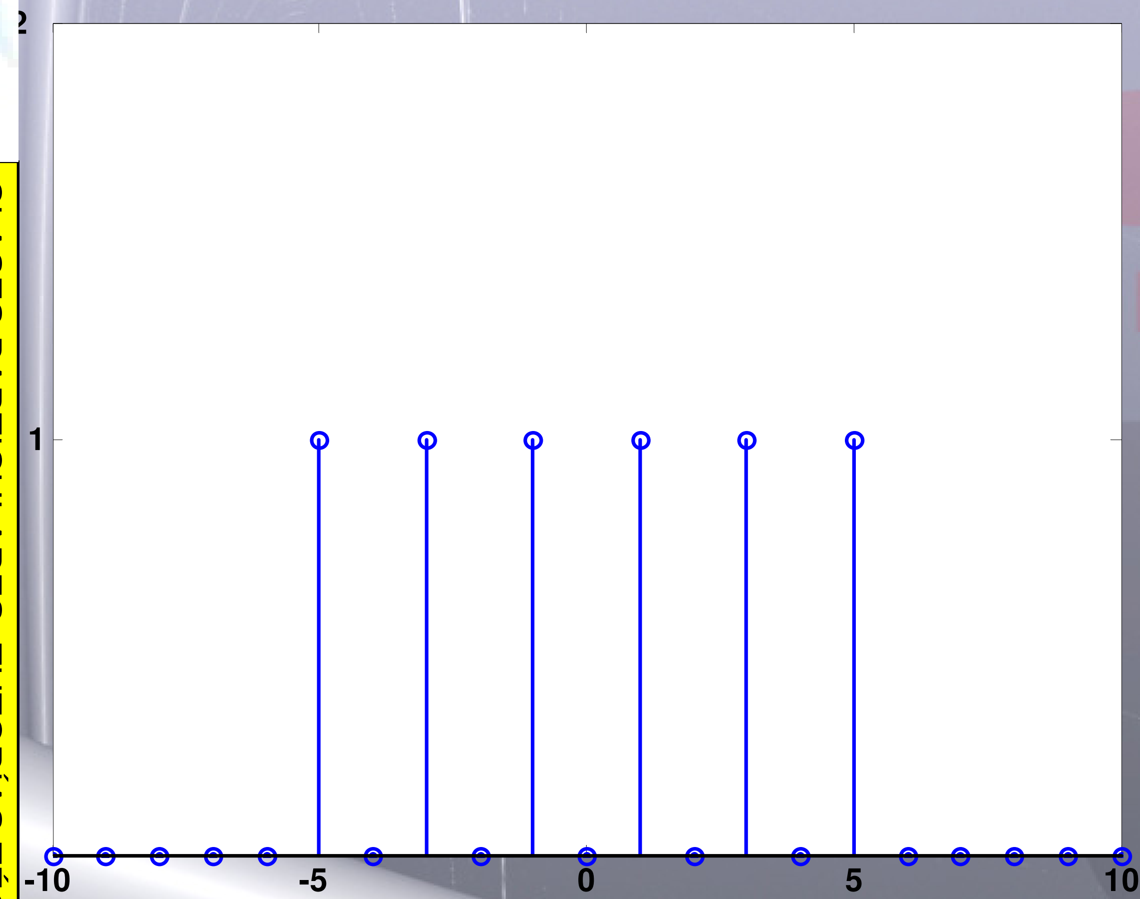
asure likeness between a given signal and a delayed n of itself we use the inner product of both signals. So, orrelation is defined that way.

keness measurement (inner product) is defined in different r power-defined and energy-defined signals

ation of inner product depends on the available information signal

- me description of realizations is available, we can compute inner product ime average
- tatistical information is available, inner product will be computed as istical average

...ple 1: Energy-Defined Signal



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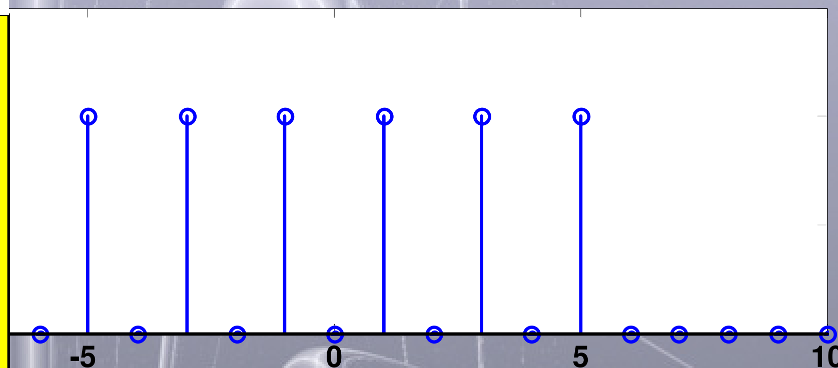
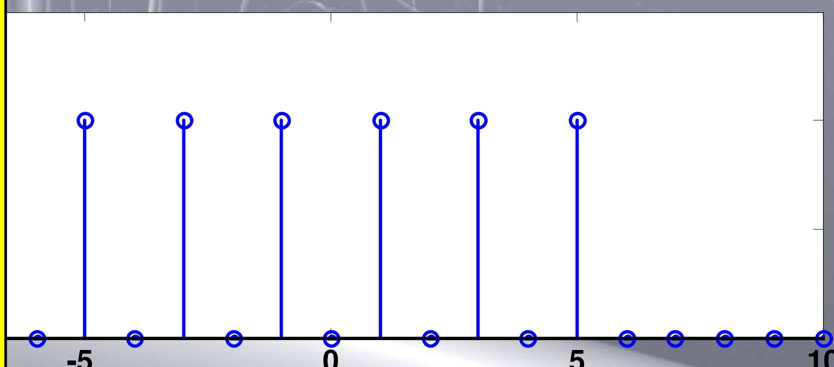
ple 1: Energy-Defined Signal





...

x(-5)x(-5) = 1

x(-4)x(-4) = 0

x(-3)x(-3) = 1

x(-2)x(-2) = 0

x(-1)x(-1) = 1

x(0)x(0) = 0

x(1)x(1) = 1

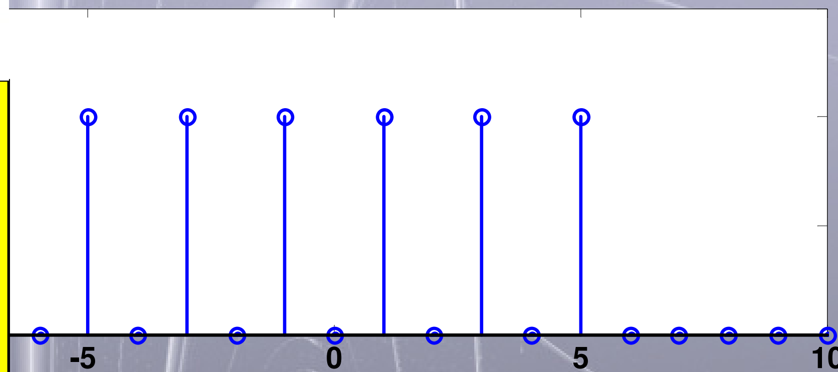
x(2)x(2) = 0

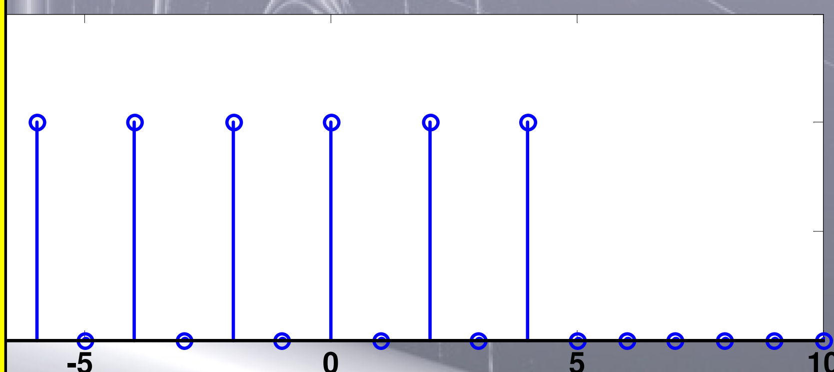
x(3)x(3) = 1

...

$\Sigma x(n)x(n) = 6$

ple 1: Energy-Defined Signal





...

$x(-5)x(-4) = 0$

$x(-4)x(-3) = 0$

$x(-3)x(-2) = 0$

$x(-2)x(-1) = 0$

$x(-1)x(0) = 0$

$x(0)x(1) = 0$

$x(1)x(2) = 0$

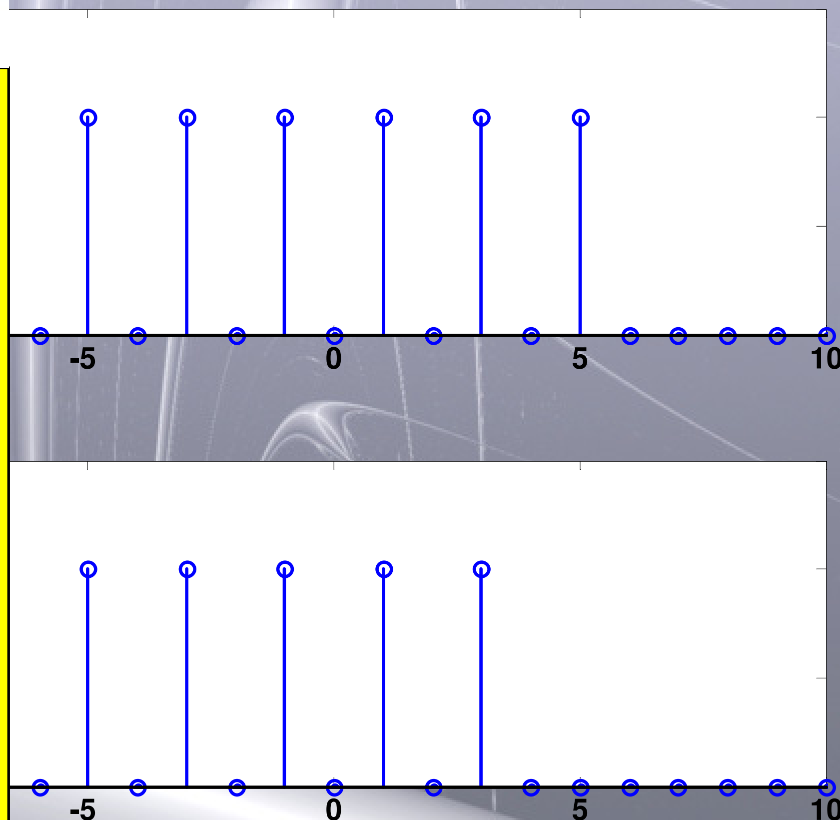
$x(2)x(3) = 0$

$x(3)x(4) = 0$

...

$\Sigma x(n)x(n+1) = 0$

ple 1: Energy-Defined Signal



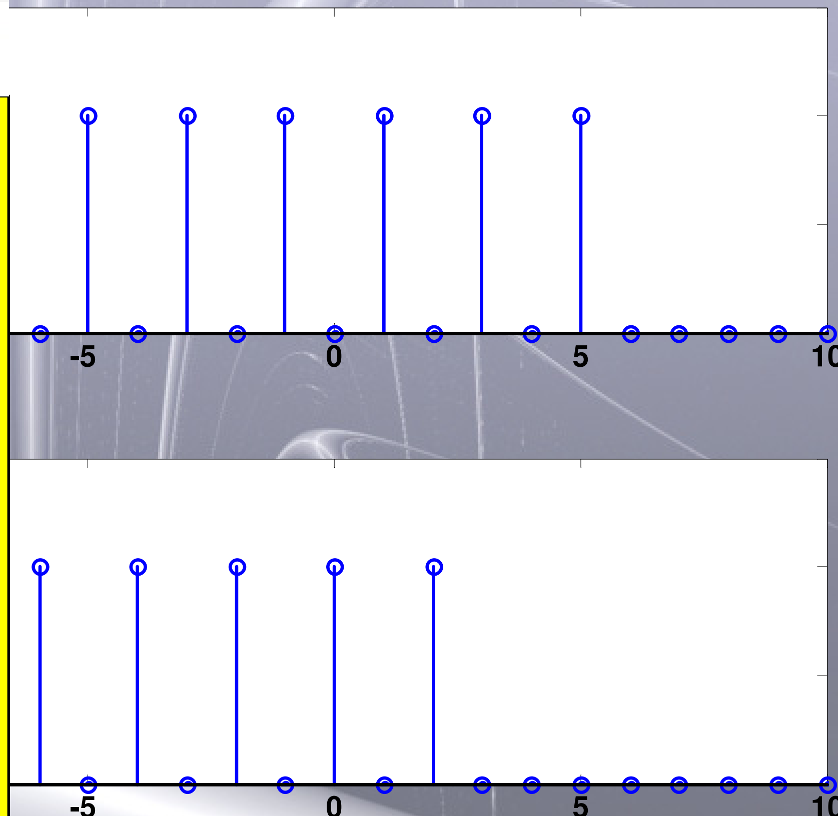
...

x(-5)x(-3)	=	1
x(-4)x(-2)	=	0
x(-3)x(-1)	=	1
x(-2)x(0)	=	0
x(-1)x(1)	=	1
x(0)x(2)	=	0
x(1)x(3)	=	1
x(2)x(4)	=	0
x(3)x(5)	=	1

...

$\Sigma x(n)x(n+2) = 5$

ple 1: Energy-Defined Signal



...
x(-5)x(-2) = 0
x(-4)x(-1) = 0
x(-3)x(0) = 0
x(-2)x(1) = 0
x(-1)x(2) = 0
x(0)x(3) = 0
x(1)x(4) = 0
x(2)x(5) = 0
x(3)x(6) = 0
...

$\Sigma x(n)x(n+3) = 0$

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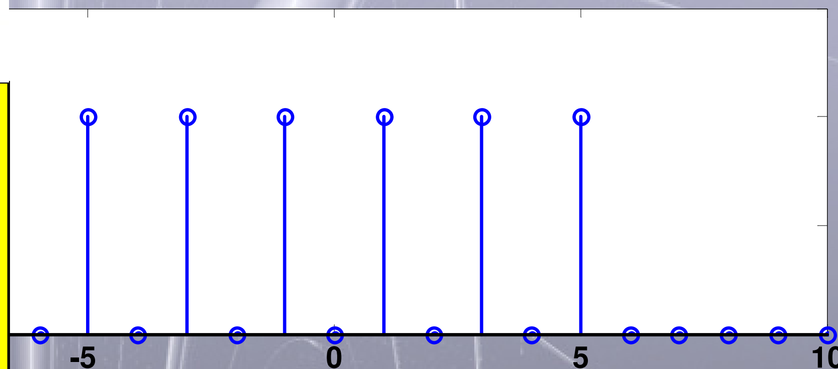
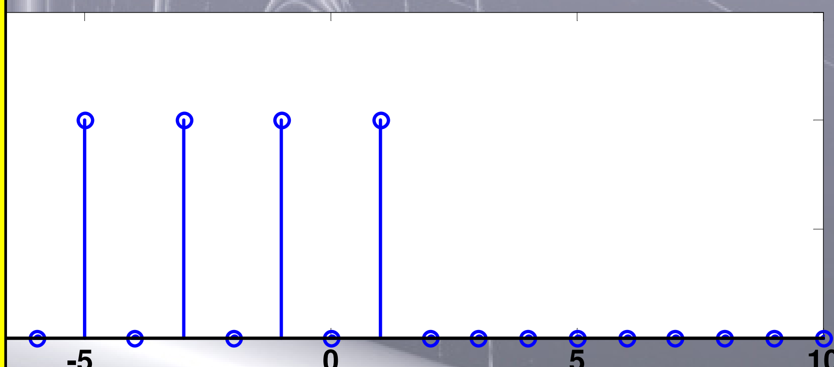
ple 1: Energy-Defined Signal





...

$x(-5)x(-1) = 1$

$x(-4)x(0) = 0$

$x(-3)x(1) = 1$

$x(-2)x(2) = 0$

$x(-1)x(3) = 1$

$x(0)x(4) = 0$

$x(1)x(5) = 1$

$x(2)x(6) = 0$

$x(3)x(7) = 0$

...

$\Sigma x(n)x(n+3) = 4$

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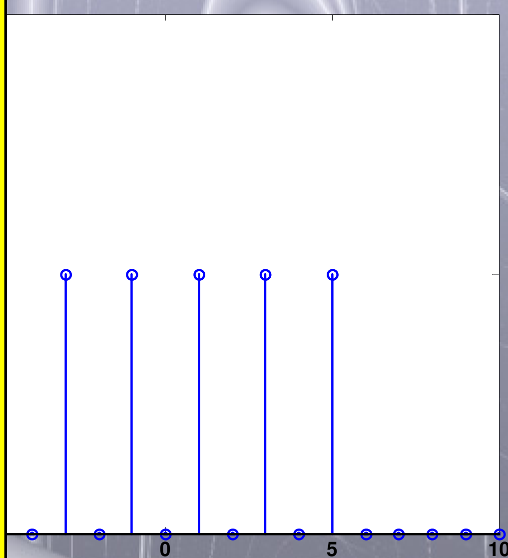
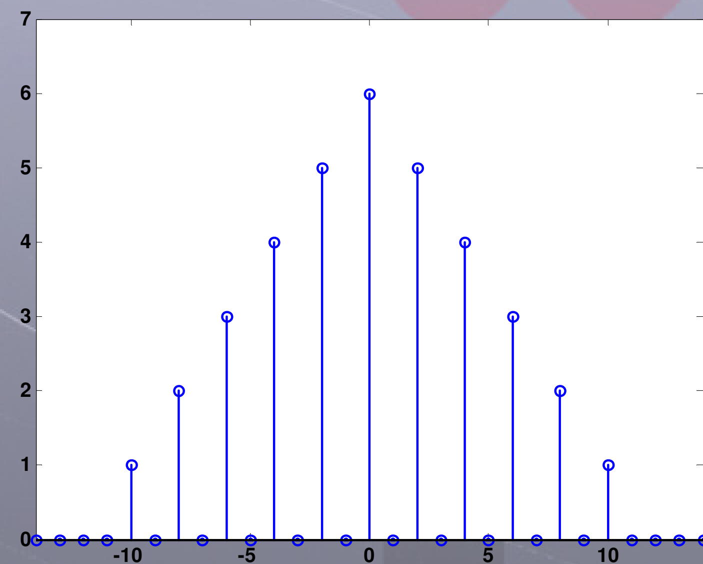
ple 1: Energy-Defined Signal

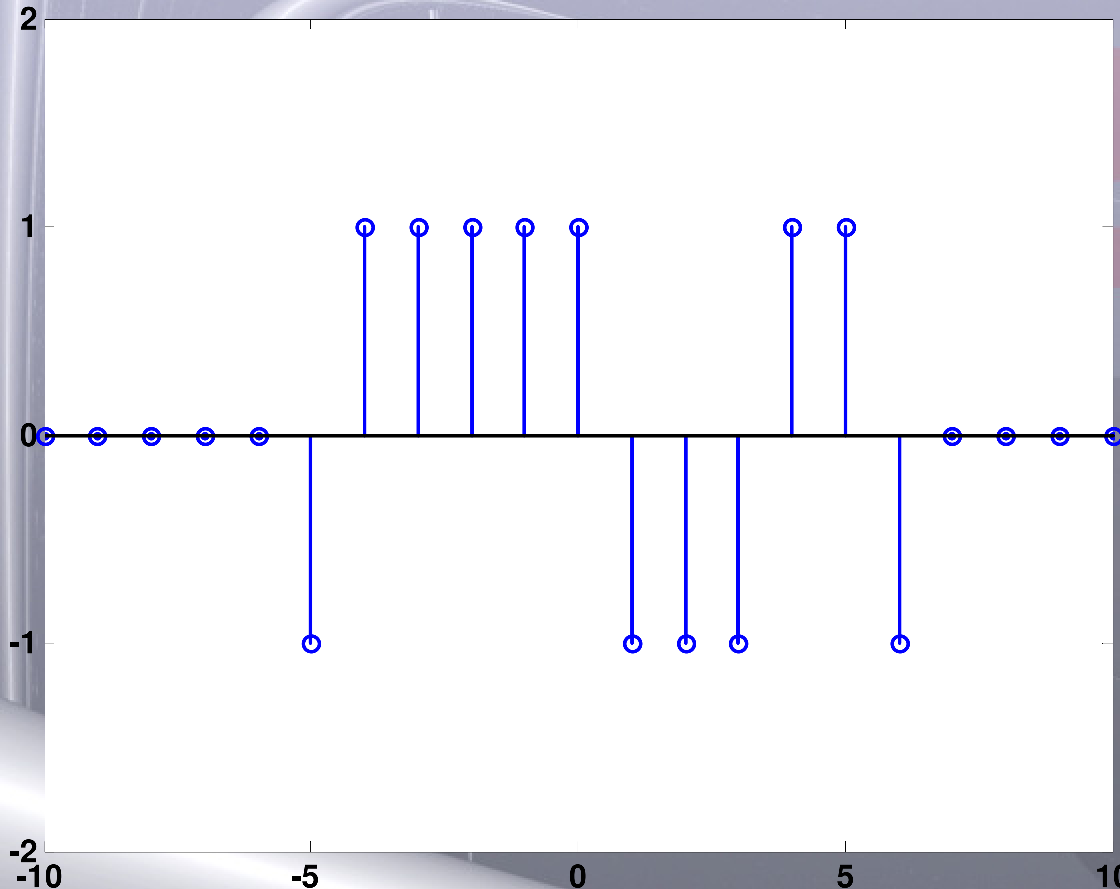


x(n)

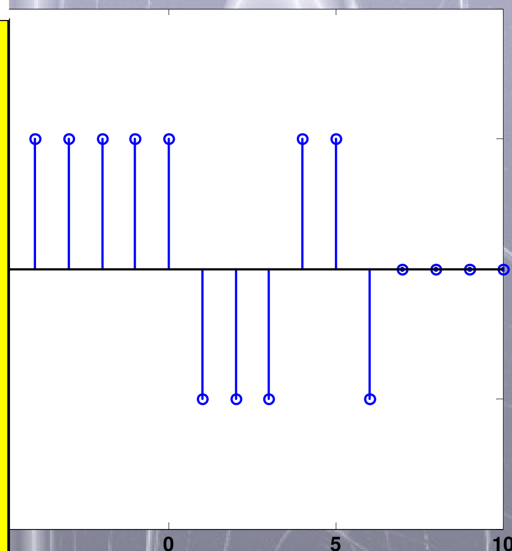


$R_x(k)=\Sigma x(n)x(n+k)$

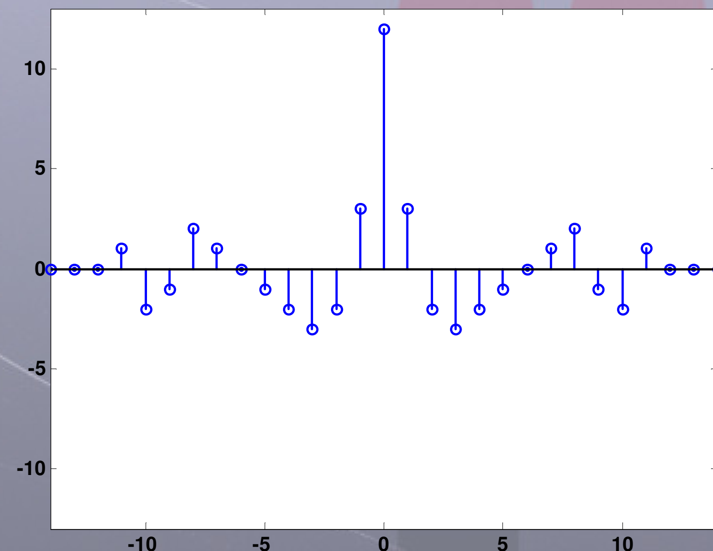
ple 2: Energy-Defined Signal



ple 2: Energy-Defined Signal



x(n)



$R_x(k)=\Sigma x(n)x(n+k)$

orrelation for Energy-Defined Signals

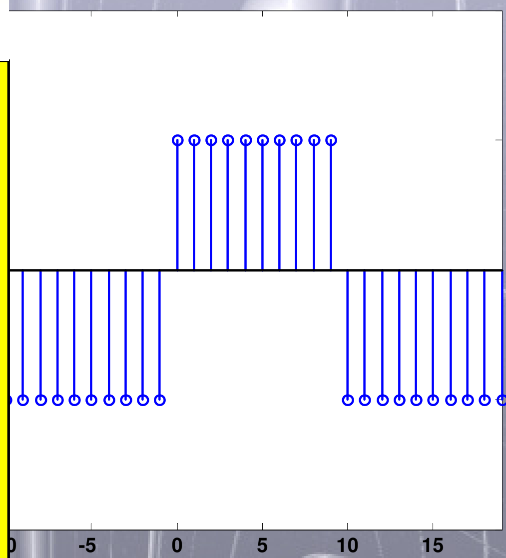
is a discrete signal energy-defined, its autocorrelation
on $R_x[k]$ is defined as:

$$R_{\mathbf{x}}[k] = \sum_{n=-\infty}^{\infty} x[n]x[n-k]$$
$$= x[k] * x[-k]$$

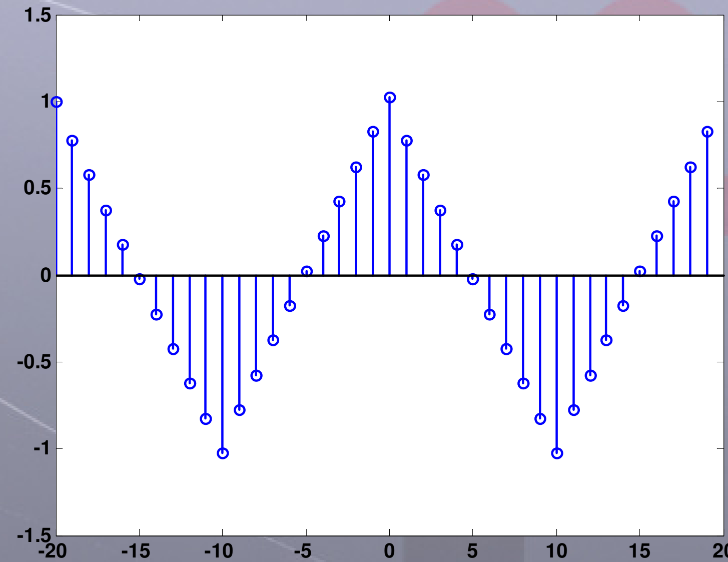
s a continuous-time signal energy-defined, its
rrelation function $R_x(t)$ is defined as:

$$R_{\mathbf{x}}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt$$
$$= x(\tau) * x(-\tau)$$

ple 3: Power-Defined Signal



x(n)



$R_{\mathbf{x}}[k]$

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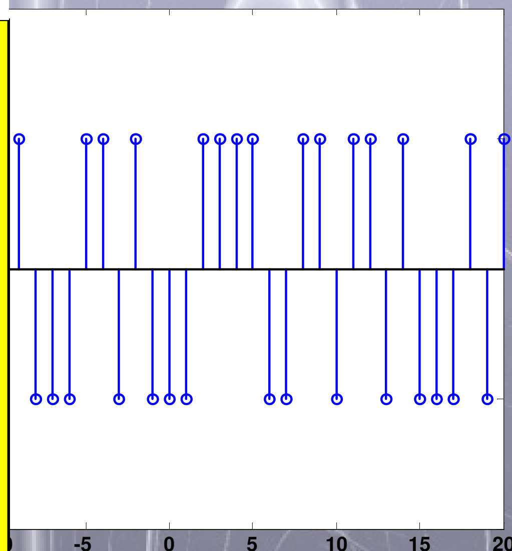
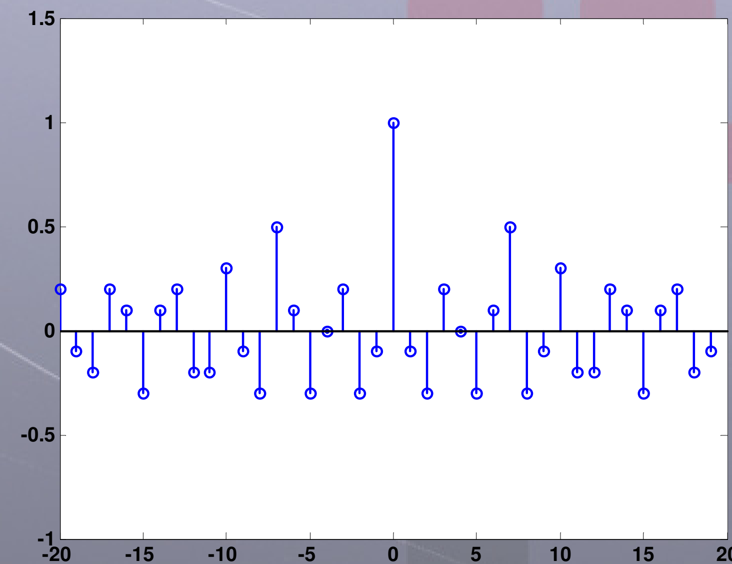
ple 4: Power-Defined Signal



x(n)



$R_x[k]$

orrelation for Power-Defined Signals

is a discrete signal power-defined, its autocorrelation
on $R_x[k]$ is defined as:

$$R_{\mathbf{x}}[k] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]x[n-k]$$

s a continuous-time signal power-defined, its autocorrelation
on $R_x(t)$ is defined as:

$$R_{\mathbf{x}}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)dt$$

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work

pute the autocorrelation of: $x(t) = A\cos(2\pi f t)$

orrelation Summary

utocorrelation function is a measurement of the self-
ss of a signal – in other words, the autocorrelation
information about the likeness of a signal with itself
elayed

fore, the autocorrelation function summarizes the time
vior of a signal

ematically, the autocorrelation function is a projection
product) of a signal against a delayed version of itself
ll possible values of delay. Consequently, the
num of the autocorrelation function is at delay equal to
- the likeness of a signal with itself – its power or
y

orrelation Properties

correlation definition is different for energy-defined
power-define signals.

th cases, autocorrelation measures signal
ess to delayed version of itself, referring this
ess to its maximum value which happens at delay
l to zero.

nergy-defined signals

$$R_{\mathbf{x}}(0) = E_x$$

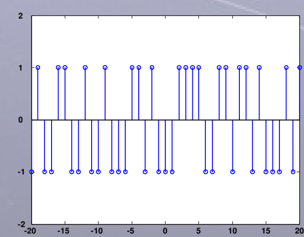
or power-defined signals

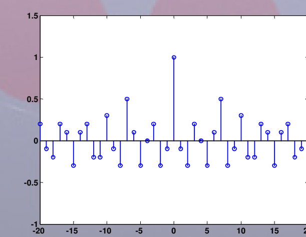
$$R_{\mathbf{x}}(0) = P_x$$

orrelation Properties

orrelation function satisfies:

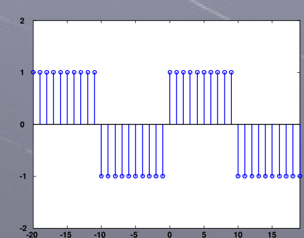
$$(\tau)\right| \leq R_{\mathbf{x}}(0)$$

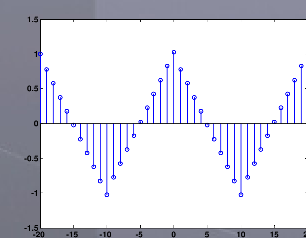




or the particular case of periodic signals, it holds that:

$$L \cdot T) = R_{\mathbf{x}}(0)$$





where L is any integer number and T is the signal period

orrelation Properties

- mmetry: $R_X(\tau)$ is and even signal: $R_X(\tau) = R_X(-\tau)$.
- aximum: $R_X(\tau)$ maximum is for $\tau = 0$ (and coincides with ergy/power of the signal), $| R_X(\tau) | \leq R_X(0)$.
- riodicity: if for a give value of T, it holds that $R_X(T) = R_X(0)$, en it also holds that $R_X(kT+\tau) = R_X(0+\tau)$ for any integer ue of k.
- egrability: the autocorrelation function of any signal (except e periodic signals) can be integrated – i.e. it is a energy-fined signal itself.

orrelation of Stochastic Processes

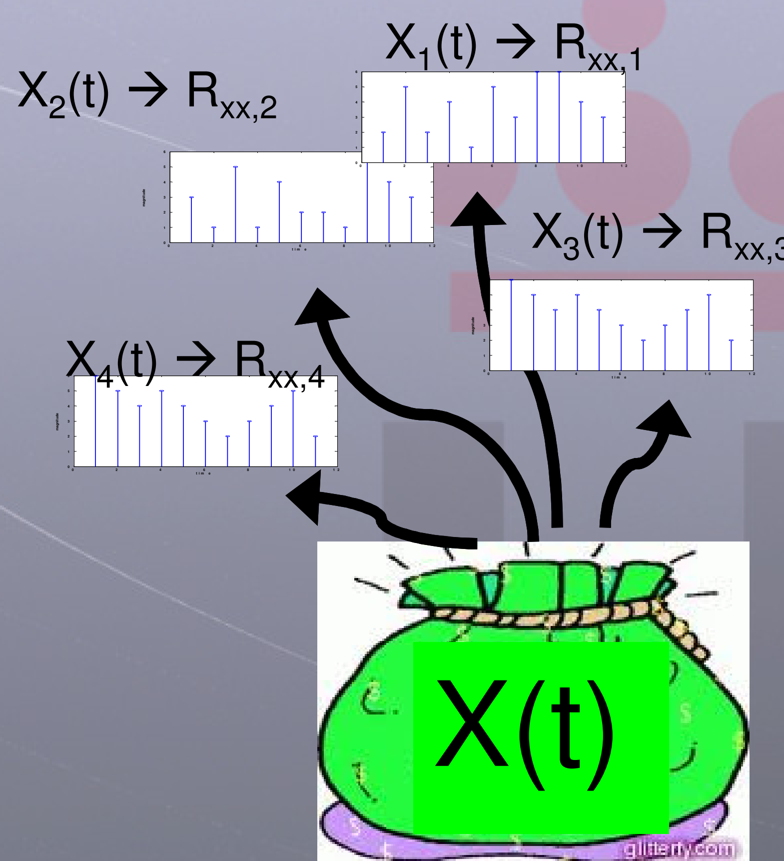
- ll the two viewpoints for SP:
  - set of signals with common properties, although signals itself e not identical point-by-point
  - physical mechanism that generates sets of signals cording to a stochastic pattern
- y case, to describe an SP the set of signals has to escribed, but a single signal can not be described.
- ow can we define the autocorrelation of a SP?

...on

...ssume, as starting point, that ...e a set of signals and we ...one of them by a random ...nism.

...tocorrelation of one of these ...tions, $x_1(t)$, can be computed ...me average using next ...sion:

$$\ldots\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x_1(t)x_1(t-\tau)dt$$



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...on

...we compute autocorrelation in such a way, we are not ...uting the autocorrelation of the SP, but the one of a ...ular signal (realization)

... order to compute the autocorrelation of the SP, we ...d average over all possible realizations:

$$R_{\mathbf{x}}(\tau) = E\left[\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t-\tau)dt\right]$$

$$= \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} x(t)x(t-\tau)dt\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\left[x(t)x(t-\tau)\right]dt$$

...on → Definition

...**expected value does not depend on the time**, then:

$$R_x(\tau) = E\left[x(t)x(t-\tau)\right]\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt$$
$$= E\left[x(t)x(t-\tau)\right]$$

...utocorrelation function for a SP is noted as $R_X(t_1,t_2)$, and its ...on is

$R_X(t_1,t_2) = E\,[X(t_1)\,X(t_2)]$:

$$R_X(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1 dx_2$$

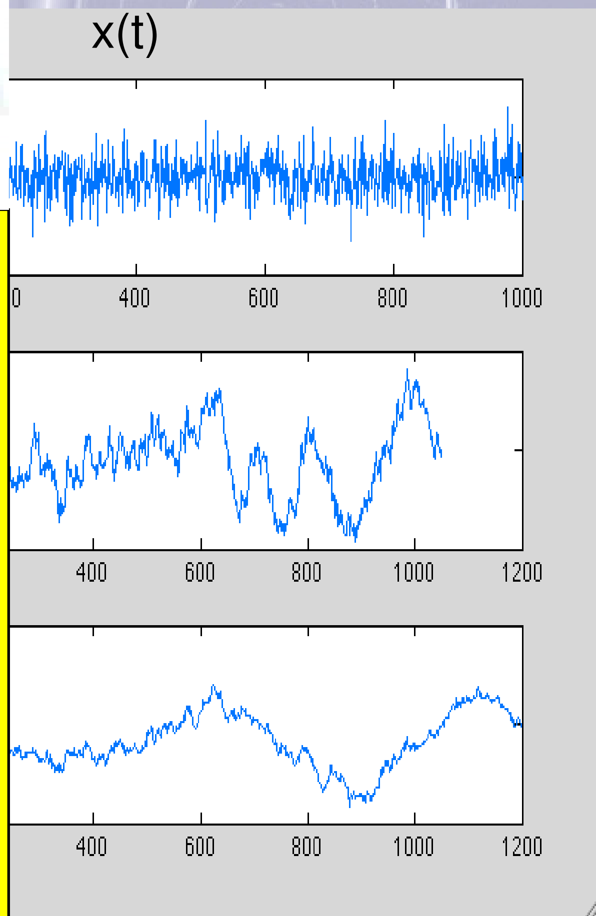
... is a measurement of the likeness between Random ...les obtained in two instants of the SP, $X(t_1)$ and $X(t_2)$. i.e. ...$_2$) is a measurement of the likeness (variation) of the signal ... different instants of time t_1 and t_2.

ple: Rxx based on one SP realization

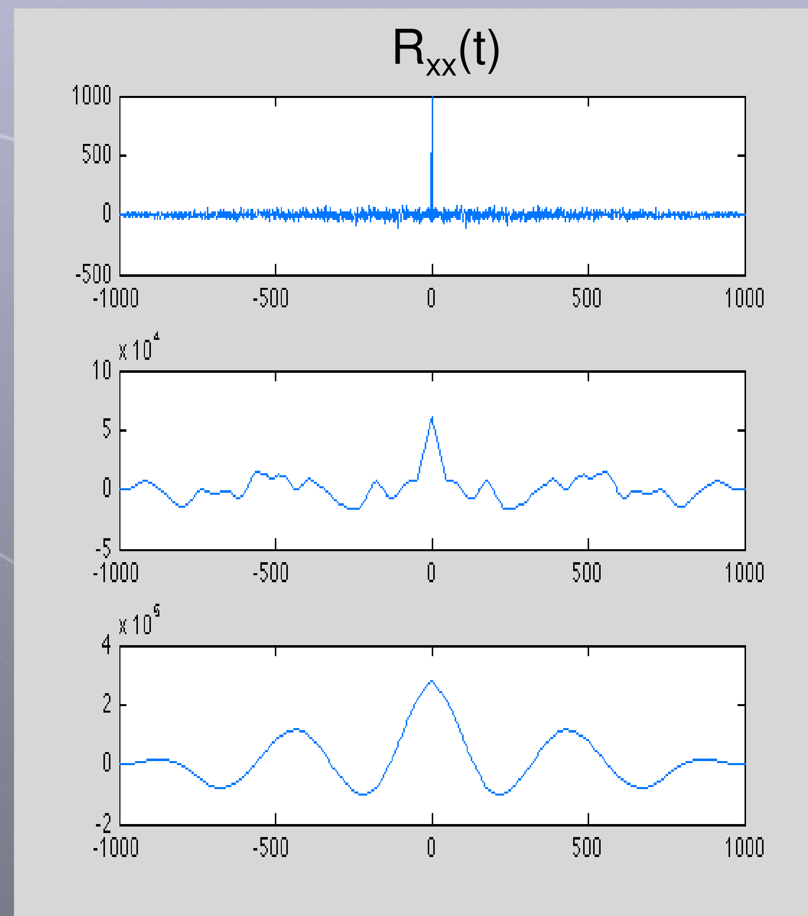
rea
of

rea
of

rea
of



3 more correlated than SP 2,
n is more correlated than SP 1
(uncorrelated)



Autocorrelations obtained using
only the single realization

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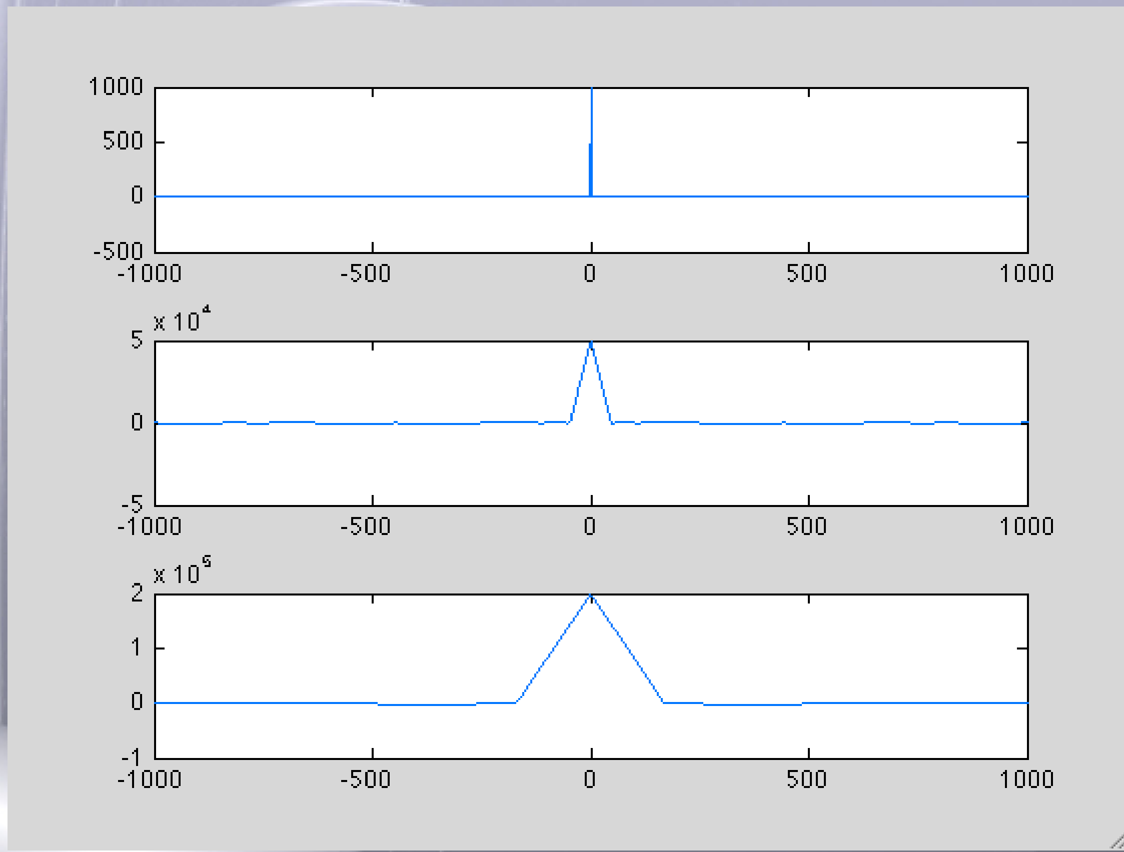
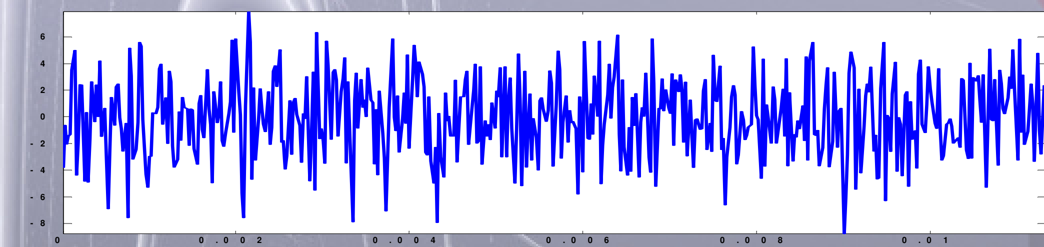
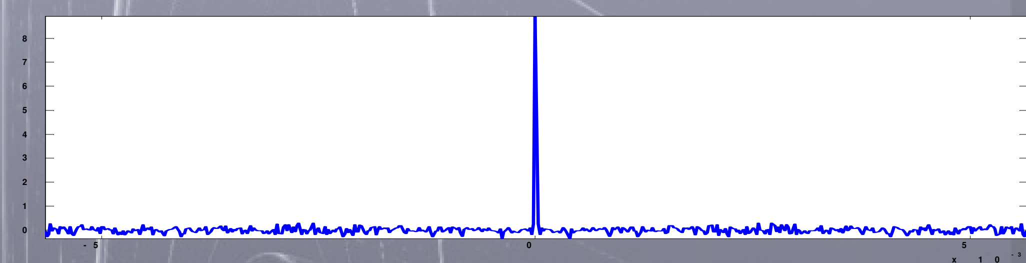
ple: Rxx averaging over “all” (many) SP ations



related Processes

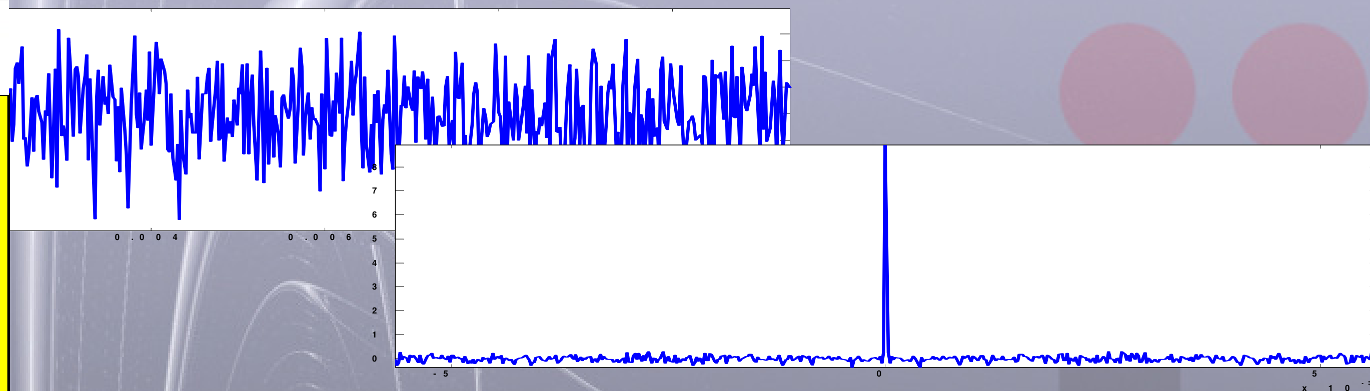
is uncorrelated if $R_X(t_1,t_2) = 0$, for any t_1 and t_2 that $t_1 \neq t_2$.

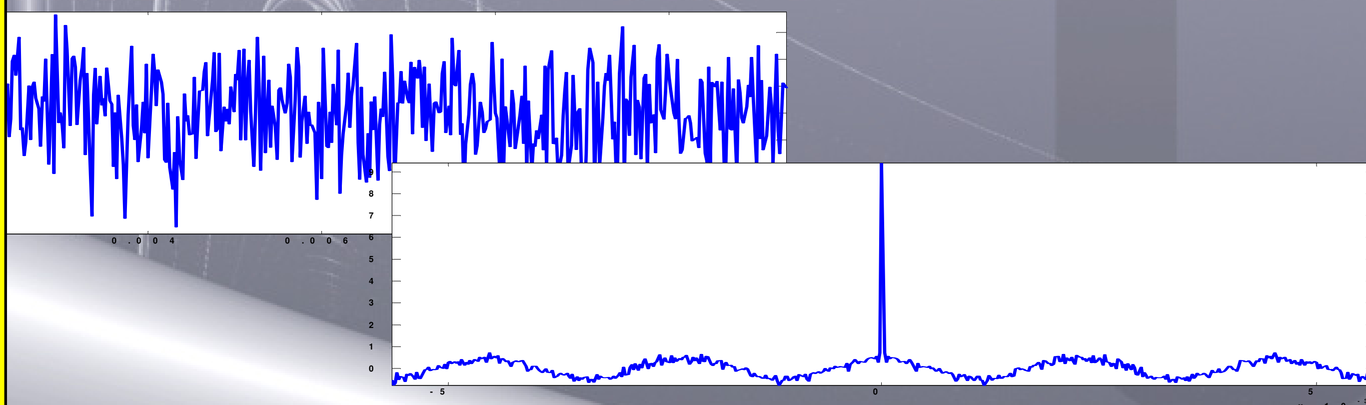




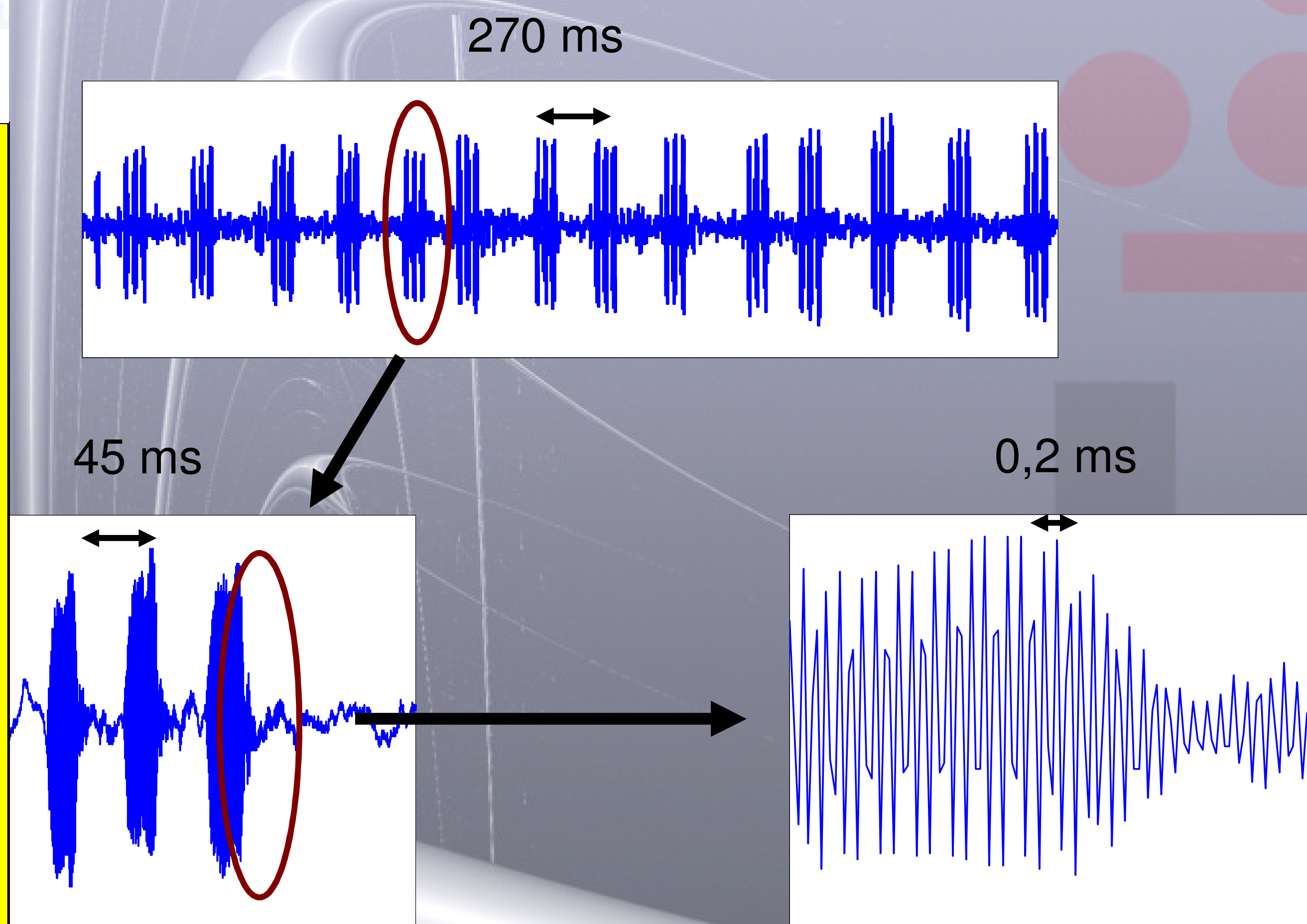
uitively, there is not “likeness” between samples at different es of the process

n Telecommunications often is Uncorrelated

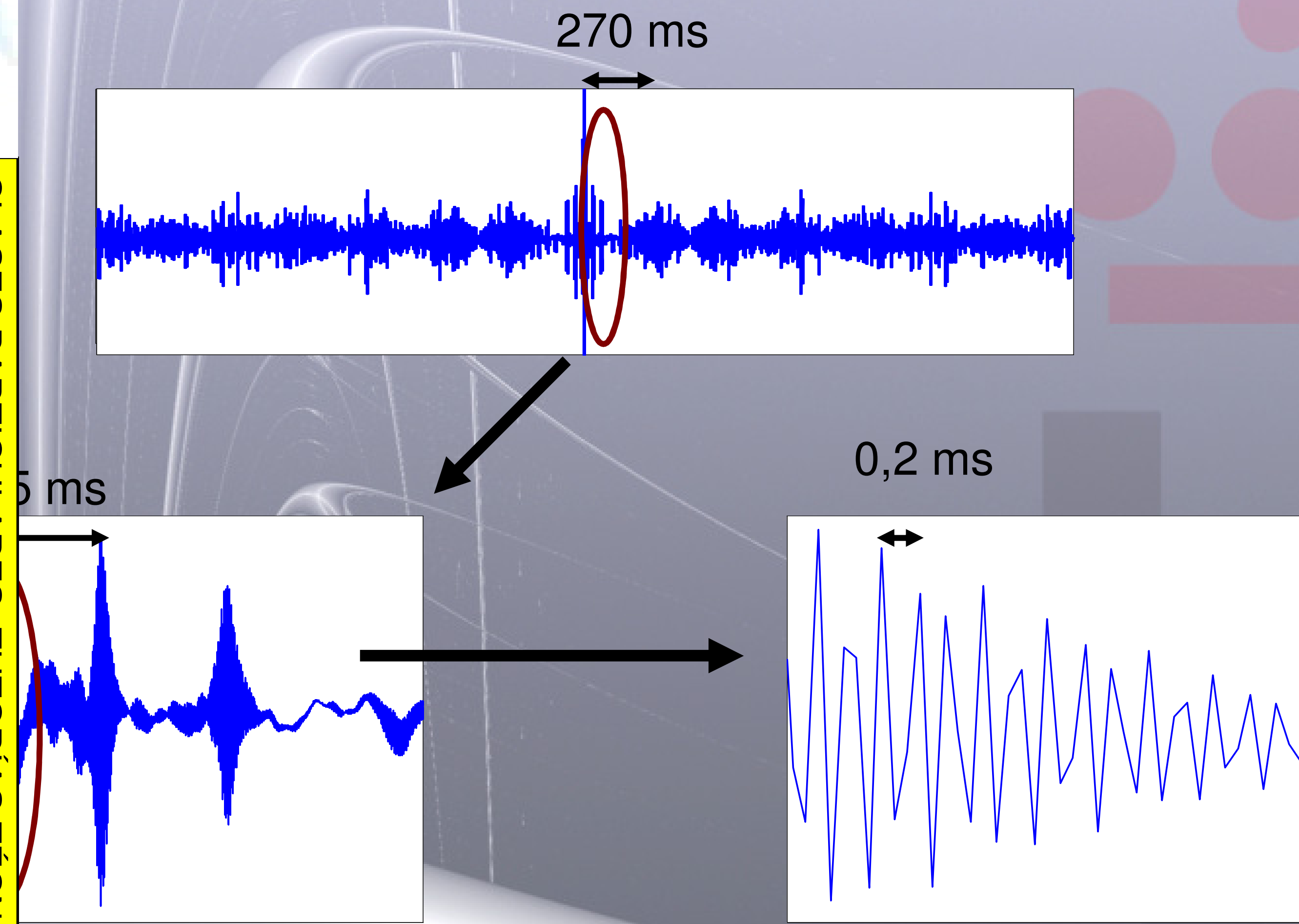




n/off wave (cricket)



n/off wave



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arizing

correlation is a measurement of the likeness
een a signal and a delayed version of itself

realization of a SP may be quite different and
puting any statistic on it will mislead to totally
rect information. Autocorrelation has to average
any) realizations, i.e. compute the expected value

values in the autocorrelation function hints about
ition patterns

tical Independence

cenarios:

lependence of the samples of a signal

lependence of two signals

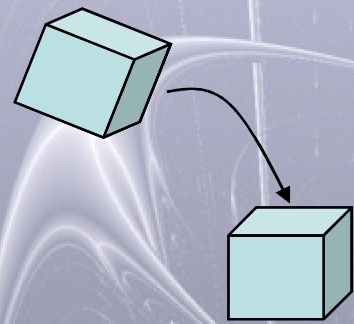
vely, two samples are independent if the mechanisms that
ates them are also independent

lly, two Random Variables, X_1 and X_2, are independent if,
nly if,

$$f_{X_1X_2}(x_1,x_2)=f_{X_1}(x_1)\cdot f_{X_2}(x_2)$$

...on for Independent Samples

dice

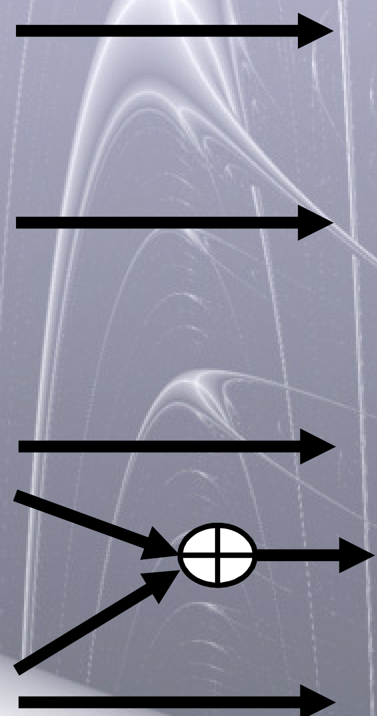


...nt: → 3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3

3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3, …

...t: → 4, 6, 6, 5, 6, 4, 3, 7, 10, 7, …

…on for Independent Signals



3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3

1, 1, 6, 4, 3, 1, 2, 4, 5, 5, 2

Independent?

…nt:

3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3

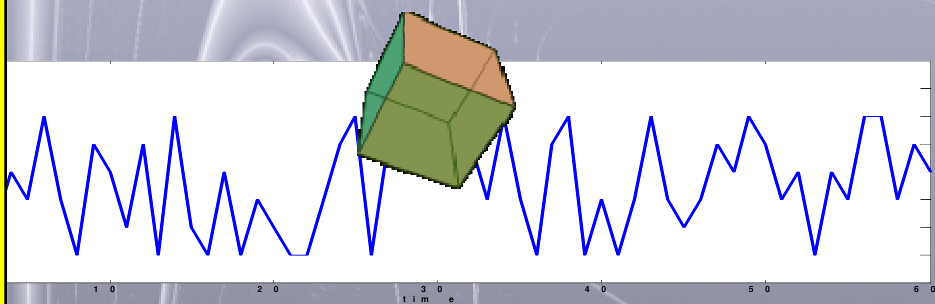
4, 2, 11, 5, 7, 3, 4, 5, 11, 9, 5

1, 1, 6, 4, 3, 1, 2, 4, 5, 5, 2

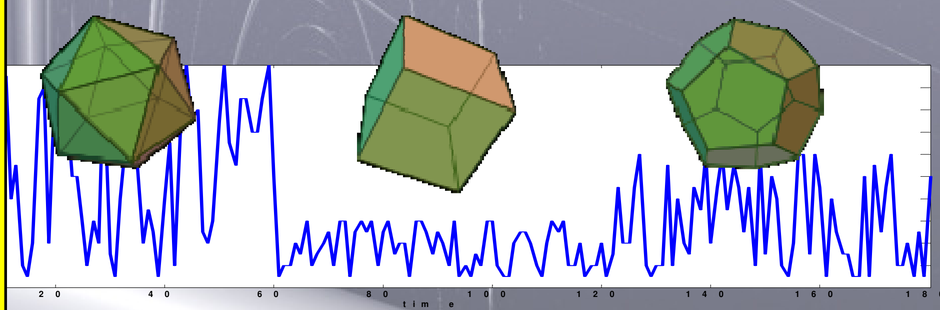
Independent?

d Stationarity

P is stationary if its statistics do not depend on



Stationary



Non-Stationary

nary Stochastic Processes

arriving to practical SP, two types of stationarity can be defined:

ct (Sense) Stationary Processes (SSP). The pdf of the process does not nge with time delay Δ

$$f_X(t_1,t_2,...,t_n) = f_X(t_1+\Delta, t_2+\Delta,...,t_n+\Delta)$$

de Sense Stationary Processes (WSSP), those that satisfy the two following trictions:

- The mean value of the process, E{X(t)}, does not varies with time

$$R_X(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$

- The autocorrelation $R_X(t_1,t_2)$ depends only on the time difference $\tau = t_1 - t_2$. Thus, for WSSP we compute the autocorrelation as $R_X(\tau)$.

$$R_{\mathbf{x}}(\tau) = E\left[x(t)x(t-\tau)\right]$$

P is also WSSP, but the opposite is not true

d Ergodicity

ochastic Process is Ergodic if its statistics can be
uted as time average of one of its realizations

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x(t))dt = E(g(X(t)))$$

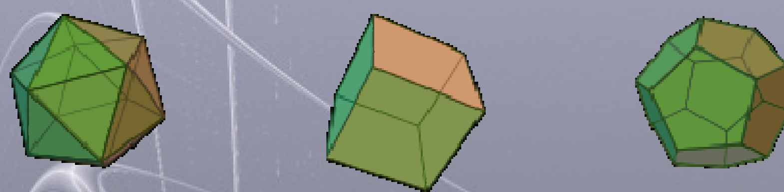
dicity implies stationarity. Any ergodic process is
nary

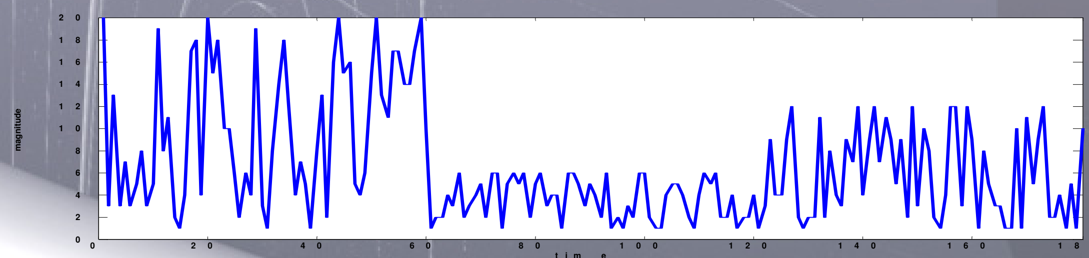
ever, stationarity does not imply ergodicity

nple: roll a dice infinite times

ple 7

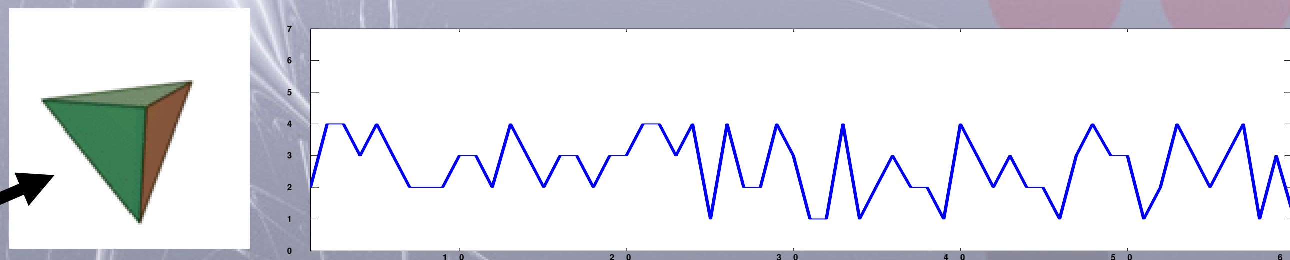
ollowing SP is not stationary, and therefore it is rgodic either.

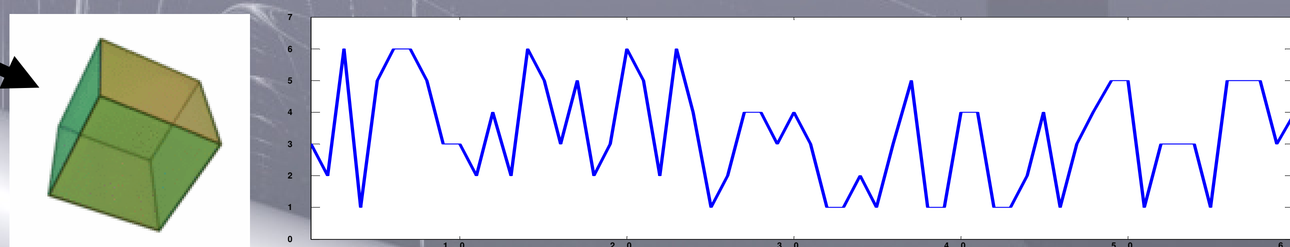




ple 8

ollowing Strict Sense Stationary Process is not dic. Why?





...veraging and Statistical Expected Value

- ... average and expected value returns the same ... only for ergodic SP

- ...xample, if we assume that human voice signal ...sponds to a ergodic SP, then we can estimate ...tics from time averaging only one recorded piece ...ce, and assume that those values coincide with ...tatistics of any voice signal

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veraging and Statistical Expected Value

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

$$\updownarrow$$

$$) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$\langle x^2(t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)\,dt$$

$$\updownarrow$$

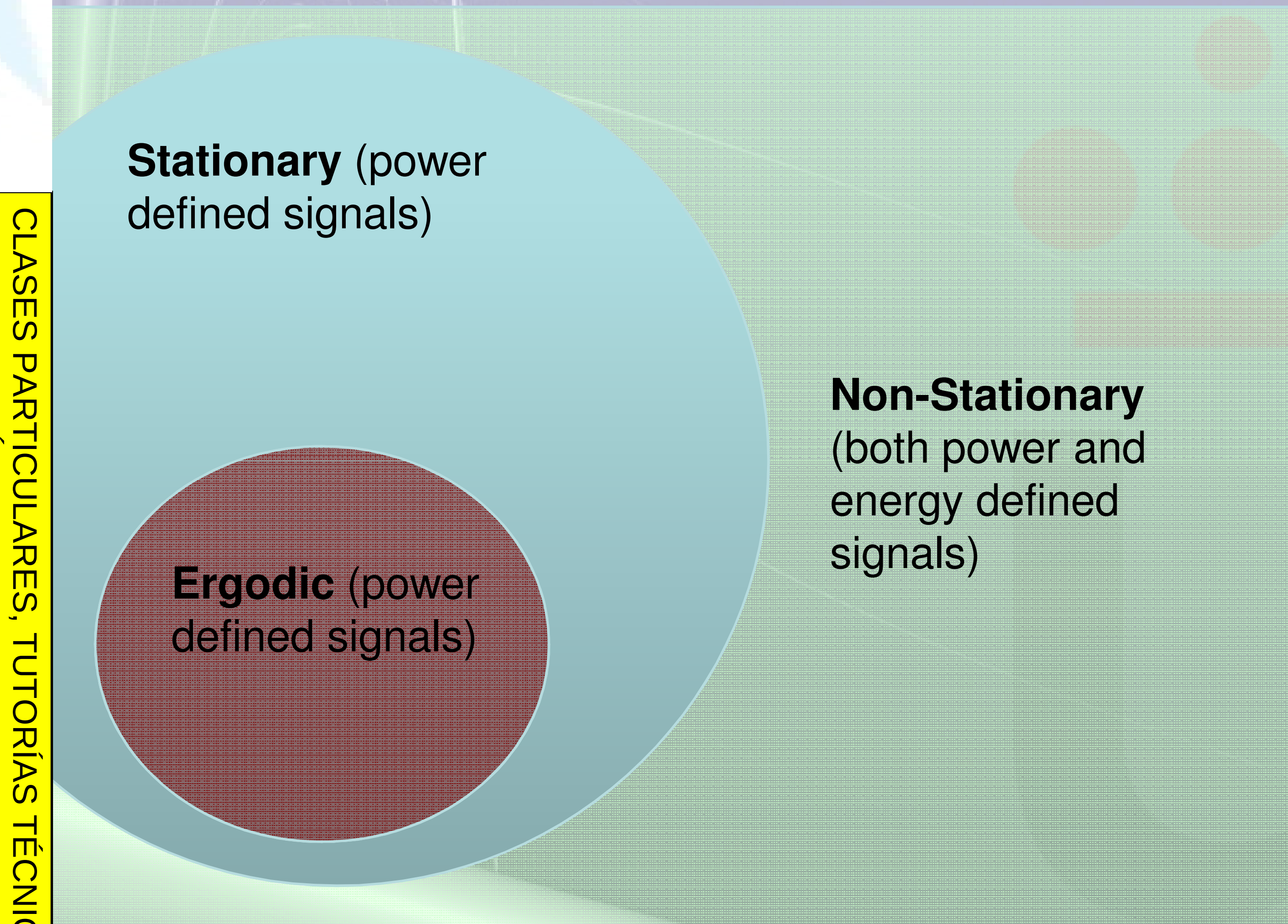
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$R_{\mathbf{x}}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t-\tau)\,dt$$

$$\updownarrow$$

$$R_X(\tau) = E(X(t)X(t-\tau))$$

ssification



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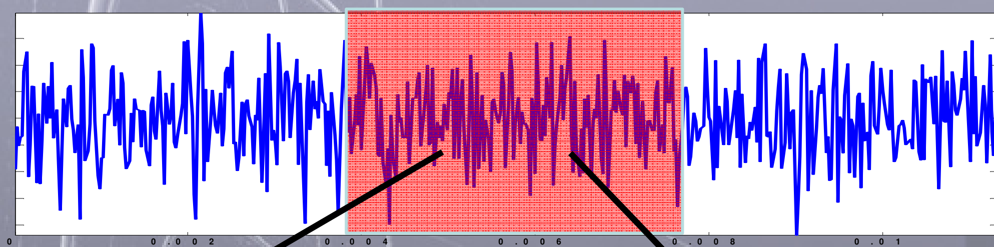
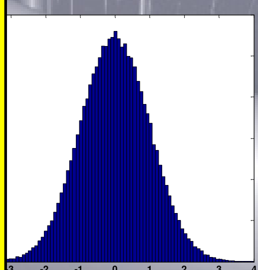
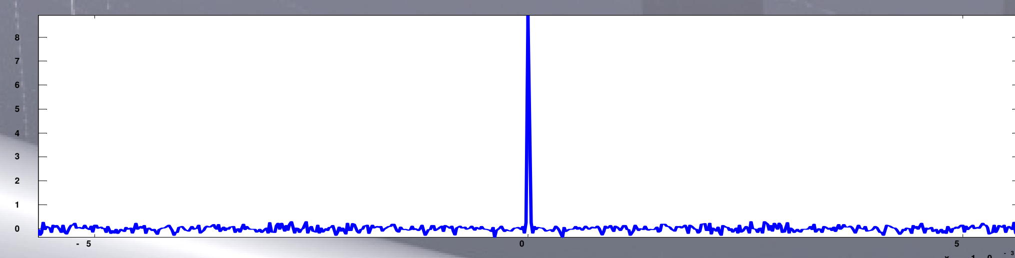
...s the practical meaning of Stationarity?

...an characterize one realization from a segment of it







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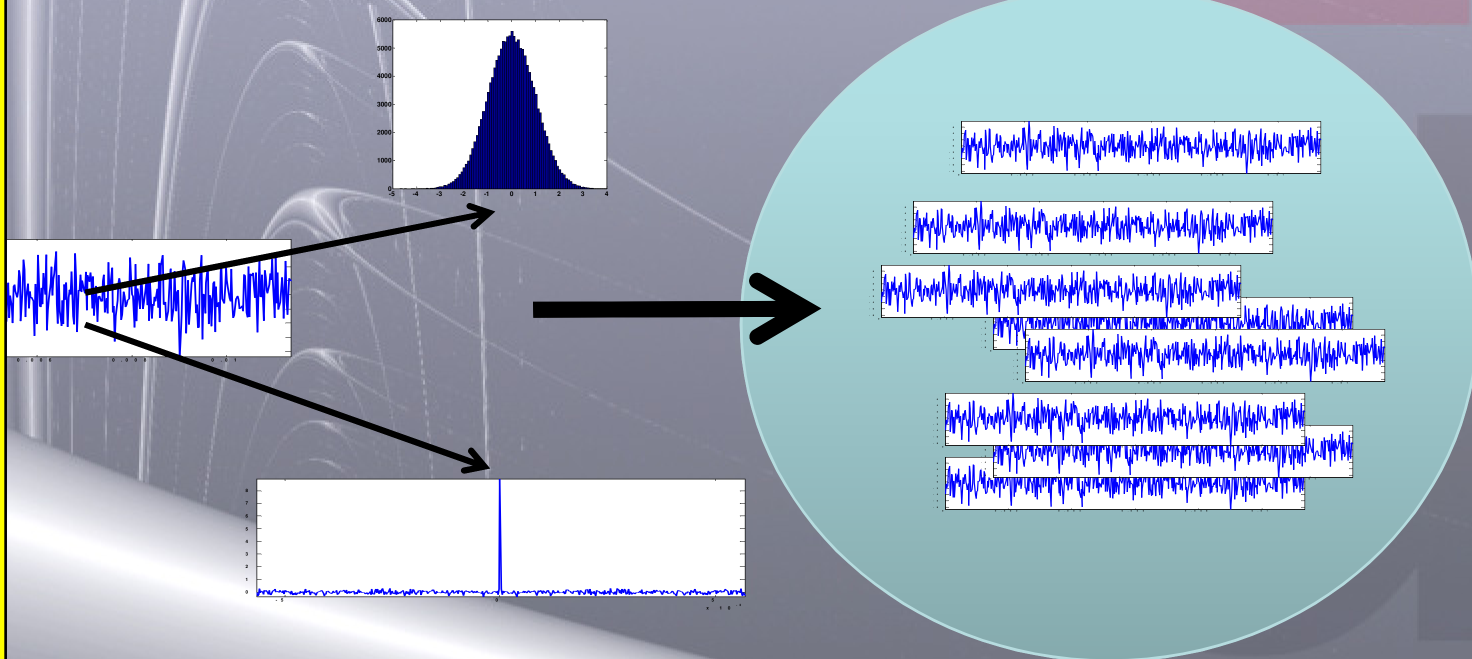
...s the practical meaning of Ergodicity?

...an characterize the SP from a realization of it



-Correlation

ocorrelation is a measurement of how a signal like itself delayed; the cross-correlation function des information about likeness of two signals ying one respect the other)

nding of the type of signals, cross-correlation can efined for:

- oss-Correlation of Energy Defined signals
- oss-Correlation of Power Defined signals
- oss-Correlation of one Energy Defined signal and one wer Defined signal

Correlation of two Energy Defined Signals

n] and y[n] be two energy defined discrete signals (**or one y defined and the other Power defined).** Their cross-ation function is defined as:

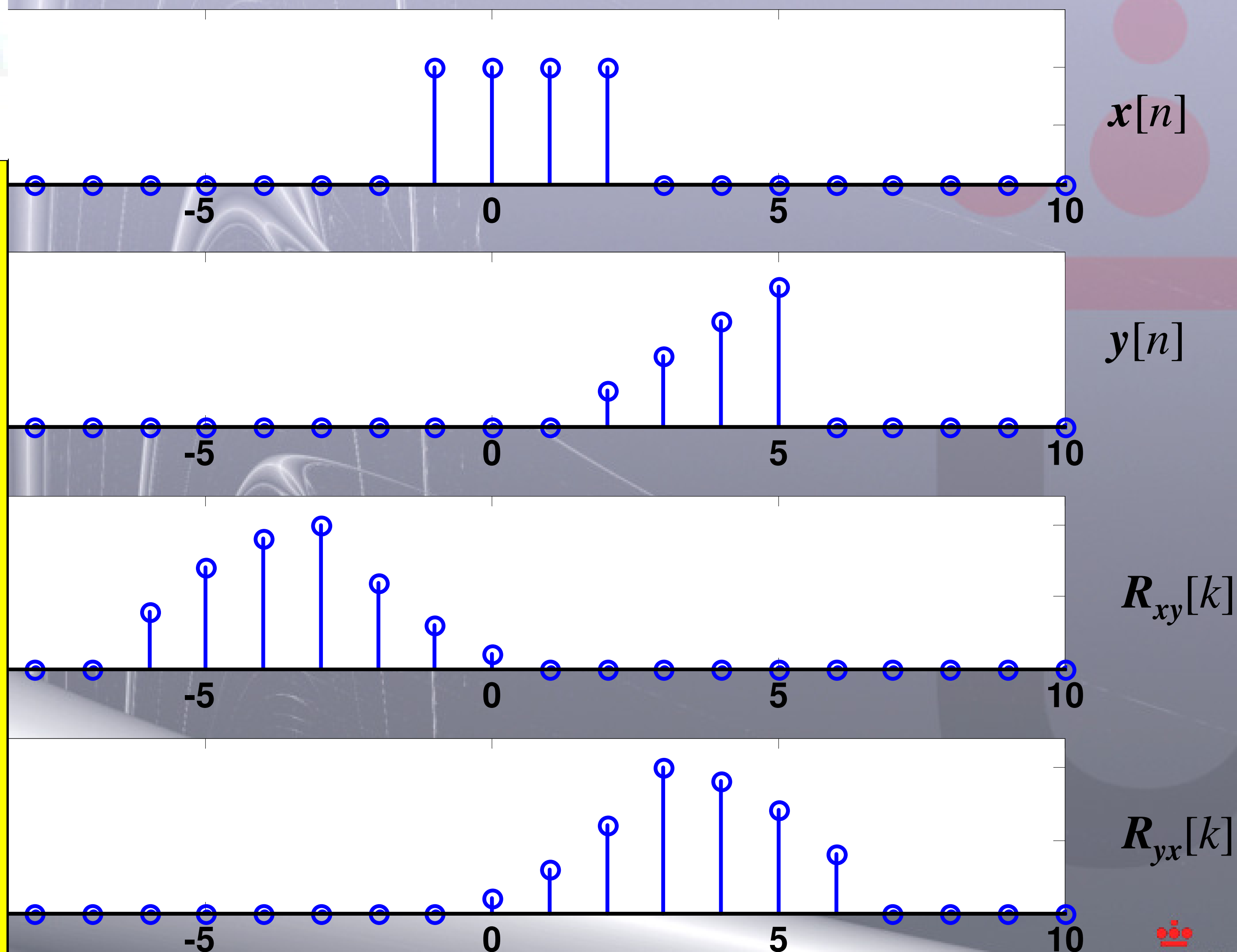
$$R_{xy}[k] = \sum_{n=-\infty}^{\infty} x[n]y[n-k]$$

$$= x[k] * y[-k]$$

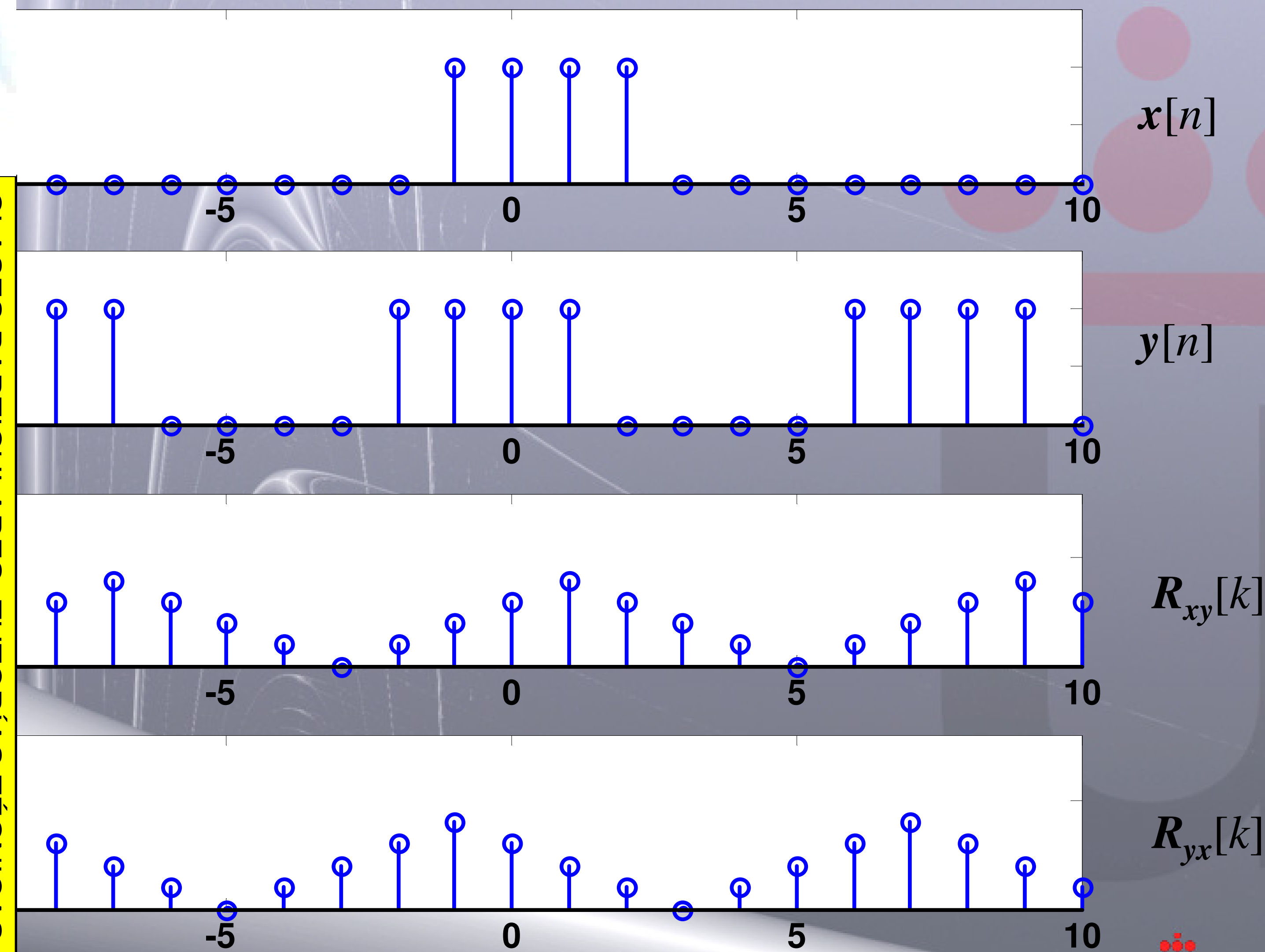
and y(t) are two energy defined continuous signals (**or one y defined and the other Power defined).** Their cross-ation function is defined as:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t-\tau)dt = x(\tau) * y(-\tau)$$

...ple 1



ple 2



Correlation of two Power Defined Signals

[n] and y[n] be two power defined discrete signals. cross-correlation function, $R_{xy}[k]$, is defined as:

$$R_{xy}[k] = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{n=-\infty}^{\infty} x[n] y[n-k]$$

(t) and $y(t)$ be two power defined continuous ls. Their cross-correlation function, $R_{xy}(\tau)$, is ed as:

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) y(t-\tau) dt$$

-Correlation from a Statistical Viewpoint

s-Correlation of two stochastic processes can be ed also using their joint-pdf:

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),Y(t_2)}(x,y)\,dxdy$$

n both processes are stationary, then it holds that:

$$R_{XY}(\tau) = E\left[X(t)Y(t-\tau)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t),Y(t-\tau)}(x,y)\,dxdy$$

-Correlation Properties

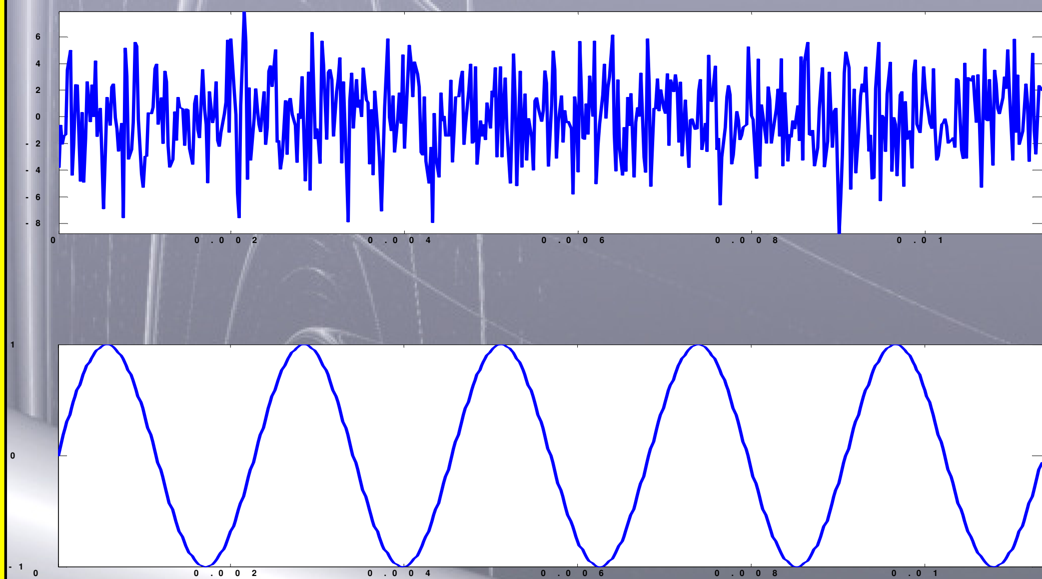
$(\tau) = R_{yx}(-\tau)$.

$= R_{xy}(0) = R_{yx}(0)$ can be understood as the cross-power between) and y(t).

e maximum of $R_{xy}(\tau)$ points to the time delay at which both nals exhibit their maximum likeness

related Processes

- P are uncorrelated if $R_{xy}(\tau) = 0$ for every value of τ
- be proven that if two SP are independent and at least one of as zero mean, then they are uncorrelated



...f Signals

...orrelation of the sum of two signal can be expressed as:

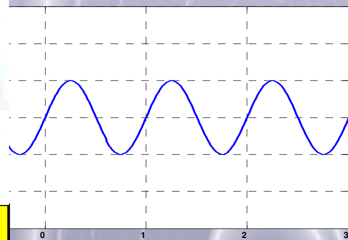
$$R_{\mathbf{x}+\mathrm{y}}(\tau) = R_{\mathbf{x}}(\tau) + R_{\mathrm{y}}(\tau) + R_{\mathbf{x}\mathrm{y}}(\tau) + R_{\mathrm{yx}}(\tau)$$

...d particularizing for τ=0, the power of the sum is:

$$\begin{aligned} P_{\mathbf{x}+y} &= R_{\mathbf{x}+\mathrm{y}}(0) \\ &= R_{\mathbf{x}}(0) + R_{\mathrm{y}}(0) + R_{\mathbf{x}\mathrm{y}}(0) + R_{\mathrm{yx}}(0) \\ &= P_{\mathbf{x}} + P_{\mathrm{y}} + P_{\mathbf{x}\mathrm{y}} + P_{\mathrm{yx}} \end{aligned}$$

...portant to note that only when the two processes are ...orrelated, the power of the sum is the sum of the ...wers

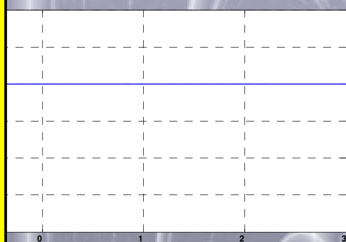
Signal plus DC



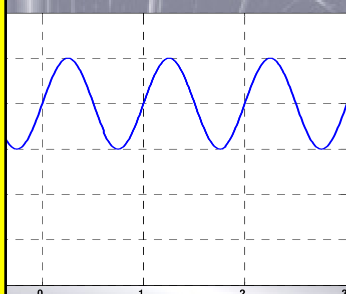
$$P_{\sin} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \sin^2(ft)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[-\frac{1}{2} A^2 \cos(2ft) - \frac{1}{2} A^2 \cos(ft - ft) \right] dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} 2T \frac{1}{2} A^2 = \frac{A^2}{2}$$



$$P_{con} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} B^2 dt = B^2$$



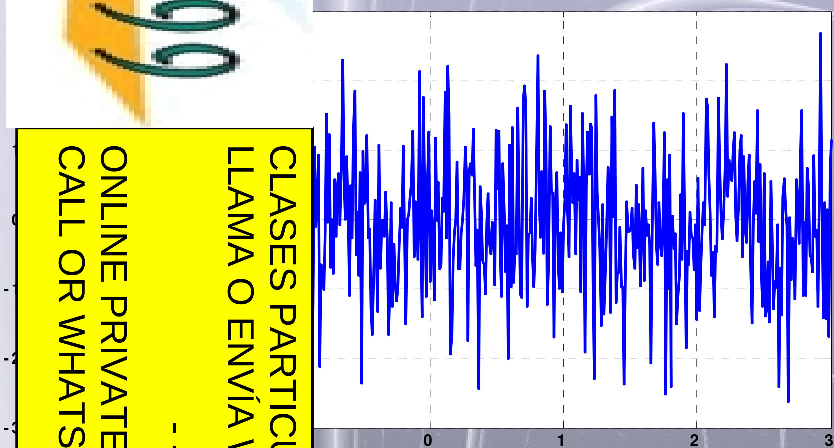
$$P_{con+\sin} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (B + A\sin(ft))^2\,dt$$

$$= \frac{A^2}{2} + B^2$$

$$= P_{con} + P_{\sin}$$

sian Noise

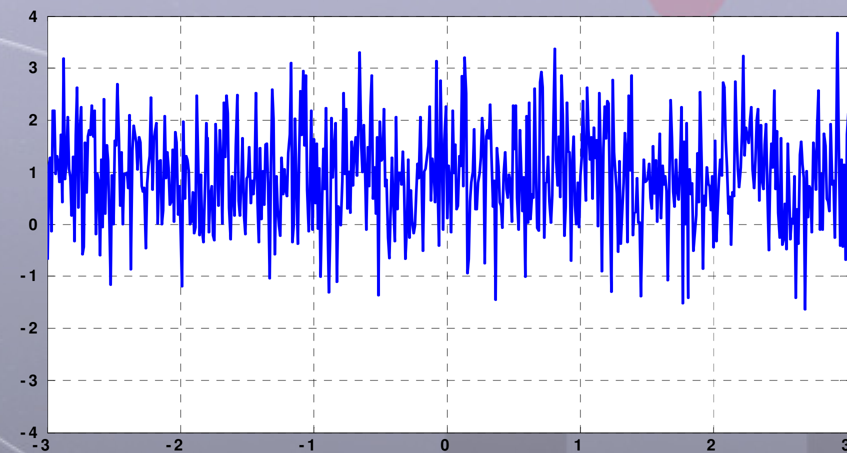
Zero mean



$$\int_{-\infty}^{\infty} x^2 f_X(x)dx$$

$$\sigma^2$$

B mena



$$P_{noise+B} = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

$$= \sigma^2 + B^2$$

$$= P_{noise} + P_{con}$$

o we want to study the Spectrum?

- c processes are quite common in nature
  - tronomy: lunar phases, planets orbits, solar storms, …
  - ology: heart beat,…
  - ysics: acoustic vibrations, electromagnetic waves, …
- mmunications, spectrum of the signal is of capital rtance when designing and analyzing systems
- gnal is deterministic, the Fourier Transform utes the spectrum, but what happen for hastic Processes?

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y Spectral Density

Energy Spectral Density (ESD) of a energy-
ed signal is calculated as the Fourier Transform of
to-correlation function:

$$
\begin{aligned}
G_X(f) &= F\{R_X(\tau)\} \\
&= F\{x(\tau) * x(-\tau)\} \\
&= |X(f)|^2
\end{aligned}
$$

r Discrete time signals ESD is defined in similar way

y Spectral Density

- Energy Spectral Density describes how the energy tributed along different frequencies
- he total signal energy can be calculated by rating the ESD for all frequencies

$$E_X = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$= \int_{-\infty}^{\infty} G_X(f) df$$

Spectral Density

Power Spectral Density (PSD) of a power-defined signal is ...ated as the Fourier Transform of its auto-correlation ...on:

$$S_X(f) = F\{R_X(\tau)\}$$

...r Discrete time signals PSD is defined in similar way

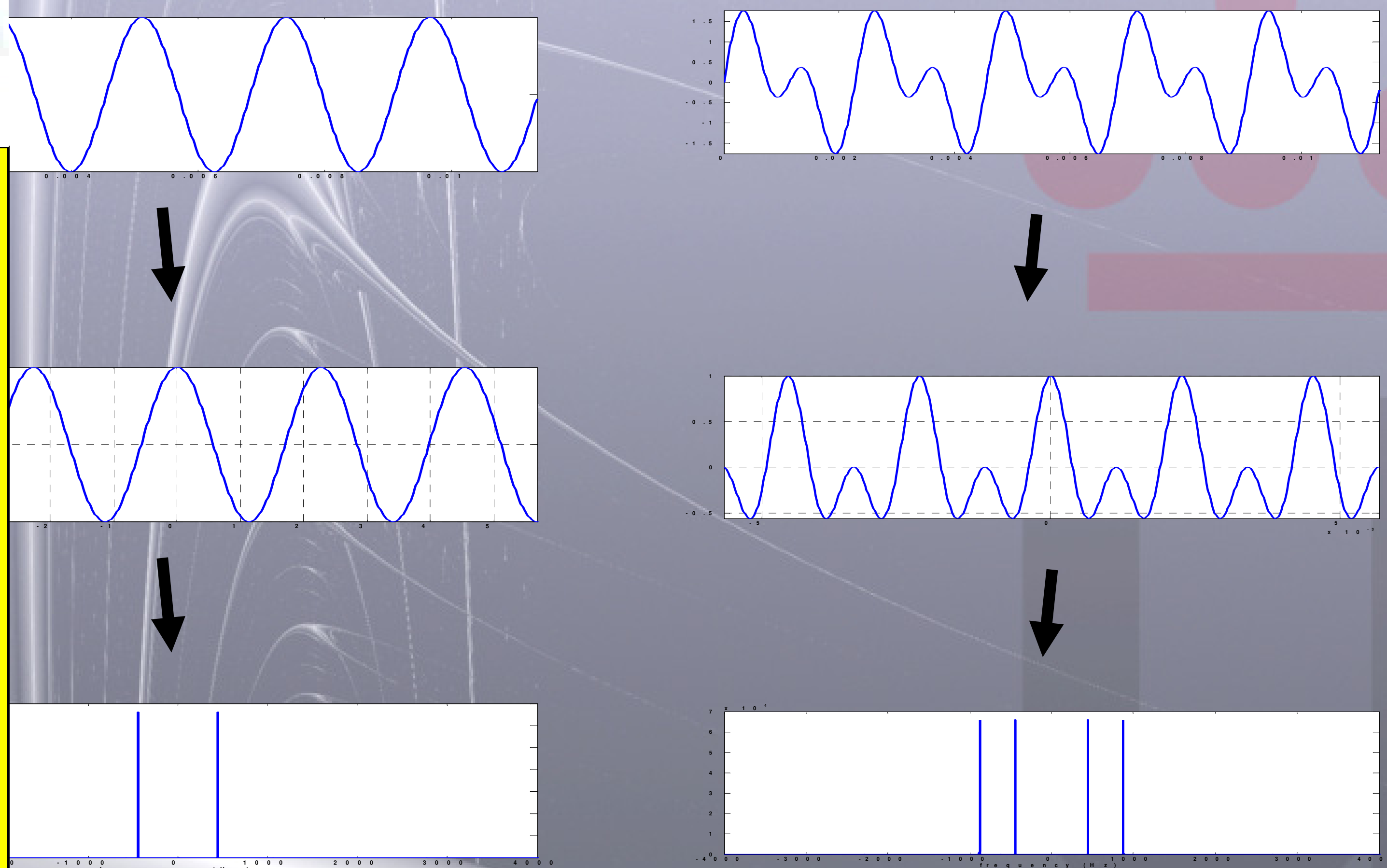
Spectral Density

- Power Spectral Density describes how the power ... tributed along different frequencies
- ...he total signal power can be calculated by ...rating the PSD for all frequencies

$$P_X = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} S_X(f) df$$

Wave and the Sum of two Sine Waves



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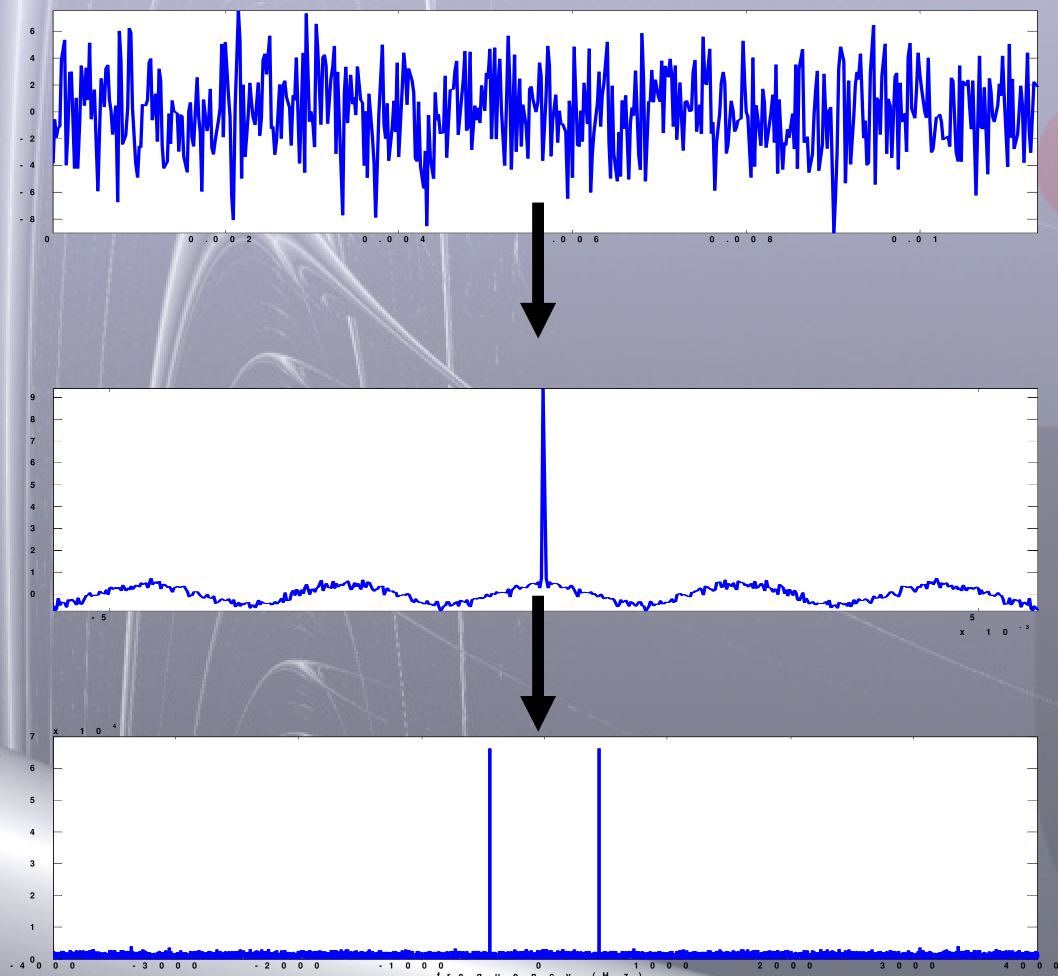
Wave and White Noise



roperties

mmetry: PSDs are even functions, $G_x(f)=G_x(-f)$ y $f)=S_x(-f)$.

n: PSD is real-valued and No-Negative for any ue of frequency f, $G_x(f)\geq 0$, $S_x(f)\geq 0$.

egrability: Energy or Power can be calculated as integral of the their Spectral Density

Spectral Density

e $x(t)$ and $y(t)$ two energy-defined signals which -correlation function is $R_{xy}(\tau)$. Their Cross tral Density is defined as the Fourier Transform of cross-correlation function:

$$G_{xy}(f) = F\{R_{xy}(\tau)\}$$

ogously, if $x(t)$ and $y(t)$ are power-defined signals cross spectral density is defined as:

$$S_{xy}(f) = F\{R_{xy}(\tau)\}$$

...f Signals

...tral Density of a sum of signals satisfies the ...ving relationship:

$$\ldots_{\mathbf{x}+\mathrm{y}}(f) = F\left\{R_{\mathbf{x}+\mathrm{y}}(\tau)\right\}$$

$$= F\left\{R_{\mathbf{x}}(\tau) + R_{\mathrm{y}}(\tau) + R_{\mathbf{x}\mathrm{y}}(\tau) + R_{\mathrm{yx}}(\tau)\right\}$$

$$= S_{\mathbf{x}}(\tau) + S_{\mathrm{y}}(\tau) + S_{\mathbf{x}\mathrm{y}}(\tau) + S_{\mathrm{yx}}(\tau)$$

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ary of Spectral Density

y (or Power) Spectral Density describes how energy
wer) is distributed along frequencies

e used for Stochastic Processes and deterministic
s

case of SP, Spectral Density represents an statistical
ge of the process. One particular realization may have
ent spectrum

ral Density of the sum of several signals can be
ated by computing the Spectral Density of each
entary signal and their cross spectral density

...ary of Concepts in this Chapter

...s chapter we learned:

- ...w to model a Stochastic Process
- ...ferent properties of SP: Independence, Stationarity, Ergodicity
- ...w to compute the autocorrelation of a SP and its physical meaning
- ...w to compute the Energy(Power) Spectral Density of a SP and its ...ysical meaning