TEMA 3 - SIGNALS AND SYSTEMS IN THE FREQUENCY DOMAIN

LINEAR SYSTEMS WITH CIRCUIT APPLICATIONS

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Biomedical Engineering Degree

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LTI systems response to sinusoidals

Motivation

- In the previous topic, the LTI systems were characterized by means of their impulse response: the time domain.
- Now we will see how to characterize the LTI systems by means of their response to sinusoids: the frequency domain.
- Usage of complex exponential functions as a mathematical tool simplifies calculations.
- The frequency domain representation is the foundation of current telecommunications systems.

Outline of this topic

1. We start seeing that the response of LTI systems to complex exponentials depends on the frequency.
2. We represent periodic signals as the sum of exponential functions: Fourier Series.
3. We represent any type of signals as the sum (by means of integration operation) of exponential functions: the Fourier Transform.
4. We study to basic applications: filtering and modulation.

Frequency response of LTI systems

The response of LTI systems to complex exponentials

- Consider a continuous time LTI system, characterized by $h(t)$.
- Suppose that the LTI system input is a complex exponential $x(t) = e^{s_0 t}$, being $s_0 = \sigma + j\omega$.
- The LTI system output is calculated by means of the convolution method:

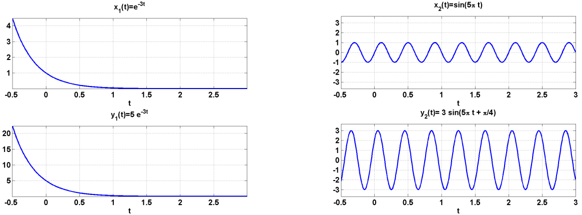
$$y(t) = x(t) * h(t) = e^{s_0 t} * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau =$$

$$= \int_{-\infty}^{\infty} e^{s_0(t-\tau)}h(\tau)d\tau = e^{s_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-s_0\tau}d\tau = x(t)H(s_0)$$

- $H(s_0)$ is a (complex) constant, that depends on the impulse response and on the exponent of the system input (the exponential function).
- Complex exponential signals are known as *eigenfunctions* of the LTI systems, as the system output to these inputs equals the input multiplied by a constant factor. Both amplitude and phase may change, but the frequency does not change.

Frequency response of LTI systems

Response to real exponential functions and to sinusoids



System function

- If we represent the factor scales for any s_0, we obtain the system function:

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

- Note that this function includes the system response to any complex exponential function.
- Also note that this function depends on the impulse response, that includes all the information related to the LTI system.

Frequency response of LTI systems

Example: output calculation using the system function

- Consider a LTI system characterized by $h(t) = u(t)$. Calculate the output when the input is:

$$x(t) = Ae^{s_1 t} + Be^{s_2 t} + Ce^{s_3 t}$$

- We start calculating the system function:

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} u(\tau)e^{-s\tau}d\tau = \int_{0}^{\infty} e^{-s\tau}d\tau =_{Real(s)>0}$$

$$= \frac{1}{-s}[e^{-s\tau}]_0^\infty = \frac{1}{-s}[0-1] = \frac{1}{s}$$

- Using the linearity property:

$$y(t) = H(s_1)Ae^{s_1 t} + H(s_2)Be^{s_2 t} + H(s_3)Ce^{s_3 t} = \frac{A}{s_1}e^{s_1 t} + \frac{B}{s_2}e^{s_2 t}\frac{C}{s_3}e^{s_3 t}$$

where we assume that $Real(s_1), Real(s_2), Real(s_3) > 0$.

Frequency response of LTI systems

System function and frequency response

- Complex exponentials with an exponent that is an imaginary number, $x(t) = e^{j\omega t}$, are always periodic signals.
- Moreover, we will see that any periodic signal can be represented as a weighted sum of this kind of signals.
- What is the LTI system response to these complex exponentials? We can perform convolution. Or we can use the system function $H(s)$ in the special case $s = j\omega$.

$$y(t) = H(s = j\omega)x(t) = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau} d\tau = H(j\omega)e^{j\omega t}$$

- The LTI system response to $H(j\omega)$ is called *frequency response* or transfer function. This function depends on the frequency of the input and it will affect (modify) different frequencies differently (amplitude and phase).

Frequency response of LTI systems

Example: calculation of the frequency response

- Given a LTI system characterized by $h(t) = e^{-t}u(t)$, calculate and plot its frequency response. Calculate the output when the input is $x(t) = 2e^{j2t} + 3e^{j\pi t}$.
- We calculate the frequency response:

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-(1+j\omega)\tau}d\tau =$$

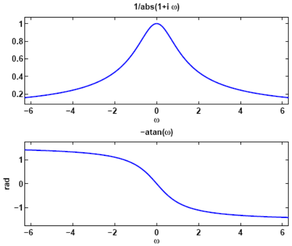
$$= \frac{-1}{1+j\omega}[e^{-(1+j\omega)\tau}]_0^{\infty} = \frac{1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2}$$

- Its modulus and phase are:

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}; \quad \angle H(j\omega) = \arctan(-\omega)$$

- The requested output is:

$$y(t) = \frac{2}{1+2j}e^{j2t} + \frac{3}{1+\pi j}e^{j\pi t}$$



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Fourier series representation for periodic signals

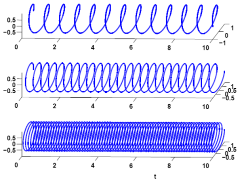
Fourier series representation

- Jean Baptiste J Fourier (advisor and soldier with Napoleon, mathematician and politician) proved in 1807 that any periodic signal with fundamental period T_0 can be represented as a linear combination (weighted sum) of complex exponential functions.

  - The set of harmonically related complex exponentials is defined as:

$$\phi_k(t) = e^{jk\frac{2\pi}{T_0}t}, \text{ con } k = 0, \pm 1, \pm 2, \ldots$$

  - With fundamental periods: $T_0, \frac{T_0}{2}, \frac{T_0}{3}, \ldots$
  - And frequencies: $f_0, 2f_0, 3f_0, \ldots$



- Then, if $x(t) = x(t + T_0)$, it may be represented using Fourier series as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Examples: demo for ECG, speech and square wave.

Fourier series representation for periodic signals

Convergence example

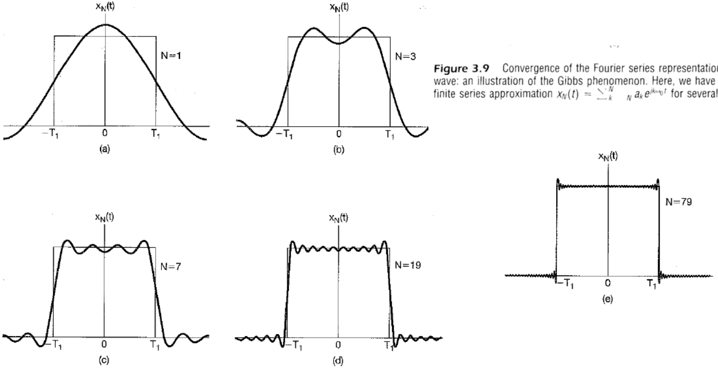


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

Fourier series representation for periodic signals

Coefficient calculation

- In order to calculate coefficients a_k, we multiply both sides by $e^{-jl\omega_0 t}$ and integrate over a T_0 period:

$$\int_0^{T_0} x(t)e^{-jl\omega_0 t}dt = \int_0^{T_0} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jl\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^{T_0} e^{jk\omega_0 t} e^{-jl\omega_0 t}dt$$

Considering that $\int_0^{T_0} e^{j(k-l)\omega_0 t}dt = \begin{cases} 0, & \text{si } k \neq l \\ T_0, & \text{si } k = l \end{cases}$ then:

$$\int_0^{T_0} x(t)e^{-jl\omega_0 t}dt = a_l T_0 \Rightarrow a_l = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jl\omega_0 t}dt$$

- Summary for the Fourier series representation for continuous-time periodic signals:

$$\text{Analysis equation: } x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\text{Synthesis equation: } a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt$$

Fourier series representation for periodic signals

Relation with the frequency response

- By means of this relation we can easily characterize the output of a LTI system to an input that is a periodic signal.
- Recall that a LTI system has a frequency response $H(j\omega)$.
- Recall that when the input of a LTI system is $x(t) = e^{j\omega_0 t}$, the output is $y(t) = H(j\omega_0)x(t) = H(j\omega_0)e^{j\omega_0 t}$.
- If $x(t) = x(t + T_0)$ (periodic), then it has the following Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Therefore, the output $y(t)$ can be calculated using the linearity property:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Questions:

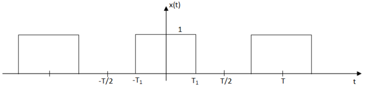
1. Run the script *demoDSF.m* and compare the Fourier series representation of the input and output signals of the given LTI system.

Fourier series representation for periodic signals

Example: calculation of Fourier series representation coefficients

- Calculate the coefficients of the Fourier series representation of $x(t)$, periodic with fundamental period T, defined by:

$$x(t) = \begin{cases} 1, & \text{si } |t| < T_1 \\ 0, & \text{si } T_1 < |t| < T/2 \end{cases}$$



- As $x(t)$ is periodic it can be represented using Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.
- Coefficient calculation:

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} 1e^{-jk\omega_0 t}dt =_{(k\neq 0)}$$

$$= \frac{1}{T}\frac{-1}{jk\omega_0}[e^{-jk\omega_0 t}]_{-T_1}^{T_1} = \frac{-1}{jk\omega_0 T}[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}] = \cdots = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

- For $k = 0$, we calculate the coefficient independently:

$$a_0 = \frac{1}{T}\int_T x(t)e^{j0\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} 1dt = \frac{2T_1}{T}$$

We can see that in this case, it corresponds with the general case of a_k for $k = 0$ by solving the indeterminate form (this is not the general case).

Fourier series representation for periodic signals

Example: calculation of Fourier series representation coefficients by inspection

- Calculate the coefficients of the Fourier series representation of $x(t) = \sin(\omega_0 t)$.
- As $x(t)$ is periodic (fundamental period $T_0 = 2\pi/\omega_0$) it can be represented using Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.
- But in this case we don't need to integrate, as:

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

- Therefore, comparing both equations:

$$k = 1 \Rightarrow a_1 = \frac{1}{2j}; \quad k = -1 \Rightarrow a_{-1} = \frac{-1}{2j}; \quad a_k = 0 \; \forall k \neq \pm 1$$

Questions:

2. Calculate the Fourier series representation of $x(t) = \cos(5\pi t + \pi/3) + \sin(10\pi t)$, without solving the analysis equation.
3. Is it possible to calculate the Fourier series representation of $x(t) = \cos(5\pi t + \pi/3) + \sin(10t)$?

Properties of the Fourier series representation

Average value

- Coefficient a_0 of any Fourier series representation is the average value of the signal, as:

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)e^{j0\omega_0 t}dt = \frac{1}{T_0}\int_{T_0} x(t)dt$$

Notation

- We consider periodic signals, $x(t) = x(t+T)$ and $y(t) = y(t+T)$, with identical fundamental period T.
- The coefficients will be $x(t) \xrightarrow{\mathcal{FS}} a_k; \quad y(t) \xrightarrow{\mathcal{FS}} b_k$.

Linearity

- If $z(t) = Ax(t) + By(t) = z(t+T)$, then:

$$z(t) \xrightarrow{DSF} c_k = Aa_k + Bb_k$$

- Proof: $z(t) = Ax(t) + By(t) = A\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + B\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} =$
$= \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k)\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$.

Properties of the Fourier series representation

Time shifting

- Consider $y(t) = x(t - t_0)$, then $y(t) = y(t + T)$, and moreover:

$$y(t) = x(t - t_0) \xrightarrow{DSF} b_k = a_k e^{-jk\omega_0 t_0}$$

- Proof: We know that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \text{ con } a_k = \frac{1}{T_0}\int_0^T x(t) e^{-jk\omega_0 t} dt$$

- As $y(t)$ is also periodic, it can be represented using Fourier series $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$, given by:

$$b_k = \frac{1}{T}\int_0^T y(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_0^T x(t - t_0) e^{-jk\omega_0 t} dt$$

Variable change: $t - t_0 = l$; $dt = dl$; $t = 0 \Rightarrow l = -t_0$; $t = T \Rightarrow l = T - t_0$. Therefore:

$$b_k = \frac{1}{T}\int_{-t_0}^{-t_0+T} x(l) e^{-jk\omega_0 (l+t_0)} dl = e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(l) e^{-jk\omega_0 l} dl = e^{-jk\omega_0 t_0} a_k$$

Properties of the Fourier series representation

Time reversal

- Consider $y(t) = x(-t)$. Then $y(t)$ is periodic and:

$$y(t) = x(-t) \xrightarrow{\mathcal{FS}} b_k = a_{-k}$$

- Proof: homework (similar to the time shifting case).

Time scaling

- Consider $y(t) = x(at)$. Then $y(t)$ is periodic, but the fundamental period is $T_1 = T/a$ and:

$$y(t) = x(at) \xrightarrow{\mathcal{FS}} b_k = a_k$$

- Note that this Fourier series representation considers different period, $\omega_1 = a\omega_0$, and:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1 t}$$

- Proof: homework.

Properties of the Fourier series representation

Multiplication

- Consider $z(t) = x(t)y(t)$ that has a fundamental period of T and:

$$z(t) = x(t)y(t) \xrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

- Proof: see Oppenheim.

Conjugation and conjugate symmetry

- Consider $y(t) = x^*(t)$ that has a fundamental period of T and:

$$y(t) = x^*(t) \xrightarrow{\mathcal{FS}} b_k = a^*_{-k}$$

- Proof: homework.
- This property is fundamental for the understanding of the utility of complex exponential functions.

Properties of the Fourier series representation

Parseval's relation

- The average power of a periodic signal $x(t)$ equals the sum of the squared module of all its Fourier series representation coefficients.

$$P_m = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Proof: homework, consider that $\int_T |x(t)|^2 dt = \int_T x(t)x^*(t)dt$.

Differentiation and integration

- We have the following:

$$y(t) = \frac{dx(t)}{dt} \xrightarrow{\mathcal{FS}} b_k = jk\omega_0 a_k$$

$$z(t) = \int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{\mathcal{FS}} c_k = \frac{1}{jk\omega_0}a_k$$

- Proofs: homework.
- Note: for the integration property, it is necessary that $a_0 = 0$ so $z(t)$ is periodic. In this case, it is easy to see that $c_0 = 0$.

Properties of the Fourier series representation

Tabla de propiedades del DSF

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\left\{\begin{array}{ll} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

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Fourier Transform for aperiodic signals

Fourier series representation for periodic square wave (I)

- Fourier also proposed a representation for aperiodic signals using complex exponentials. This representation uses the limit and integral concepts (instead of sums).
- We begin with a square wave, where $a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$ and $a_0 = \frac{2T_1}{T}$, with $\omega_0 = \frac{2\pi}{T}$.
- For fixed T_1 and for increasing T, we can see how the Fourier series representation coefficients vary. For that, we can express these coefficients as:

$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}_{|\omega=k\omega_0}$$

- We plot for $T = 4T_1$, $T = 8T_1$ and $T = 16T_1$.

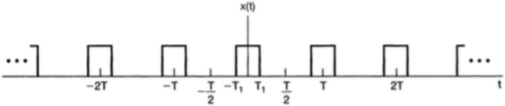


Figure 4.1 A continuous-time periodic square wave.

Fourier Transform for aperiodic signals

Fourier series representation for periodic square wave (II)

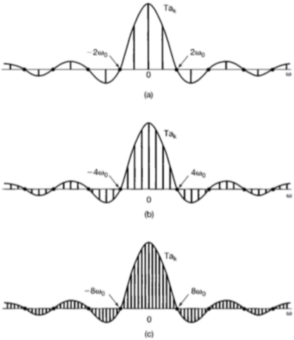


Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

- In this case,

$$lim_{T\to\infty} x(t) = \Pi\left(\frac{t}{T_1}\right)$$

  the Fourier series representation coefficients become more and more closely spaced samples of the envelope, that is a sinc function.

Fourier Transform for aperiodic signals

Fourier series representation for aperiodic signals (I)

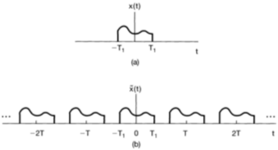


Figure 4.3 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

- In general, any finite-time aperiodic signal $x(t)$ can be represented as:

$$x(t) = lim_{T\to\infty}\tilde{x}(t) = lim_{T\to\infty}\sum_{k=-\infty}^{\infty} x(t-kT)$$

- Signal $\tilde{x}(t)$ is periodic with fundamental period T, and it admits a Fourier series representation:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \text{ con } a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt$$

Fourier Transform for aperiodic signals

Fourier series representation for aperiodic signals (II)

- We can calculate the Fourier series representation coefficients as:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-jk\omega_0 t}dt$$

- We define the *Fourier Transform* of $x(t)$ as the envelope of Ta_k:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Therefore, we can write the coefficients as $a_k = \frac{1}{T}X(jk\omega_0)$, and then:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t} =$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

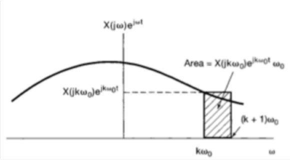


Figure 4.4 Graphical interpretation of eq. (4.7).

Fourier Transform for aperiodic signals

Fourier series representation for aperiodic signals (III)

- Calculating the limit $lim_{T\rightarrow\infty}$ in the previous equation, we obtain:

$$\tilde{x}(t) \rightarrow x(t); \quad k\omega_0 \rightarrow \omega \text{ (it is a continuous variable)}$$

$$\sum \rightarrow \int; \quad \omega_0 \rightarrow d\omega \text{ (infinitesimally close)}$$

  and the obtained equation is the Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

- Even if this demo is performed for finite-time signals, it is also suitable for all energy-defined signals (more precisely when the Dirichlet boundary conditions are fulfilled).
- Summary for the Fourier Transform:

$$\text{Analysis equation: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\text{Synthesis equation: } X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform for aperiodic signals

Example: Calculation of the Fourier Transform of a positive exponential function

- Calculate the Fourier Transform of $x(t) = e^{-at}u(t)$, being $a > 0$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \cdots = \frac{1}{a + j\omega}$$

- Higher values are localized at low frequencies.

Example: Calculation of the Fourier Transform of the unit impulse

- Calculate the Fourier Transform of $x(t) = \delta(t)$.

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

- The unit impulse has a Fourier Transform consisting of equal contributions at all frequencies.

Questions:

4 Calculate the Fourier Transform of $x(t) = e^{-a|t|}$.

Fourier Transform for aperiodic signals

Example: Calculation of the Fourier Transform of the rectangular pulse signal

- Calculate the Fourier Transform of $x(t) = \Pi\left(\frac{t}{T_1}\right)$ (rectangular pulse between $-T_1$ y T_1).

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \cdots = \frac{2T_1 \sin(\omega T_1)}{\omega T_1} = 2T_1 sinc(\omega T_1)$$

- The Fourier Transform of a rectangular pulse is the sing function. Their width are inversely proportional.

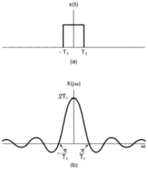


Figure 4.8 (a) The rectangular pulse signal of Example 4.4 and (b) its Fourier transform.

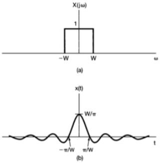


Figure 4.9 Fourier transform pair of Example 4.5: (a) Fourier transform for Example 4.5 and (b) the corresponding time function.

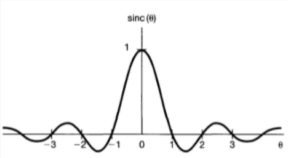


Figure 4.10 The sinc function.

Fourier Transform for aperiodic signals

Example: rectangular pulse and sinc

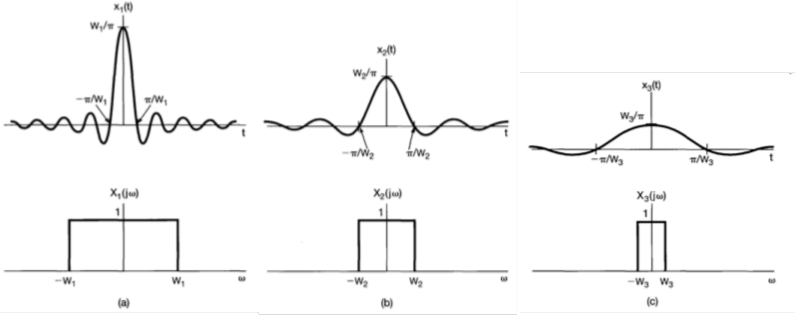


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of W.

Properties of the Fourier Transform

Properties of the Fourier Transform (I)

- We use the notation $x(t) \xrightarrow{\mathcal{FT}} X(j\omega)$.
- Linearity: $z(t) = ax(t) + by(t) \xrightarrow{\mathcal{FT}} Z(j\omega) = aX(j\omega) + bY(j\omega)$.
- Time shifting: $y(t) = x(t - t_0) \xrightarrow{\mathcal{FT}} Y(j\omega) = e^{-j\omega t_0} X(j\omega)$.
- Conjugation and Conjugate Symmetry: $y(t) = x^*(t) \xrightarrow{\mathcal{FT}} Y(j\omega) = X^*(-j\omega)$.
- Differentiation and Integration:

$$y(t) = \frac{dx(t)}{dt} \xrightarrow{\mathcal{FT}} Y(j\omega) = j\omega X(j\omega)$$

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{\mathcal{FT}} Y(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Questions:

5 Prove these properties.

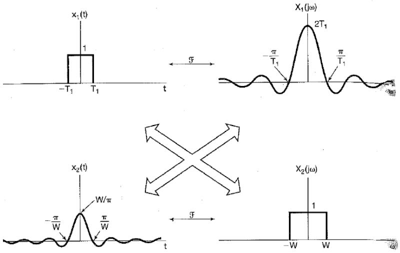
Properties of the Fourier Transform

Properties of the Fourier Transform (II)

- Time scaling: $y(t) = x(at) \xrightarrow{\mathcal{FT}} Y(j\omega) = \frac{1}{|a|}X(j\frac{\omega}{a})$.
- Time reversing: $y(t) = x(-t) \xrightarrow{\mathcal{FT}} Y(j\omega) = X(-j\omega)$.
- Duality:

$$g(t) \xrightarrow{\mathcal{FT}} f(\omega)$$

$$f(t) \xrightarrow{\mathcal{FT}} 2\pi g(-\omega)$$



Properties of the Fourier Transform

Questions:

6. Prove the previous properties.
7. Show that the property holds by using that the Fourier Transform of a sinc is a rectangular pulse and viceversa.

Example: Duality property

- We know that $x(t) = e^{-2|t|} \xrightarrow{\mathcal{FT}} X(j\omega) = \frac{2}{1+\omega^2}$.
- We want to calculate the Fourier Transform of $y(t) = \frac{2}{1+t^2}$.
- By using the duality property, $Y(j\omega) = 2\pi e^{-2|\omega|}$.

Properties of the Fourier Transform

Properties of the Fourier Transform (III)

- It also worth mention that:

$$y(t) = -jtx(t) \xrightarrow{\mathcal{FT}} Y(j\omega) = \frac{dX(j\omega)}{d\omega}$$

$$y(t) = e^{j\omega_0 t}x(t) \xrightarrow{\mathcal{FT}} Y(j\omega) = X(j(\omega - \omega_0))$$

$$y(t) = -\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xrightarrow{\mathcal{FT}} Y(j\omega) = \int_{-\infty}^{\omega} X(j\eta)d\eta$$

Parseval's Relation

- The energy of signal $x(t)$ can be calculated in the frequency domain as:

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Questions:

8 Prove the previous properties.

Properties of the Fourier Transform

The convolution property

- For a LTI system, characterized in the time domain by $h(t)$ and in the frequency domain by $H(j\omega)$:

$$y(t) = x(t) * h(t) \xrightarrow{\mathcal{FT}} Y(j\omega) = X(j\omega)H(j\omega)$$

- Proof:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t}dt =$$

$$= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dtd\tau =_{(t-\tau=u)} \cdots =$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(u)e^{-j\omega u}du\right)e^{-j\omega\tau}d\tau = H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{j\omega\tau}d\tau =$$

$$= H(j\omega)X(j\omega)$$

- It also worth mention that:

$$z(t) = x(t)y(t) \xrightarrow{\mathcal{FT}} Z(j\omega) = \frac{1}{2\pi}X(j\omega) * Y(j\omega)$$

Properties of the Fourier Transform

Questions:

9 Consider the Fourier Transform of a rectangular pulse. Calculate the Fourier Transform of:

$$y(t) = u(t-1) + 0.5u(t-2) - 0.5u(t-3) - u(t-4)$$

Basic Fourier Transform pairs

Fourier Transform of a pure imaginary exponential function

- The Fourier Transform of a pure imaginary exponential function is an impulse.

$$x(t) = e^{j\omega_0 t} \xrightarrow{\mathcal{FT}} X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

- Proof: as $X(j\omega) = 2\pi\delta(\omega - \omega_0)$, then:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} \delta(j(\omega - \omega_0))e^{j\omega t}d\omega =$$

$$= \int_{-\infty}^{\infty} \delta(j(\omega - \omega_0))e^{j\omega_0 t}d\omega = e^{j\omega_0 t}$$

- However, this proof is only valid for energy-defined signals.
- Note that the Fourier Transform can be also calculated for power-defined signals. In this case we obtain impulse functions in the transformed signal.

Basic Fourier Transform pairs

Fourier Transform for periodic signals

- Using previous transform pair, we can obtain the Fourier Transform for any periodic signal $x(t) = x(t+T)$, using the linearity property:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xrightarrow{\mathcal{FT}} X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Fourier Transform of a cosine signal

- We can express $x(t) = \cos(\omega_0 t)$ as $x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$. Therefore, its coefficients are $a_1 = a_{-1} = 1/2$, y $a_k = 0$ for $k \neq 0$.
- Its Fourier Transform is:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = 2\pi \left(\frac{1}{2}\delta(\omega - \omega_0) + \frac{1}{2}\delta(\omega + \omega_0) \right)$$

- Therefore:

$$x(t) = \cos(\omega_0 t) \xrightarrow{\mathcal{FT}} X(j\omega) = \pi \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right)$$

Basic Fourier Transform pairs

Fourier Transform of a sine signal

- For $x(t) = \sin(\omega_0 t)$ we can obtain the Fourier Transform in a similar way:

$$x(t) = \sin(\omega_0 t) \xrightarrow{\mathcal{FT}} X(j\omega) = \frac{\pi}{j}\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$

Fourier Transform of a constant

- The Fourier Transform of signal $x(t) = 1$ can be calculated considering $x(t)$ as a periodic signal with fundamental period T, where $a_0 = 1$ y $a_k = 0$ for $k \neq 0$. Then:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = 2\pi\delta(\omega)$$

Basic Fourier Transform pairs

Fourier Transforms considering train of impulses

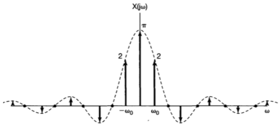


Figure 4.12 Fourier transform of a symmetric periodic square wave.

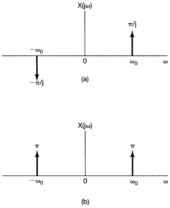


Figure 4.13 Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

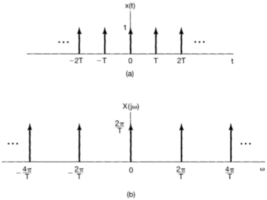


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

Questions

10 Calculate the Fourier Transform of the train of impulses $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$.

11 Calculate the Fourier Transform of $x(t) = \frac{\sin(Wt)}{\pi t}$.

12 Calculate the Fourier Transform of $y(t) = u(t)$.

13 Calculate the Fourier Transform of $x(t) = \delta(t - t_0)$.

14 Calculate the Fourier Transform of $x(t) = te^{-at}u(t)$, with $a > 0$.

Summary of the Fourier Transform

Properties of the Fourier Transform

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Summary of the Fourier Transform

Basic Fourier Transform pairs

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Symmetries

Cuestiones

15 The Fourier Transform of any real signal is a Hermitian function (the magnitude is an even function of frequency and the phase is an odd function of frequency or equivalently the real part is an even function of frequency and the imaginary part is an odd function of frequency). Prove this symmetry property graphically using the signal $x(t) = e^{-at}u(t)$, with $a > 0$.

16 The Fourier Transform for any real and even signal is also a real and even function with the frequency. Prove this symmetry property graphically using the signal $x(t) = e^{-a|t|}$, with $a > 0$.

17 The Fourier Transform of the real part of a real signal $x(t)$ is the real part of $X(j\omega)$. Calculate, using the symmetry property, the Fourier Transform of $x(t) = e^{-a|t|}$, with $a > 0$.

18 Prove the Conjugation property. Using this property, prove that if $x(t)$ is a real signal, its Fourier Transform is a Hermitian Function. Moreover, prove that is spectrum $|X(j\omega)|$ is an even function of frequency.

Index

Problems

Problem 1 $(*)$

Let be $x(t)$ a periodic real signal with fundamental period $T = 8$ s. The non-zero coefficientes of the Fourier Series of $x(t)$ are $a_1 = a_{-1} = 2$, $a_3 = a_{-3}^* = 4j$. Express $x(t)$ in the following way:

$$x(t) = \sum_{k=0}^{\infty} A_k \cos\left(\omega_k t + \phi_k\right)$$

Problem 2 $(*)$

Compute the Fourier Series coefficients a_k of the following periodic signal with $\omega_0 = 2\pi$.

$$x(t) = \begin{cases} 0.5, & 0 \leq t < 0.5 \\ -0.5, & 0.5 \leq t < 1 \end{cases}$$

Problems

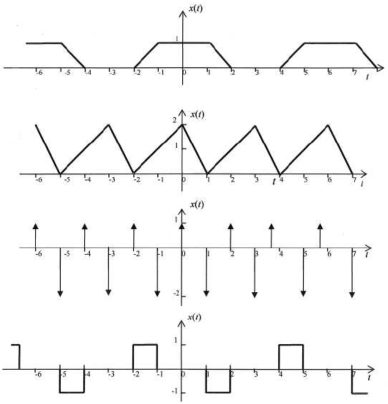
Problem 3

Consider the each of the following signals:
$x(t) = \cos(4\pi t)$; $y(t) = \sin(4\pi t)$; $z(t) = x(t)y(t)$.

1. Determine the FS coefficients of $x(t)$.
2. Determine the FS coefficients of $y(t)$.
3. Determine the coefficients of $z(t)$ using the direct expression of the multiplication of both signals (without using properties).

Problem 4 (*)

Determine FS for each of the following signals.



Problems

Problem 5 (*)

Let be $X(j\omega)$ the Fourier Transform of the signal $x(t)$. Use FT properties to obtain the following transforms:

1. $x_1(t) = x(1-t) + x(-1-t)$
2. $x_2(t) = x(3t-6)$
3. $x_3(t) = \frac{d^2x(t-1)}{dt^2}$

Problem 6

Considere the following signal:

$$x(t) = \begin{cases} 0, & |t| > 1 \\ (t+1)/2, & |t| \leq 1 \end{cases}$$

1. Determine the expression of $X(j\omega)$.
2. Considering the real part of $X(j\omega)$, show that is the FT of the even part of $x(t)$.
3. Which is the FT of the odd part of $x(t)$?

Problems

Problem 7 (*)

Let's suppose we know a given signal and its FT:

$$e^{-|t|} \leftrightarrow \frac{2}{1+\omega^2}$$

1. Use FT properties to compute the FT of $te^{-|t|}$.
2. Apply duality property to obtain the FT of $\frac{4t}{(1+t^2)^2}$.

Problem 8

Let be a signal with FT $X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$ and let be $h(t) = u(t) - u(t-2)$.

1. Is $x(t)$ periodic?
2. Is $x(t) * h(t)$ periodic?
3. Can be periodic the convolution of two periodic signals?

Problems

Problem 9 $(*)$

Let $h(t)$ the impulse response of a causal LTIS, with FT:

$$H(j\omega) = \frac{1}{j\omega + 3}$$

For a given input $x(t)$, the systems produces the output $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$. Determine $x(t)$.

Problema11 $(*)$

Compute the convolution of the signals $x(t)$ and $h(t)$, by first computing their FT, and applying then the convolution property of the FT and FT^{-1}:

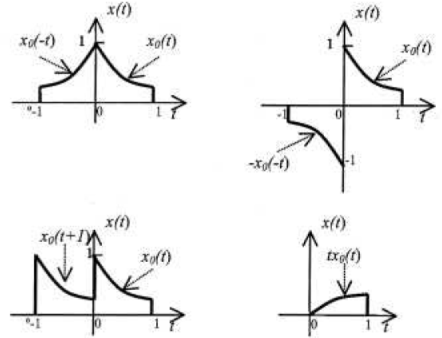
1. $x(t) = te^{-2t}u(t)$ with $h(t) = e^{-4t}u(t)$
2. $x(t) = te^{-2t}u(t)$ with $h(t) = te^{-4t}u(t)$
3. $x(t) = e^{-t}u(t)$ with $h(t) = e^{t}u(-t)$

Problem 10

Given the following signal:

$$x_0(t) = \begin{cases} e^{-t}, & 0 \leq t \leq 1 \\ 0, & resto \end{cases}$$

Determine the FT for each of the following signals. (Note: begin by determining the FT of $x_0(t)$ and use properties).



Problems

Problem 12

Let be $x(t) = e^{-(t-2)}u(t-2)$ and $h(t) = u(t+1) - u(t-3)$. Verify that the FT of the convolution is then same as the product of each FT.

Problema 13

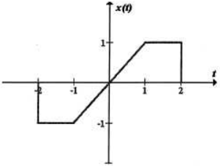
Let be $H(j\omega)$ the FT of the impulse response for a particual LTIS, compute $h(t)$ in the following cases:

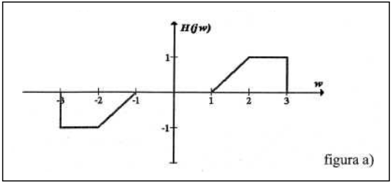
1. $H(j\omega) = 2\left(\delta(\omega-1) - \delta(\omega+1)\right) + 3\left(\delta(\omega-2\pi) - \delta(\omega+2\pi)\right)$.
2. $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$, con $|H(j\omega)| = 2\left(u(\omega+3) - u(\omega-3)\right)$ y $\angle H(j\omega) = -\frac{3}{2}\omega + \pi$.
3. $H(j\omega) = \frac{\sin^2(3\omega)\cos(\omega)}{\omega^2}$.

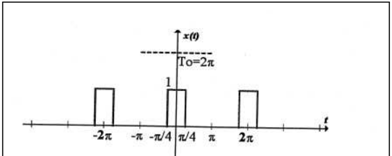
Problems

Problema 14 (*)

Compute the FT of the following signals.







Problem 15 (*)

Considere a LTIS with a FT of the impulse respones given by the figure (a). Considere also the periodic signal in figure (b)

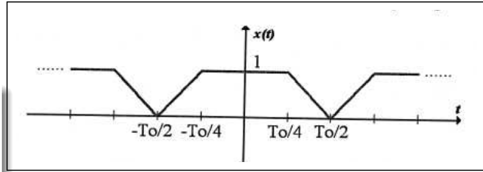
1. Find the impulse response $h(t)$.
2. Compute the FT of $x(t)$.
3. Compute the FS coefficients for $x(t)$.
4. What is the power of the signal$x(t)$? What percentage of this power is in the output?
5. Compute the expression of the output signal in the time domain.

Problems

Problem 16

Considere the periodic signal, with period $T_=0$ sketched in the figure.

1. Find the FS coefficients.
2. Compute its FT and sketch it (signal spectrum)
3. This signal is the input for a system with a FT of the impulse response $H(j\omega) = u(\omega + 4\pi/T_0) - u(\omega - 4\pi/T_0)$. What percentage of the input signal power is findiing in the output of the system?
4. Compute and sketch the output signal in the time domain.



Problem 17 $(*)$

Let be $x(t)$ the input of a LTIS with the following impulse response:

$$h(t) = \frac{2W_1W_2}{\pi} sinc\left(\frac{W_1 t}{\pi}\right) sinc\left(\frac{W_2 t}{\pi}\right)$$

where $W_1 > W_2$. Compute the output $y(t)$, when the input is:

$$x(t) = \frac{(W_1 - W_2)^2}{2\pi} sinc^2\left(\frac{W_1 - W_2}{2\pi} t\right)$$

Problems

Problema 18 $(*)$

Let be $X(j\omega)$ the FT of $x(t)$, according to the figure..

1. Find $\angle X(j\omega)$.
2. Find $X(j0)$.
3. Find $\int_{-\infty}^{\infty} X(j\omega)d\omega$.
4. Evaluate $\int_{-\infty}^{\infty} \|X(j\omega)\|^2 d\omega$
5. Sketch the inverse FT of $Real\{X(j\omega)\}$.