PROBLEMA

Sean las funciones de onda

$$\phi_1(x) = A_1 e^{-\left(\frac{x}{a}\right)^2}, \ A_1>0 \ ; \ \phi_2(x) = A_2 e^{-2\left(\frac{x}{a}\right)^2}, \ A>0$$

1) Calcúlese A_1 y A_2 de modo que ϕ_1 y ϕ_2 tengan norma igual a 1.

2) Tómese la superposición siguiente de las ondas anteriores

$$\phi(x) = A\left(\phi_1(x) + \sqrt{2}\, e^{i\alpha}\phi_2(x)\right)$$

con $A>0$ y $\alpha \in \mathbb{R}$. Calcúlese $\langle p^2\rangle_\phi$ y pruébese los efectos físicos del valor de α. ¿Qué valores de α hay que tomar para que $\langle p^2\rangle_\phi = \frac{5}{3}\frac{\hbar^2}{a^2}$

$$\|\phi_1\| = \sqrt{(\phi_1,\phi_1)} \ ; \ (\phi_1,\phi_1) = \int_{-\infty}^{+\infty} dx\, \phi_1^*(x)\phi_1(x)$$

$$A_1 \ / \ 1 = (\phi_1,\phi_1) = A_1^2 \int_{-\infty}^{+\infty} dx\, e^{-2\left(\frac{x}{a}\right)^2} = \left(\frac{2}{\pi a^2}\right)^{-1/2} \cdot A_1^2$$

$$\Rightarrow \boxed{A_1 = \left(\frac{2}{\pi a^2}\right)^{1/4}}$$

Analogamente

$$A_2 \ / \ 1 = (\phi_2,\phi_2) = A_2^2 \int_{-\infty}^{+\infty} dx\, e^{-4\left(\frac{x}{a}\right)^2} = \left(\frac{4}{\pi a^2}\right)^{-1/2} A_2^2 \Rightarrow$$

$$\boxed{A_2 = \left(\frac{4}{\pi a^2}\right)^{1/4}}$$

Fijemos ahora A de modo que $\|\phi\| = 1$

$$\|\phi\| = \sqrt{(\phi,\phi)}$$

Calculemos $\langle p^2 \rangle_\phi$. Por definición,

$$\langle p^2 \rangle_\phi = (\phi, P^2\phi) = (P^\dagger\phi, P\phi) = (P\phi, P\phi) -$$

ya que $P^\dagger = P$.

$$P\phi_1 = -i\hbar \frac{d\phi_1(x)}{dx} = i\hbar \frac{2x}{a^2} \phi_1(x)$$

$$P\phi_2 = -i\hbar \frac{d\phi_2(x)}{dx} = i\hbar \frac{2x}{a^2} 2\phi_2(x)$$

$$P\phi = A(P\phi_1 + \sqrt{2}e^{i\alpha} P\phi_2) = Ai\hbar \frac{2x}{a^2}(\phi_1 + 2\sqrt{2}e^{i\alpha}\phi_2) \equiv \hat{\phi}(x)$$

$$\langle p^2 \rangle_\phi = \int_{-\infty}^{+\infty} dx \, (\hat{\phi}(x))^* \hat{\phi}(x) =$$

$$= A^2 \hbar^2 \frac{4}{a^2} \int_{-\infty}^{+\infty} dx\, x^2 (\phi_1 + 2\sqrt{2}e^{i\alpha}\phi_2)^* (\phi_1 + 2\sqrt{2}e^{i\alpha}\phi_2) =$$

$$= A^2\hbar^2 \frac{4}{a^2} \Big\{ \int_{-\infty}^{+\infty} dx\, x^2 \phi_1^2 + 8\int_{-\infty}^{+\infty} dx\, x^2 \phi_2^2$$

$$+ 2\sqrt{2} \int_{-\infty}^{+\infty} dx\, x^2 \phi_1\phi_2 \,(e^{i\alpha} + e^{-i\alpha}) \Big\} =$$

$$= A^2\hbar^2 \frac{4}{a^2} \Big\{ \int_{-\infty}^{+\infty} dx\, x^2 \phi_1^2 + 8 \int_{-\infty}^{+\infty} dx\, x^2 \phi_2^2$$

$$+ 4\sqrt{2}\cos\alpha \int_{-\infty}^{+\infty} dx\, x^2 \phi_1 \phi_2 \Big\}$$

Teniendo en cuenta que

$$\int_{-\infty}^{+\infty} dx\, x^2 \phi_1^2 = \frac{a^2}{4} \; ; \int_{-\infty}^{+\infty} dx\, x^2 \phi_2^2 = \frac{a^2}{8} \, ,$$

$$\int_{-\infty}^{+\infty} dx\, x^2 \phi_1 \phi_2 = \frac{a^2}{3 \cdot 2^{1/4}\sqrt{3}} \, ,$$ se llega a que

A / $1 = (\phi,\phi) = A^2 \left(\phi_1 + \sqrt{2}e^{i\alpha}\phi_2 , \phi_1 + \sqrt{2}e^{i\alpha}\phi_2 \right) =$

$= A^2 \Big(\overbrace{(\phi_1,\phi_1)}^{=1} + (\sqrt{2}e^{i\alpha}\phi_2 , \sqrt{2}e^{i\alpha}\phi_2) +$

$+ (\phi_1, \sqrt{2}e^{i\alpha}\phi_2) + (\sqrt{2}\, e^{i\alpha}\phi_2, \phi_1) \Big) =$

$= A^2 \Big(1 + \sqrt{2}e^{-i\alpha}\sqrt{2}e^{i\alpha}\overbrace{(\phi_2,\phi_2)}^{=1} + \sqrt{2}e^{i\alpha}(\phi_1,\phi_2)$

!!!

$+ \sqrt{2}\, e^{-i\alpha}(\phi_2,\phi_1) \Big) = A^2\big(1 + 2\overbrace{(\phi_2,\phi_2)}^{=1} + \sqrt{2}(\phi_1,\phi_2)\, 2\cos\alpha\big)$

$(\phi_1,\phi_2) = (\phi_2,\phi_1)$ ya que ϕ_1 y ϕ_2 son reals

$\sqrt{2}e^{i\alpha}(\phi_1,\phi_2) + \sqrt{2}e^{-i\alpha}(\phi_2,\phi_1) = \sqrt{2}(\phi_1,\phi_2)\underbrace{(e^{i\alpha}+e^{-i\alpha})}_{2\cos\alpha}$

$$1 = A^2 (1 + 2 + 2\sqrt{2}(\phi_1,\phi_2)\cos\alpha)$$

$$(\phi_1,\phi_2) = A_1 A_2 \int_{-\infty}^{+\infty} e^{-3\left(\frac{x}{a}\right)^2} = \frac{2^{3/4}}{\sqrt{3}}$$

$$1 = A^2\left(3 + 2\sqrt{2}\,\frac{2^{3/4}}{\sqrt{3}}\cos\alpha\right) = A^2\left(3 + \frac{4\cdot 2^{1/4}\cos\alpha}{\sqrt{3}}\right)$$

$$\boxed{A = \frac{1}{\sqrt{3 + \frac{4\cdot 2^{1/4}\cos\alpha}{\sqrt{3}}}}}$$

$$\langle p^2\rangle_4 = A^2\hbar^2\frac{4}{a^4}\left\{\frac{a^2}{4}+\frac{8a^2}{8}+4\sqrt{2}\cos\alpha\,\frac{a^2}{3\cdot 2^{1/4}\sqrt{3}}\right\} =$$

$2^{1/4}$

$$= \frac{\hbar^2}{a^2}A^2\,4\left\{\frac{1}{4}+1+4\,2^{1/4}\frac{1}{3\sqrt{3}}\cos\alpha\right\}$$

$$= \frac{\hbar^2}{a^2}A^2\left\{1+4+16\,2^{1/4}\frac{1}{3\sqrt{3}}\cos\alpha\right\} =$$

$$= \frac{\hbar^2}{a^2}A^2\left\{5+16\,2^{1/4}\frac{1}{3\sqrt{3}}\cos\alpha\right\} =$$

$$= \frac{\hbar^2}{a^2}\,\frac{1}{3+\dfrac{4\cdot 2^{1/4}\cos\alpha}{\sqrt{3}}}\left\{5+16\,2^{1/4}\frac{1}{3\sqrt{3}}\cos\alpha\right\} =$$

$$= \frac{\hbar^2}{a^2}\left(\frac{3}{3}\right)\left(\frac{1}{3+\dfrac{4\cdot 2^{1/4}\cos\alpha}{\sqrt{3}}}\right)\left(\frac{3}{3}\right)\left(5+16\cdot 2^{1/4}\frac{1}{3\sqrt{3}}\cos\alpha\right)\Big\} =$$

$$= \frac{\hbar^2}{a^2}\,\frac{1}{9+4\,2^{1/4}\sqrt{3}\cos\alpha}\left(15+16\cdot 2^{1/4}\frac{\sqrt{3}}{3}\cos\alpha\right) =$$

$$= \frac{\hbar^2}{a^2}\,\frac{1}{3}\,\frac{45+16\cdot 2^{1/4}\sqrt{3}\cos\alpha}{9+4\,2^{1/4}\sqrt{3}\cos\alpha} = \frac{\hbar^2}{a^2}\left(\frac{45+16z}{9+4z}\right)$$ con $z = 2^{1/4}\sqrt{3}\cos\alpha$

$$= \frac{\hbar^2}{a^2}\frac{1}{3}\left(\frac{9+36+4\cdot 4z}{9+4z}\right) = \frac{\hbar^2}{a^2}\frac{1}{3}\left(\frac{9+9\cdot 4+4\cdot 4z}{9+4z}\right) =$$

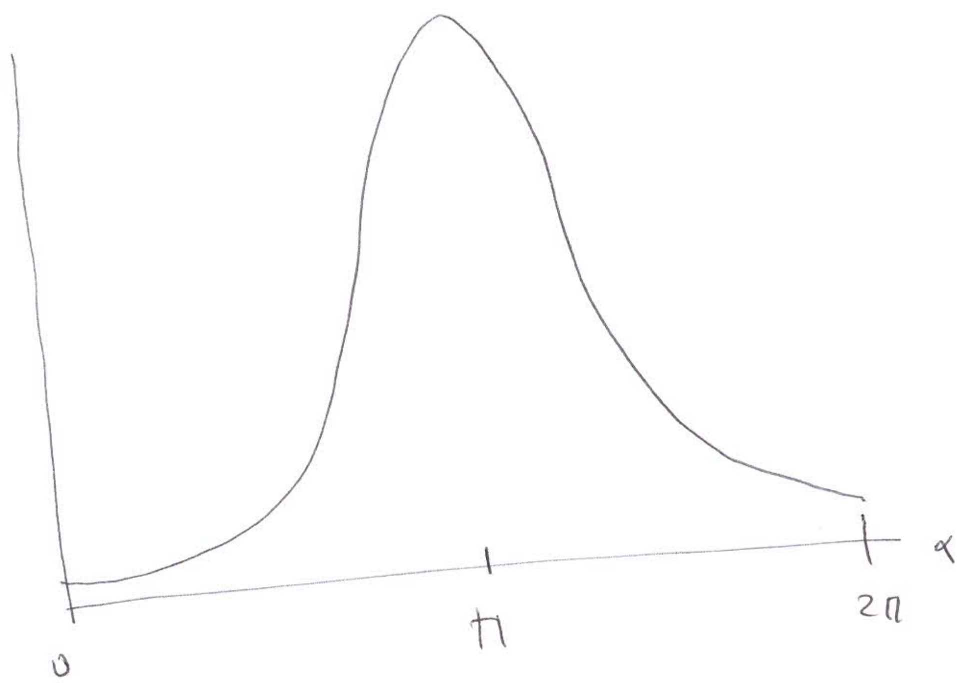
$$= \frac{\hbar^2}{a^2}\,\frac{1}{3}\,\frac{9+4(9+4z)}{9+4z} = \frac{\hbar^2}{a^2}\frac{1}{3}\left(4+\frac{9}{9+4z}\right)$$

$$\boxed{\langle p^2\rangle_4 = \frac{\hbar^2}{a^2}\,\frac{1}{3}\left(4+\frac{9}{9+4\cdot 2^{1/4}\sqrt{3}\cos\alpha}\right)}$$

$$\langle p^2 \rangle_\phi = \frac{\hbar^2}{a^2} f(\alpha)$$

$$f(\alpha) = \frac{1}{3}\left(4 + \frac{9}{9 + 4 \cdot 2^{1/4}\sqrt{3}\cos\alpha}\right)$$

Si representamos $f(\alpha)$ obtenemos



Es decir $\langle p^2 \rangle_\phi$ depende de α.

$\alpha = \frac{\pi}{2}, \frac{2\pi}{3}$ da $\langle p^2 \rangle_\phi = \frac{5}{3}\frac{\hbar^2}{a^2}$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{3}(4+1) = \frac{5}{3}$$

$$f\left(\frac{3\pi}{2}\right) = \frac{1}{3}(4+1) = \frac{5}{3}$$