Problema

Una partícula libre de masa m se encuentra, en t=0, en un estado cuya función de onda es

$$\phi(x) = \frac{A}{(x^2+a^2)(x-2ai)} \quad , \quad a>0$$

1) Calcular A de modo que ϕ esté normalizada a 1.

2) Calcular $\tilde{\phi}(p)$, la función de onda en la representación de momentos.

3) Calcular la probabilidad de que la partícula esté fuera del intervalo $[-a, a]$

4) Calcular la probabilidad de que el momento esté en el intervalo $[-\frac{\hbar}{a}, \frac{\hbar}{a}]$

5) obtener $\Delta_\phi X$

6) obtener $\Delta_\phi P$

7) ¿Se cumple $\Delta_\phi X \Delta_\phi P = \frac{\hbar}{2}$? ¿Porqué no?

8) ¿Cuál es la probabilidad de que $\phi(x) = \frac{A}{(x^2+a^2)(x-2ai)}$ esté en $\psi(x) = \sqrt{\frac{2a^3}{\pi}} \frac{1}{x^2+a^2}$?

9) ¿Cuánto vale $\phi(x,t)$?

$\phi(x,t)$ la función de onda libre en $t>0$ con $\phi(x,t=0) = \phi(x)$.

① $$\phi(x) = \frac{A}{(x^2+a^2)(x-2ai)} \; : \; \phi^*(x) = \frac{A^*}{(x^2+a^2)(x+2ai)}$$ ②

$$|\phi(x)|^2 = |A|^2 \frac{1}{(x^2+a^2)^2(x^2+4a^2)}$$

$$1 = \|\phi\| = \left(\int_{-\infty}^{+\infty} dx\, |\phi|^2 \right)^{1/2} = |A|^2 \int_{-\infty}^{+\infty} dx \frac{1}{(x^2+a^2)^2(x^2+4a^2)}$$

$$\Rightarrow \quad A = e^{i\theta} \left(\int_{-\infty}^{+\infty} dx \frac{1}{(x^2+a^2)^2(x^2+4a^2)} \right)^{-1/2} =$$

$$= \frac{3a^{5/2}}{\pi^{1/2}} e^{i\theta}$$ Tomamos $\underline{\theta = 0}$.

$$\phi(x) = \frac{3a^{5/2}}{\pi^{1/2}} \frac{1}{(x^2+a^2)(x-2ai)}$$

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②

$$\tilde{\phi}(p) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi\hbar}} \phi(x)\, e^{-\frac{ipx}{\hbar}} = \frac{3a^{5/2}}{\pi^{1/2}} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx \frac{e^{-i\frac{px}{\hbar}}}{(x^2+a^2)(x-2ai)}$$ (2.2)

$\underline{\underline{p \geq 0}}$

$$C = \Gamma_-^{(R)} + C_R^- : \; \Gamma_-^{(R)} = \{ z = x,\; x \in [R, -R] \}$$

$$C_R^- = \{ z = Re^{i\theta} ;\; \theta \in [-\pi, \pi] \}$$

$$e^{-ip\frac{z}{\hbar}} = e^{-i\frac{px}{\hbar}} e^{\frac{p}{\hbar}y} \to 0 \text{ cuando } y \to -\infty \Rightarrow$$

$\Rightarrow e^{-ipz/\hbar} \to 0$ cuando $R \to +\infty$ con $z \in C_R^-$

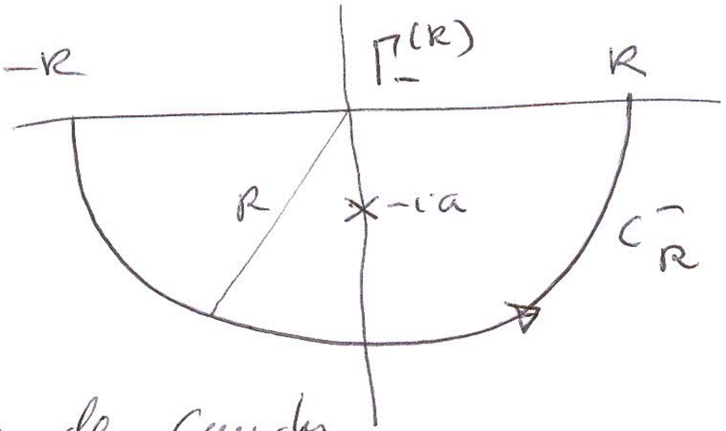
Sea $z \in C_R^-$, entonces,

$$\left| \frac{1}{(z^2+a^2)(z-2ai)} \right| = \frac{1}{|z^2+a^2|} \frac{1}{|z-2ai|} \leq \frac{1}{\big||z|^2 - |a|^2\big|} \frac{1}{\big||z| - |2ai|\big|}$$

$$= \frac{1}{|R^2-a^2|} \frac{1}{|R-2a|} \to 0 \text{ cuando } R \to \infty$$

Por tanto, de acuerdo con el lema de Jordan

$$\lim_{R\to\infty}\int_{C_R^-} dz\,\frac{1}{(z^2+a^2)(z-2ai)}\,e^{-\frac{ipz}{\hbar}} = 0 \qquad (3.1)$$



Por el Th de Cauchy

$$\oint_C dz\,\frac{1}{(z^2+a^2)(z-2ai)}\,e^{-ipz/\hbar} = 2\pi i \sum_{\substack{\text{polos en el}\\ \text{interior de } C}} \text{Res}\left\{\frac{e^{-\frac{ipz}{\hbar}}}{(z^2+a^2)(z-2ai)}\right\}$$

El único polo en el interior de C es el polo de

$$\frac{1}{(z^2+a^2)(z-2ia)} = \frac{1}{(z+ia)(z-ia)(z-2ia)}$$

en $z=-ia$. $z=-ia$ es un polo simple, así que

$$\underset{z=-ia}{\text{Res}}\,\frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} = \lim_{z\to -ia}\,(z+ia)\,\frac{e^{-ipz/\hbar}}{(z+ia)(z-ia)(z-2ai)} =$$

$$= -\frac{1}{6a^2}\,e^{-\frac{pa}{\hbar}}$$. Entonces

$$\oint_C dz\,\frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} = 2\pi i\left(-\frac{1}{6a^2}\right)e^{-\frac{pa}{\hbar}} =$$

$$= -\frac{\pi i}{3a^2}\,e^{-\frac{pa}{\hbar}} \qquad (3.2)$$

$$\oint_C dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} = \int_{\underline{\Gamma}(R)} dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)}$$

$$+ \int_{C_R^-} dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)}$$

Teniendo en cuenta (3.1), se llega a la conclusión de que

$$\lim_{R\to+\infty} \oint_C dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} = \lim_{R\to\infty} \int_{\underline{\Gamma}(R)} dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)}$$

con $z = x \quad x \in [R, -R) \Rightarrow dz = dx$ en $\underline{\Gamma}(R)$,

tendremos

$$\lim_{R\to+\infty} \oint_C dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} = -\int_{-\infty}^{+\infty} dx \, \frac{e^{-ipx/\hbar}}{(x^2+a^2)(x-2ai)}$$

Utilizando (3.2), obtenemos que

$$\int_{-\infty}^{\infty} dx \, \frac{e^{-ipx/\hbar}}{(x^2+a^2)(x-2ai)} = -\lim_{R\to+\infty} \oint_C dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} =$$

$$= \frac{\pi i}{3a^2} e^{-\frac{pa}{\hbar}}$$

Substituyendo este resultado en (2.2) se obtiene

$$\underline{\geq 0} \Rightarrow \tilde{f}(p) = \frac{\pi i}{3a^2} e^{-\frac{pa}{\hbar}} \frac{3a^{3/2}}{\pi^{1/2}} \frac{1}{\sqrt{2\pi\hbar}} =$$

$$= i\sqrt{\frac{a}{2\hbar}} \, e^{-pa/\hbar} \qquad (4.1)$$

Calculemos ahora

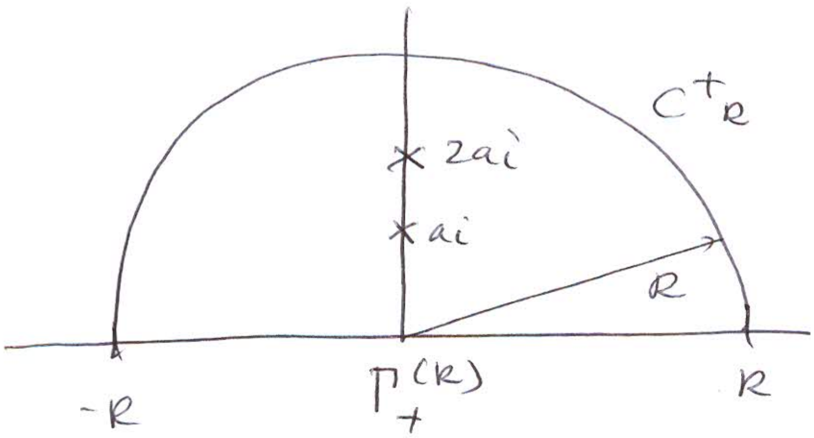
$$\int_{-\infty}^{+\infty} dx \, \frac{1}{(x^2+a^2)(x-2ai)} \, e^{-i\frac{px}{\hbar}}$$

cuando $p \leq 0$.

$z = x + iy \Rightarrow e^{-i\frac{pz}{\hbar}} = e^{-i\frac{px}{\hbar}} \, e^{\frac{py}{\hbar}}$. Como $p \leq 0$, $e^{\frac{py}{\hbar}}$ tiende a cero si $y > 0$, y tiende a infinito si $y < 0$.

Por tanto, el circuito del th de Cauchy hay que cerrarlo por arriba: hay que tomar

$$C^+_R = \{ z = Re^{i\theta}, \ \theta \in [0, \pi] \}$$



$$C' = \Gamma^{(R)}_+ + C^+_R$$

$$\Gamma^{(R)}_+ = \{ z = x, \ x \in [-R, R] \}$$

$$C^{(R)}_+ = \{ z = Re^{i\theta}; \ \theta \in [0, \pi] \}$$

Como

$$\left| \frac{1}{z^2+a^2\,(z-2ai)} \right| \to 0 \quad \text{cuando } z \in C_R^+ \text{ con } R \to +\infty,$$

podemos aplicar el lema de Jordan to conclude that

$$\lim_{R\to\infty} \int_{C_R^+} dt\, \frac{e^{-ipt/\hbar}}{(z^2+a^2)(z-2ai)} = 0 \quad \text{si } \underline{\underline{p \leq 0}}. \quad (6.1)$$

Ahora $\dfrac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)}$ tiene dos polos simples en el Dominio cuya frontera es C':

$$z = 2ai \quad y \quad z = ai$$

$$\frac{1}{(z^2+a^2)(z-2ai)} = \frac{1}{(z+ia)(z-ia)(z-2ai)} \to$$

Entonces,

$$\mathrm{Res}_{z=ai} \frac{e^{-ipt/\hbar}}{(z^2+a^2)(z-2ai)} = \lim_{z\to ai} (z-ai)\, \frac{e^{-ipt/\hbar}}{(z^2+a^2)(z-2ai)} =$$

$$= \lim_{z\to ai} \frac{e^{-ipz/\hbar}}{(z+ai)(z-2ai)} = \frac{e^{pa/\hbar}}{(2ai)(-ai)} = \frac{1}{2a^2}\, e^{-\frac{|p|a}{\hbar}} \quad (6.2)$$

$$\mathrm{Res}_{z=2ai} \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} = \left. \frac{e^{+2pa/\hbar}}{(z^2+a^2)\cancel{(z-2ai)}}\,\cancel{(z-2ai)} \right|_{t=2ai}$$

$$= -\frac{1}{3a^2}\, e^{2pa/\hbar} \quad (6.3)$$

De acuerdo con el th de Cauchy

$$\oint_{C'} dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} = 2\pi i \sum_{\substack{\text{polos} \\ \text{dentro de } C'}} \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} =$$

$$= 2\pi i \left\{ \operatorname{Res}_{z=ai} \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} + \operatorname{Res}_{z=2ai} \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} \right\} =$$

$$= \frac{2\pi i}{a^2} \left(\frac{e^{-|p|a/\hbar}}{2} - \frac{e^{-2|p|a/\hbar}}{3} \right)$$

. Teniendo en cuenta (6.1), se concluye que si $p \leq 0$, se tiene

$$\int_{-\infty}^{+\infty} dx \, \frac{e^{-ipx}}{(x^2+a^2)(x-2ai)} = \lim_{R\to\infty} \oint_{C'} dz \, \frac{e^{-ipz/\hbar}}{(z^2+a^2)(z-2ai)} =$$

$$= \frac{2\pi i}{a^2} \left(\frac{e^{-|p|a/\hbar}}{2} - \frac{e^{-2|p|a/\hbar}}{3} \right)$$

Substituyendo esta expresión en (2.1), concluimos que

$$p \leq 0 \Rightarrow \tilde{f}(p) = \frac{3a^{3/2}}{\cancel{\pi}^{1/2}} \frac{1}{\sqrt{2\cancel{\pi}\hbar}} \frac{2\cancel{\pi} i}{a^2} \left(\frac{e^{-|p|a/\hbar}}{2} - \frac{e^{-2|p|a/\hbar}}{3} \right)$$

$\sqrt{2}$

a

$$= \frac{a}{\sqrt{2\hbar}} \, i \left(3 e^{-|p|a/\hbar} - 2 e^{-2|p|a/\hbar} \right) \quad (7.1)$$

(4.1) y (7.1) conducen a

8

$$\tilde{\phi}(p) = \begin{cases} i\sqrt{\frac{a}{2\hbar}}\, e^{-\frac{pa}{\hbar}} , & \text{si } p \geq 0 \\ i\sqrt{\frac{a}{2\hbar}}\left(3e^{-|p|a/\hbar} - 2e^{-2|p|a/\hbar}\right), & \text{si } p \leq 0 \end{cases} \qquad (8.1)$$

Nótese que $\tilde{\phi}(p)$ es continua en $p=0$

3

$$P(\text{partícula esté fuera de } [-a,a]) = 1 - P(\text{partícula esté en } [-a,a])$$

$$= 1 - \int_{-a}^{a} dx\, |\phi(x)|^2 = 1 - |A|^2 \int_{-a}^{a} dx \frac{1}{(x^2+a^2)^2(x^2+4a^2)}$$

$$= 1 - \frac{6+\pi+4\,\text{ArcCotg}[2]}{4\pi} = \frac{3\pi - 6 - 4\,\text{ArcCotg}[2]}{4\pi}$$

Recuérdese que $\text{Arctg}[\frac{1}{2}] = \text{ArcCotg}[2]$

4

$$P(\text{momento esté en } [-\tfrac{\hbar}{a}, \tfrac{\hbar}{a}]) = \int_{-\hbar/a}^{\hbar/a} dp\, |\tilde{\phi}(p)|^2 =$$

$$= \int_0^{\hbar/a} dp\, |\tilde{\phi}(p)|^2 + \int_{-\hbar/a}^{0} dp\, |\tilde{\phi}(p)|^2 =$$

$$= \int_0^{\hbar/a} dp \frac{a}{2\hbar} e^{-\frac{2pa}{\hbar}} + \frac{a}{2\hbar}\int_{-\hbar/a}^{0} dp \left(3e^{-|p|a/\hbar} - 2e^{-2|p|a/\hbar}\right)$$

$$= \frac{1}{4}\left(1 - \frac{1}{e^2}\right) + \frac{-2+8e-9e^2+3e^4}{4e^4}$$

5 $\quad \Delta_\phi X = \sqrt{\langle x^2\rangle_\phi - (\langle x\rangle_\phi)^2}$

$$\langle x\rangle_\phi = (\phi, X\phi) = \int_{-\infty}^{+\infty} dx\, \phi^* X\phi = \int_{-\infty}^{+\infty} dx\, x\,|\phi(x)|^2 = 0$$

ya que $|\phi(x)|^2$ es par.

$$\langle x^2\rangle_\phi = (\phi, x^2\phi) = \int_{-\infty}^{+\infty} dx\, \phi^* \overbrace{X^2\phi}^{x^2\phi} = \int_{-\infty}^{+\infty} dx\, \phi^* x^2\phi =$$

$$= \int_{-\infty}^{+\infty} dx\, x^2 |\phi(x)|^2 = |A|^2 \int_{-\infty}^{+\infty} dx \frac{x^2}{(x^2+a^2)^2 (x^2+4a^2)} = \frac{a^2}{2}$$

Entonces

$$\Delta_\phi X = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}}$$

6 $\quad \Delta_\phi P = \sqrt{\langle p^2\rangle_\phi - (\langle p\rangle_\phi)^2}$

$$\langle p\rangle_\phi = (\tilde\phi, P\tilde\phi) = \int_{-\infty}^{+\infty} dp\, \tilde\phi^*(p) \underbrace{(P\tilde\phi)(p)}_{p\tilde\phi(p)} = \int_{-\infty}^{+\infty} dp\, p\,|\tilde\phi(p)|^2 = 0$$

ya que $|\tilde\phi(p)|^2$ es par

$$\langle p^2\rangle_\phi = (\tilde\phi, P^2\tilde\phi) = \int_{-\infty}^{+\infty} dp\, \tilde\phi^*(p) \underbrace{(P^2\tilde\phi)(p)}_{p^2\tilde\phi(p)} = \int_{-\infty}^{+\infty} dp\, p^2 |\tilde\phi(p)|^2$$

$$= \int_0^{\infty} dp\, p^2 |\tilde\phi(p)|^2 + \int_{-\infty}^{0} dp\, p^2 |\tilde\phi(p)|^2 \overset{(5.1)}{=}$$

$$= \frac{a}{2\hbar}\int_0^{\infty} dp\, p^2 e^{-pa/\hbar} + \frac{a}{2\hbar}\int_{-\infty}^{+\infty} dp\, p^2 \left(3e^{-|p|a/\hbar} - 2e^{-2|p|a/\hbar}\right)^2$$

$$= \frac{125}{144}\frac{\hbar^2}{a^2} \Rightarrow \boxed{\Delta_\phi P = \sqrt{\frac{125}{144}}\,\frac{\hbar}{a}}$$

7

$$\Delta_\phi x \, \Delta_\phi P = \frac{\cancel{\alpha}}{\sqrt{2}} \cdot \sqrt{\frac{125}{144}} \frac{\hbar}{\cancel{\alpha}} = \sqrt{\frac{125}{2\times 144}} \hbar > \frac{\hbar}{2}$$

$$\sqrt{\frac{125}{2\times 144}} = 0.65 > 0.5 \ \frac{1}{2}$$

$\phi(x)$ no es mínimo $\Rightarrow$ $\Delta_\phi x \, \Delta_\phi P > \frac{\hbar}{2}$.

8

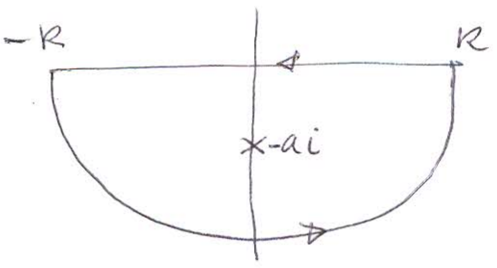
Probabilidad de que $\phi(x)$ esté en $\psi(x) \equiv |(\phi,\psi)|^2$

$$(\phi,\psi) = \int_{-\infty}^{+\infty} dx \, \phi^*_{(x)} \psi(x) = A\sqrt{\frac{2a^3}{\pi}} \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2(x-2ai)} =$$

$$= A\sqrt{\frac{2a^3}{\pi}} \int_{-\infty}^{+\infty} \frac{dx}{(x+ia)^2 (x-ia)^2 (\cancel{x-2ai})} = \frac{2i\sqrt{2}}{3} ; \quad A = \frac{3a^{5/2}}{\pi^{1/2}}$$

$$|(\phi,\psi)|^2 = \frac{2\sqrt{2}}{3}$$

La integral $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2(x-2ai)}$ se obtiene aplicando el th de Cauchy al recinto. En el examen se daría como dato!!!



9

$$\Phi(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, \tilde{\Phi}(p)\, e^{-i/\hbar (Et - px)} \qquad \text{con } E = \frac{p^2}{2m}$$

$\tilde{\Phi}(p)$ está dada en (8.1)

$$\Phi(x,t) = \int_0^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, i\sqrt{\frac{a}{2\hbar}}\, e^{-\frac{pa}{\hbar}}\, e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)}$$

$$+ \int_{-\infty}^{0} \frac{dp}{\sqrt{2\pi\hbar}}\, i\sqrt{\frac{a}{2\hbar}} \left(3 e^{-\frac{|p|a}{\hbar}} - 2 e^{-2|p|a/\hbar}\right) e^{-i/\hbar\left(\frac{p^2 t}{2m} - px\right)}$$

$$= i\sqrt{\frac{a}{4\pi\hbar^2}} \int_0^{\infty} dp\; e^{-ip^2\frac{t}{2m\hbar} - p\left(\frac{a}{\hbar} - \frac{ix}{\hbar}\right)}$$

$$+ \int_0^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, i\sqrt{\frac{a}{2\hbar}} \left(3 e^{-\frac{pa}{\hbar}} - 2 e^{-\frac{2pa}{\hbar}}\right) e^{-i\frac{p^2 t}{2m\hbar} + \frac{ipx}{\hbar}}$$

$$= i\sqrt{\frac{a}{4\pi\hbar^2}} \sqrt{\frac{2m\hbar}{t}} \int_0^{\infty} dq\; e^{-iq^2 - q\left(\frac{a - ix}{\hbar}\right)\sqrt{\frac{2m\hbar}{t}}}$$

$$+ i\sqrt{\frac{a}{4\pi\hbar^2}} \sqrt{\frac{2m\hbar}{t}} \left\{ 3\int_0^{\infty} dq\; e^{-iq^2 - q\left(\frac{a + ix}{\hbar}\right)\sqrt{\frac{2m\hbar}{t}}} \right.$$

$$\left. - 2\int_0^{\infty} dq\; e^{-iq^2 - q\left(\frac{2a + ix}{\hbar}\right)\sqrt{\frac{2m\hbar}{t}}} \right\} =$$

$$= i\sqrt{\frac{ma}{2\pi\hbar t}}\;\frac{1+i}{\sqrt{2}}\left\{ F\left[\frac{1+i}{2}\sqrt{\frac{m}{\hbar t}}\,(a-ix)\right] + 3F\left[\frac{1+i}{2}\sqrt{\frac{m}{\hbar t}}\,(a+ix)\right] - 2F\left[\frac{1+i}{2}\sqrt{\frac{m}{\hbar t}}\,(2a+ix)\right]\right\}$$

$F[u]$ es la función de Dawson

$$\boxed{\begin{aligned}\Phi(x,t) = i\sqrt{\frac{ma}{\pi\hbar t}}\;\frac{1+i}{2}\Bigg\{ & F\left[\frac{1+i}{2}\sqrt{\frac{m}{\hbar t}}\,(a-ix)\right] \\ & + 3F\left[\frac{1+i}{2}\sqrt{\frac{m}{\hbar t}}\,(a+ix)\right] - 2F\left[\frac{1+i}{2}\sqrt{\frac{m}{\hbar t}}\,(2a+ix)\right]\Bigg\}\end{aligned}}$$