Appendix C

Review of Complex Numbers

C.1 REPRESENTATION OF COMPLEX NUMBERS

The complex number z can be expressed in several ways.

Cartesian or rectangular form:

$$z = a + jb \tag{C.1}$$

where $j = \sqrt{-1}$ and a and b are real numbers referred to the *real part* and the *imaginary part* of z. a and b are often expressed as

$$a = \text{Re}\{z\} \qquad b = \text{Im}\{z\} \tag{C.2}$$

where "Re" denotes the "real part of" and "Im" denotes the "imaginary part of."

Polar form:

$$z = re^{j\theta} \tag{C.3}$$

where $r > 0$ is the *magnitude* of z and θ is the *angle* or *phase* of z. These quantities are often written as

$$r = |z| \qquad \theta = \angle z \tag{C.4}$$

Figure C-1 is the graphical representation of z. Using Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{C.5}$$

or from Fig. C-1 the relationships between the cartesian and polar representations of z are

$$a = r\cos\theta \qquad b = r\sin\theta \tag{C.6a}$$

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a} \tag{C.6b}$$

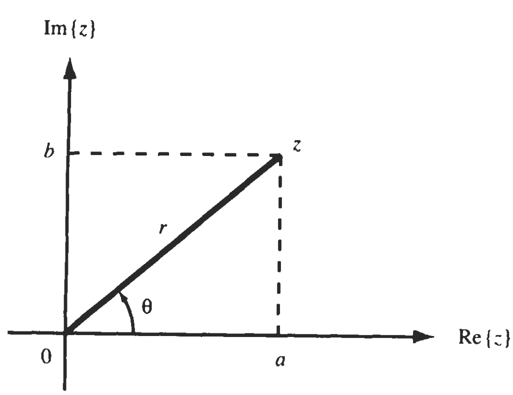

Fig. C-1

C.2 ADDITION, MULTIPLICATION, AND DIVISION

If $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \tag{C.7}$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \tag{C.8}$$

$$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + j(-a_1 b_2 + b_1 a_2)}{a_2^2 + b_2^2} \tag{C.9}$$

If $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{j(\theta_1 + \theta_2)} \tag{C.10}$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1 - \theta_2)} \tag{C.11}$$

C.3 THE COMPLEX CONJUGATE

The *complex conjugate* of z is denoted by z^* and is given by

$$z^* = a - jb = re^{-j\theta} \tag{C.12}$$

Useful relationships:

1. $zz^* = r^2$
2. $\dfrac{z}{z^*} = e^{j2\theta}$
3. $z + z^* = 2\,\mathrm{Re}\{z\}$
4. $z - z^* = j2\,\mathrm{Im}\{z\}$
5. $(z_1 + z_2)^* = z_1^* + z_2^*$
6. $(z_1 z_2)^* = z_1^* z_2^*$
7. $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$

C.4 POWERS AND ROOTS OF COMPLEX NUMBERS

The nth power of the complex number $z = re^{j\theta}$ is

$$z^n = r^n e^{jn\theta} = r^n(\cos n\theta + j \sin n\theta) \tag{C.13}$$

from which we have DeMoivre's relation

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta \tag{C.14}$$

The nth root of a complex z is the number w such that

$$w^n = z = re^{j\theta} \tag{C.15}$$

Thus, to find the nth root of a complex number z we must solve

$$w^n - re^{j\theta} = 0 \tag{C.16}$$

which is an equation of degree n and hence has n roots. These roots are given by

$$w_k = r^{1/n} e^{j[\theta + 2(k-1)\pi]/n} \qquad k = 1, 2, \ldots, n \tag{C.17}$$