• Tarea 6

① Tablero de Butcher ¿Implícito?

. $c_1 = 1/3, \; c_2 = 2/3$

- Sacamos el tablero de Butcher con el método de colocación:

$$q(\tau) = (\tau - 1/3)(\tau - 2/3)$$

$$q_1(\tau) = (\tau - 2/3) \longrightarrow q_1(c_1) = -1/3$$

$$q_2(\tau) = (\tau - 1/3) \longrightarrow q_2(c_2) = 1/3$$

$$a_{11} = \int_0^{1/3} \frac{q_1(\tau)}{q_1(c_1)} d\tau = -3 \left[\frac{(\tau - 2/3)^2}{2} \right]_0^{1/3} = -3 \cdot \frac{1/9 - 4/9}{2} = \frac{1}{2}$$

$$a_{12} = \int_0^{1/3} \frac{q_2(\tau)}{q_2(c_2)} d\tau = 3 \left[\frac{(\tau - 1/3)^2}{2} \right]_0^{1/3} = 3 \cdot \frac{-1/9}{2} = -\frac{1}{6}$$

$$a_{21} = \int_0^{2/3} \frac{q_1(\tau)}{q_1(c_1)} d\tau = -3 \left[\frac{(\tau - 2/3)^2}{2} \right]_0^{2/3} = -3 \cdot \frac{-4/9}{2} = \frac{2}{3}$$

$$a_{22} = \int_0^{2/3} \frac{q_2(\tau)}{q_2(c_2)} d\tau = 3 \left[\frac{(\tau - 1/3)^2}{2} \right]_0^{2/3} = 3 \cdot \frac{1/9 - 1/9}{2} = 0$$

$$b_1 = \int_0^{1} \frac{q_1(\tau)}{q_1(c_1)} d\tau = -3 \left[\frac{(\tau - 2/3)^2}{2} \right]_0^{1} = -3 \cdot \frac{1/9 - 4/9}{2} = \frac{1}{2}$$

$$b_2 = \int_0^{1} \frac{q_2(\tau)}{q_2(c_2)} d\tau = 3 \left[\frac{(\tau - 1/3)^2}{2} \right]_0^{1} = 3 \cdot \frac{4/9 - 1/9}{2} = \frac{1}{2}$$

• Tablero de Butcher:

$$\begin{array}{c|cc} 1/3 & 1/2 & -1/6 \\ 2/3 & 2/3 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

- Como $c_1 \neq 0 \implies$ El método es implícito

② Función de estabilidad:

$$y_{n+1} = y_n + h b_1 f(t_n + c_1 h, \xi_1) + h b_2 f(t_n + c_2 h, \xi_2)$$

donde $\begin{cases} \xi_1 = y_n + h a_{11} f(t_n + c_1 h, \xi_1) + h a_{12} f(t_n + c_2 h, \xi_2) \\ \xi_2 = y_n + h a_{21} f(t_n + c_1 h, \xi_1) + h a_{22} f(t_n + c_2 h, \xi_2) \end{cases}$

Tomamos $f(t, y) = \lambda y$:

$$\Longrightarrow \begin{cases} \xi_1 = y_n + \frac{h}{2}\lambda \xi_1 - \frac{h}{6}\lambda \xi_2 \\ \xi_2 = y_n + \frac{2}{3} h \lambda \xi_1 \end{cases}$$

$$\xrightarrow[\text{en } 1^a \text{ igualdad}]{\text{Sustituimos } \xi_2} \xi_1 = y_n + \frac{h}{2}\lambda \xi_1 - \frac{h}{6}\lambda y_n - \frac{h^2\lambda^2}{9}\xi_1$$

$$\Longrightarrow \xi_1 = \frac{1 - \frac{1}{6}h\lambda}{1 - \frac{1}{2}h\lambda + \frac{1}{9}h^2\lambda^2} y_n$$

$$\Longrightarrow \xi_2 = \frac{1 - \frac{1}{2}h\lambda + \cancel{\frac{1}{9}h^2\lambda^2} + \frac{2}{3}h\lambda - \cancel{\frac{1}{9}h^2\lambda^2}}{1 - \frac{1}{2}h\lambda + \frac{1}{9}h^2\lambda^2} y_n = \frac{1 + \frac{1}{6}h\lambda}{1 - \frac{1}{2}h\lambda + \frac{1}{9}h^2\lambda^2} y_n$$

Como $y_{n+1} = y_n + \frac{h\lambda}{2}\xi_1 + \frac{h\lambda}{2}\xi_2$

$$\Longrightarrow y_{n+1} = \frac{1 - \cancel{\frac{1}{2}h\lambda} + \frac{1}{9}h^2\lambda^2 + \frac{h\lambda}{2} - \cancel{\frac{1}{12}h^2\lambda^2} + \cancel{\frac{h\lambda}{2}} + \cancel{\frac{1}{12}h^2\lambda^2}}{1 - \frac{1}{2}h\lambda + \frac{1}{9}h^2\lambda^2} y_n$$

$$\Longrightarrow y_{n+1} = \frac{1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{9}}{1 - \frac{h\lambda}{2} + \frac{h^2\lambda^2}{9}} y_n$$

$$\cancel{\Longrightarrow R(z) = \frac{1+z}{}}$$

$$\Longrightarrow R(z) = \frac{1 + \frac{z}{2} + \frac{z^2}{9}}{1 - \frac{z}{2} + \frac{z^2}{9}} \quad \text{con } z = h\lambda$$

$C_1 = \frac{1}{4} \quad C_2 = \frac{3}{4}$

1) $q(\tau) = (\tau - \frac{1}{4})(\tau - \frac{3}{4})$

$$q_1(\tau) = \frac{(\tau - \cancel{\frac{1}{4}})(\tau - \frac{3}{4})}{(\tau - \cancel{\frac{1}{4}})} = \tau - \frac{3}{4}$$

$$q_2(\tau) = \frac{(\tau - \frac{1}{4})(\cancel{\tau - \frac{3}{4}})}{(\cancel{\tau - \frac{3}{4}})} = \tau - \frac{1}{4}$$

El método de colocación nos permite calcular los a_{ij} y b_j $i = 1, 2$ a partir de los $q_i(\tau)$ $i=1,2$

$$a_{11} = \int_0^{\frac{1}{4}} \frac{\tau - \frac{3}{4}}{\frac{1}{4} - \frac{3}{4}} d\tau = -2 \int_0^{\frac{1}{4}} \tau - \frac{3}{4} \, d\tau = -2 \left[\frac{\tau^2}{2} - \frac{3}{4}\tau \right] \Bigg|_0^{\frac{1}{4}} =$$

$$= -2 \cdot \left[\frac{1}{32} - \frac{3}{16} \right] = \cancel{\frac{1}{4}} \; \boxed{\frac{5}{16}}$$

$$a_{12} = \int_0^{\frac{1}{4}} \frac{\tau - \frac{1}{4}}{\frac{3}{4} - \frac{1}{4}} d\tau = 2 \cdot \left[\frac{\tau^2}{2} - \frac{\tau}{4} \right] \Bigg|_0^{\frac{1}{4}} = 2 \cdot \left[\frac{1}{32} - \frac{1}{16} \right] = \boxed{-\frac{1}{16}}$$

$$a_{21} = \int_0^{\frac{3}{4}} \frac{\tau - \frac{3}{4}}{\frac{1}{4} - \frac{3}{4}} d\tau = -2 \left[\frac{\tau^2}{2} - \frac{3}{4}\tau \right] \Bigg|_0^{\frac{3}{4}} = -2 \left[\frac{9}{32} - \frac{9}{16} \right] = \boxed{\frac{9}{16}}$$

$$a_{22} = \int_0^{\frac{3}{4}} \frac{\tau - \frac{1}{4}}{\frac{3}{4} - \frac{1}{4}} d\tau = 2 \cdot \left[\frac{\tau^2}{2} - \frac{\tau}{4} \right] \Bigg|_0^{\frac{3}{4}} = 2 \cdot \left[\frac{9}{32} - \frac{3}{16} \right] = \boxed{\frac{3}{16}}$$

$$b_1 = \int_0^1 \frac{\tau - \frac{3}{4}}{\frac{1}{4} - \frac{3}{4}} d\tau = -2 \cdot \left[\frac{\tau^2}{2} - \frac{3}{4}\tau \right] \Bigg|_0^1 = -2 \cdot \left[\frac{1}{2} - \frac{3}{4} \right] = \boxed{\frac{1}{2}}$$

$$b_2 = \int_0^1 \frac{\tau - \frac{1}{4}}{\frac{3}{4} - \frac{1}{4}} d\tau = 2 \cdot \left[\frac{\tau^2}{2} - \frac{1}{4}\tau \right] \Bigg|_0^1 = 2 \cdot \left[\frac{1}{2} - \frac{1}{4} \right] = \boxed{\frac{1}{2}}$$

Por lo que el tablero queda de la siguiente manera:

$$\begin{array}{c|cc} \frac{1}{4} & \frac{5}{16} & -\frac{1}{16} \\ \frac{3}{4} & \frac{9}{16} & \frac{3}{16} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Es implícito, ya que $c_1 \neq 0$

2) Desarrollamos $\mathcal{E}_1$ y $\mathcal{E}_2$ con los datos del apartado anterior y sabiendo que $f(t,Y) = \lambda Y$. Tomamos además $h\lambda = z$

$$\left\{\begin{array}{l} \mathcal{E}_1 = Y_n + \frac{5}{16} z \mathcal{E}_1 + \frac{1}{16} z \mathcal{E}_2 \\ \mathcal{E}_2 = Y_n + \frac{9}{16} z \mathcal{E}_1 + \frac{3}{16} z \mathcal{E}_2 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \mathcal{E}_1(16-5z) = 16 Y_n - z\mathcal{E}_2 \\ \mathcal{E}_2(16-3z) = 16 Y_n + 9z\mathcal{E}_1 \end{array}\right. \Rightarrow$$

$$\Rightarrow \mathcal{E}_2 = \frac{1}{z} \cdot \left[16 Y_n - \mathcal{E}_1(16-5z)\right]$$ Sustituimos en la segunda ecuación:

$$\frac{1}{z} \cdot \left[16 Y_n - \mathcal{E}_1(16-5z)\right](16-3z) = 16 Y_n + 9z\mathcal{E}_1$$

$$256 Y_n - 48z Y_n - \mathcal{E}_1(256 - 48z - 80z + 15z^2) = 16z Y_n + 9z^2 \mathcal{E}_1$$

$$(256 - 64z) Y_n = \mathcal{E}_1(256 + 24z^2 - 128z)$$

$$\Rightarrow \boxed{\mathcal{E}_1 = \frac{(256-64z) Y_n}{256 + 24z^2 - 128z}} = \boxed{\frac{(32-8z) Y_n}{32 + 3z^2 - 16z}}$$

$$\boxed{\mathcal{E}_2 = \frac{1}{z} \cdot \left[16 Y_n - \frac{(32-8z)(16-5z) Y_n}{32+3z^2-16z}\right]} = \frac{1}{z} \cdot \left[\frac{(\cancel{512} + 48z^2 - 256z - \cancel{512} + 160z + 128z - 40z^2) Y_n}{32 + 3z^2 - 16z}\right]$$

$$= \frac{1}{z} \left[\frac{8z^2 + 32z}{32+3z^2-16z}\right] = \boxed{\frac{8z+32}{32+3z^2-16z}}$$

Sustituyendo en la expresión de Y_{n+1} (usando los b_i y que $f(t,Y) = \lambda Y$)

$$Y_{n+1} = Y_n + \frac{z}{2}\mathcal{E}_1 + \frac{z}{2}\mathcal{E}_2 = Y_n + \frac{(32z - 8z^2) Y_n}{2(32+3z^2-16z)} + \frac{(8z^2+32z) Y_n}{2\cdot(32+3z^2-16z)} =$$

$$= \left(1 + \frac{64z}{2\cdot(32+3z^2-16z)}\right) Y_n = \left(1 + \frac{32z}{32+3z^2-16z}\right)$$

La función de estabilidad es la obtenida de la forma ~~$R(z) \Rightarrow$~~ $Y_{n+1} = R(z) Y_n$

$$\Rightarrow \boxed{R(z) = 1 + \frac{32z}{32 + 3z^2 - 16z}}$$