WORKSHEET 3: Differentiation of functions of one variable

1. Find all points where the graph of the following functions have a horizontal tangent line.

a) $f(x) = x^3 + 1$ b) $f(x) = 1/x^2$ c) $f(x) = x + \operatorname{sen} x$
d) $f(x) = \sqrt{x-1}$ e) $f(x) = e^x - x$ f) $f(x) = \operatorname{sen} x + \cos x$

a) x=0; b) never; c) $x = \pi + 2k\pi$.d) never; e) $x = 0$; f) $x = \frac{\pi}{4} + k\pi$.

2. (*)Prove that the tangent lines to the graphs of $y = x$ and $y = 1/x$ at their intersection points are perpendicular to each other.

3. In which point is the tangent line to the curve $y^2 = 3x$ parallel to the straight line $y = 2x$?

The point of the curve is $(\frac{3}{16}, \frac{3}{4})$.

4. (*)Calculate the intersection point with the x-axis of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$.

The intersection point is $x = 1/2$.

5. Calculate the value of a so that the tangent to the graph of $f(x) = a/x + 1$ at the point $(1, f(1))$ intersects the horizontal axis at $x = 3$.

So the intersection point will be x=3 when a=1.

6. Calculate the angle of intersection of the curves $y = \frac{1}{2}(x^2 - 1)$ and $y = \frac{1}{2}(x^3 - x)$.

In $x = -1$ the angle is $\frac{\pi}{2}$. In $x = 1$ the angle is 0.

7. (*)Given $f(x) = 2[\ln(1 + g^2(x))]^2$, use $g(1) = g'(1) = -1$, to find $f'(1)$.

$f\,'(1) = 4\ln(2)$.

8. (*)Knowing that $a^b = e^{b\ln a}$, calculate the derived function of $f(x) = x^{\operatorname{sen} x}$ and $g(x) = (\sqrt{x})^x$.

$f\,'(x) = x^{\operatorname{sen} senx}(\cos x. \ln x + senx/x)$.

$g'(x) = (\sqrt{x})^x(\ln x + 1)/2$.

9. (*)Let $f(x) = \ln(1+x^2)$ and $g(x) = e^{2x}+e^{3x}$ be two real functions. Calculate $h(x) = f(g(x))$, $v(x) = g(f(x))$, $h'(0)$ and $v'(0)$.

$h(x) = \ln(1 + e^{4x} + e^{6x} + 2e^{5x}), h'(0) = 4$

$v(x) = (1 + x^2)^2+(1 + x^2)^3, v'(0) = 0$.

10. Let $f : [-2, 2] \to [-2, 2]$ be a continuous and bijective function.

a) Suppose that $f(0) = 0$ and $f'(0) = \alpha$, $\alpha \neq 0$. Find $\left(f^{-1}\right)'(0)$.

b) Suppose now that $f(0) = 1$ and $f'(0) = \alpha$, $\alpha \neq 0$. Find$\left(f^{-1}\right)'(1)$.

c) Finally, suppose that $f(1) = 0$ and $f'(1) = \alpha$, $\alpha \neq 0$. Find$\left(f^{-1}\right)'(0)$.

a) $\left(f^{-1}\right)'(0) = \frac{1}{\alpha}$.

b) $\left(f^{-1}\right)'(1) = \frac{1}{\alpha}$

c) $\left(f^{-1}\right)'(0) = 1$

12. (*)Find a and b so that the function $f(x)=\begin{cases} 3x+2 & \text{if } x\geq 1 \\ ax^2+bx-1 & \text{if } x<1 \end{cases}$ is differentiable everywhere.

f differentiable in 1 is equivalent to $a=-3, b=9$.

13. Apply the Mean Value Theorem (Lagrange's Theorem) to f in the given interval and find the x-coordinate values of the points that satisfy the thesis of the theorem.

a) $f(x)=x^2$ in $[-2,1]$ b) $f(x)=-2\operatorname{sen} x$ in $[-\pi,\pi]$
c) $f(x)=x^{\frac{2}{3}}$ in $[0,1]$ d) $f(x)=2\operatorname{sen} x+\operatorname{sen} 2x$ en $[0,\pi]$

14. (*)Let $f(x)=x^3-3x+3$, $f:[-3,2]\to\mathbb{R}$. Find the globlal extrema.

The minimum is reached in -3 and the maximum is reached in -1 and in 2.

15. Calculate the following limits:

a)(*) $\lim\limits_{x\to\infty}(1+x)^{1/x}$ b) $\lim\limits_{x\to 0^+} x\ln x$ c)(*) $\lim\limits_{x\to\infty} x^{1/x}$ d)(*) $\lim\limits_{x\to 1^+}\left(\dfrac{1}{\ln x}-\dfrac{2}{x-1}\right)$

16. Find all the asymptotes of the following functions:

a)(*) $f(x)=\dfrac{2x^3-3x^2-8x+4}{x^2-4}$ b) $f(x)=\dfrac{x^3}{x^3+x^2+x+1}$ c)(*) $f(x)=2x+e^{-x}$
d) $f(x)=\dfrac{\operatorname{sen} x}{x}$ e)(*) $f(x)=\dfrac{x-2}{\sqrt{4x^2+1}}$ f) $f(x)=\dfrac{3x^2-x+2\operatorname{sen} x}{x-7}$
g)(*) $f(x)=\dfrac{e^x}{x}$ h)(*) $f(x)=xe^{1/x}$ i)(*) $f(x)=\dfrac{x}{e^x-1}$

a) Vertical asymptotes in $x=2$ and in $x=-2$.

On the other hand, the oblique asymptote in ∞ and in $-\infty$ is $y=2x-3$.

c) $y=2x$ is the oblique asymptote in ∞.

e) $\lim\limits_{x\to\infty}\frac{x-2}{\sqrt{4x^2+1}}=\frac{1}{2}$, $\lim\limits_{x\to-\infty}\frac{x-2}{\sqrt{4x^2+1}}=-\frac{1}{2}$. There are no more asymptotes.

g) $\lim\limits_{x\to 0^+}\frac{e^x}{x}=\infty$, $\lim\limits_{x\to 0^-}\frac{e^x}{x}=-\infty$, and there are no more vertical asymptotes.

On the other hand, $y=0$ is horizontal asymptote in $-\infty$, and there are no horizontal, nor oblique asymptote in ∞.

h) $\lim\limits_{x\to 0^+} xe^{1/x}=\infty$, and there are no more vertical asymptotes.

On the other hand, $y=x+1$ is the oblique asymptote in ∞, and also in $-\infty$.

i) There is no vertical asymptote.

On the other hand, $y=0$ is the horizontal asymptote in ∞. Finally, the line $y=-x$ is the oblique asymptote in $-\infty$.