Exercises Chapter VI

Mathematical Methods of Bioengineering

May 17, 2021

This is a guide to the exercises that you can solve. If you fall short, there are more similar exercises in the books of the subject. You can ask me questions at the end of class or in tutoring.

The idea is that you solve the exercises and do them on the blackboard in class. Each time you go to the board, it will count to the 5% of the final mark. You must participate at least 3 times in order to get the full 5% and at least 6 times to raise the final grade by +0.5 points.

1 Vectors

2 Differentiation in Several Variables

3 Vector Valued Functions

4 Maxima and Minima in Several Variables

5 Multiple Integration

6 Line Integrals

6.1 Scalar and Vector Line Integrals

6.2 Green's Theorem

6.3 Conservative Vector Fields

1. Consider the line integral $\int_C z^2 dx + 2y\ dy + xz\ dz$.

   (a) Evaluate this integral, where C is the line segment from (0,0,0) to (1,1,1).

   (b) Evaluate this integral, where C is the path from $(0,0,0)$ to $(1,1,1)$ parametrized by $\mathbf{x}(t) = (t, t^2, t^3),\ 0 \leq t \leq 1$.

   (c) Is the vector field $\mathbf{F} = z^2\mathbf{i} + 2y\mathbf{j} + xz\mathbf{k}$ conservative? Why or why not?

   SOL:

   a) $\mathbf{c}(t) = (t, t, t)$ $t \in [0,1]$, so $\int_C \mathbf{F} d\mathbf{s} = \int_0^1 (t^2 + 2t + t^2) dt = 5/3$

b) $\int_C \mathbf{F} d\mathbf{s} = 11/7$

c) Because results in a) and b) are distinct, the integral is not path-independent so can't be conservative (Theorem 3.3).

2. Determine whether the given vector field $\mathbf{F}$ is conservative. If it is, find a scalar potential function for $\mathbf{F}$.

   (a) $\mathbf{F} = e^{x+y}\mathbf{i} + e^{xy}\mathbf{j}$.

   (b) $\mathbf{F} = 2x\sin y\mathbf{i} + x^2\cos y\mathbf{j}$.

   (c) $\mathbf{F} = (e^{-y} - y\sin xy)\mathbf{i} - (xe^{-y} + x\sin xy)\mathbf{j}$.

   (d) $\mathbf{F} = (6xy^2 + 2y^3)\mathbf{i} + (6x^2y - xy)\mathbf{j}$.

   (e) $\mathbf{F} = (6xy^2 - 3x^2)\mathbf{i} + (y^2 + 6x^2y)\mathbf{j}$.

   SOL:

   Criterion: $\dfrac{\partial N}{\partial x}? = \dfrac{\partial M}{\partial y}$

   a) F not conservative.

   b) Yes. $f(x, y) = x^2 \cdot sin(y)$

   c) Yes. $f(x, y) = xe^{-y} + cos(xy)$

   d) F not conservative.

   e) Yes. $f(x, y) = (3x^2y^2 - x^3 + 1/3y^3)$

3. Of two vector fields

$$\mathbf{F} = xy^2z^3\mathbf{i} + 2x^2y\mathbf{j} + 3x^2y^2z^2\mathbf{k}$$

   and

$$\mathbf{G} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$$

   one is conservative and one is not. Determine which is which, and, for the conservative field, find a scalar potential function.

   SOL:

   Criterion: $curl(\mathbf{F}) = (0, 0, 0) \iff \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y};\ \dfrac{\partial P}{\partial x} = \dfrac{\partial M}{\partial z};\ \dfrac{\partial P}{\partial y} = \dfrac{\partial M}{\partial z}$

   It results that F is not conservative and that G is. A scalar potential function is $f(x, y, z) = x^2y + y^2z$.

4. For what values of the constants a and b will the vector field

$$\mathbf{F} = (3x^2 + 3y^2z\sin xz)\mathbf{i} + (ay\cos xz + bz)\mathbf{j} + (3xy^2\sin xz + 5y)\mathbf{k}$$

   be conservative?

   SOL:

   $a = -6,\ b = 5.$

5. Let $\mathbf{F} = x^2\mathbf{i} + \cos y \sin z\mathbf{j} + \sin y \cos z\mathbf{k}$.

   (a) Show that $\mathbf{F}$ is conservative and find a scalar potential function f for $\mathbf{F}$.
   (b) Evaluate $\int_x \mathbf{F} \cdot d\mathbf{s}$ along the path $\mathbf{x} : [0,1] \to \mathbb{R}^3,\ \mathbf{x}(t) = (t^2+1, e^t, e^{2t})$.

   SOL:

   a) Criterion: $curl(\mathbf{F}) = (0,0,0) \iff \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y};\ \ \dfrac{\partial P}{\partial x} = \dfrac{\partial M}{\partial z};\ \ \dfrac{\partial P}{\partial y} = \dfrac{\partial M}{\partial z}$ is satisfied. A scalar potential is: $f(x,y,z) = x^3/3 + \sin y \sin z$.

   b) $\int_{\mathbf{x}} \mathbf{F} d\mathbf{s} = f(2,e,e^2) - f(1,1,1) = 7/3 + \sin e \sin e^2 - \sin^2 1$.

6. Show that the line integrals are path independent, and evaluate them along the given oriented curve and also by means of *Theorem* 3.3:

   (a) $\displaystyle\int_C (3x-5y)dx + (7y-5x)dy$; C is the line segment from (1,3) to (5,2).
   (b) $\displaystyle\int_C \frac{x\ dx + y\ dy}{\sqrt{x^2+y^2}}$; C is the semicircular arc of $x^2+y^2=4$, from (2,0) to (-2,0).

   SOL:

   a) $(7y-5x)_x = -5(3x-5y)_y$, so conservative. A parametrization of the path is: $\mathbf{c}(t) = (4t+1, -t+3)\ \ t \in [0,1]$. The result of the given line integral is -33/2.

   b) The domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{0,0\}$. Remember that the equivalency: path-independent $\iff$ vector field conservative (theorem 3.3) applies only to simply connected regions. So we have to avoid the origin, since there must not be “holes”. For example we can take the semi-annular region bounded by the upper half circle of radius 3 and the one of radius 1. The field is conservative and the results of the integral is 0.

7. Find the work done by the given vector field $\mathbf{F}$ in moving a particle from the point A=$(0,0)$ to the point $B = (2,1)$.

   $$\mathbf{F} = (3x^2y - y^2))\mathbf{i} + (x^3 - 2xy)\mathbf{j}$$

   SOL:

   F is conservative so we can find the work in simple manner by: $\int_C \mathbf{F}\ d\mathbf{s} = f(2,1) - f(0,0) = 6 - 0$, where C is any piecewise C^1 path joining A and B and $f(x,y,) = x^3y - xy^2$.