Exercises Chapter VI

Mathematical Methods of Bioengineering

May 17, 2021

This is a guide to the exercises that you can solve. If you fall short, there are more similar exercises in the books of the subject. You can ask me questions at the end of class or in tutoring.

The idea is that you solve the exercises and do them on the blackboard in class. Each time you go to the board, it will count to the 5% of the final mark. You must participate at least 3 times in order to get the full 5% and at least 6 times to raise the final grade by +0.5 points.

1 Vectors

2 Differentiation in Several Variables

3 Vector Valued Functions

4 Maxima and Minima in Several Variables

5 Multiple Integration

6 Line Integrals

6.1 Scalar and Vector Line Integrals

1. Let $f(x,y) = x + 2y$. Evaluate the scalar line integral $\int_{\mathbf{x}} f\, ds$ over the given path $\mathbf{x}$.

   (a) $\mathbf{x}(t) = (2 - 3t, 4t - 1)$, $0 \leq t \leq 2$

   (b) $\mathbf{x}(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$

2. Calculate the line integral of $f(x,y,z) = x + y + z$ over $\mathbf{x}$.

$$\mathbf{x}(t) = \begin{cases} (2t, 0, 0) & t \in [0,1) \\ (2, 3t-3, 0) & t \in [1,2) \\ (2, 3, 2t-4) & t \in [2,3] \end{cases}$$

3. Calculate the line integral of $f(x,y,z) = 2x - y^{1/2} + 2z^2$ over:

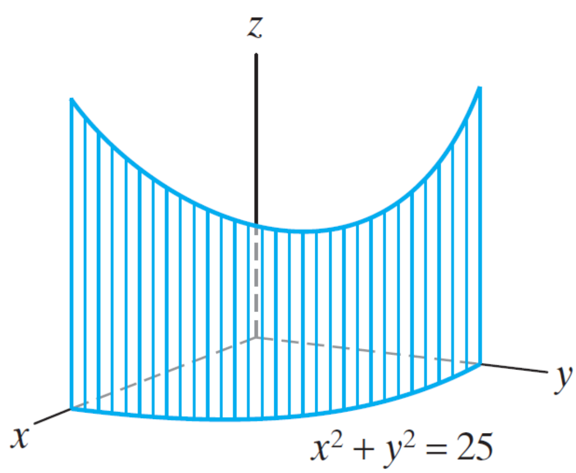
$$\mathbf{x}(t) = \begin{cases} (t, t^2, 0) & t \in [0,1) \\ (1, 1, t-1) & t \in [1,3) \end{cases}$$

4. Calculate the vector line integral $\int_{\mathbf{x}} \mathbf{F}\, d\mathbf{s}$ for:

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{x}(t) = (2t+1, t, 3t-1),\ 0 \le t \le 1$
   (b) $\mathbf{F} = (y-x)\mathbf{i} + x^4y^3\mathbf{j}$, $\mathbf{x}(t) = (t^2, t^3),\ -1 \le t \le 1$
   (c) $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}$, $\mathbf{x}(t) = (3\cos t, 2\sin t, 5t),\ 0 \le t \le 2\pi$

5. Find the work done by the force field $\mathbf{F} = 2x\mathbf{i} + \mathbf{j}$ when a particle moves along the path $\mathbf{x}(t) = (t, 3t^2, 2)$, $0 \le t \le 2$.

6. Tom Sawyer is whitewashing a picket fence. The bases of the fenceposts are arranged in the xy-plane as the quarter circle $x^2+y^2 = 25$, x, y ≥ 0, and the height of the fencepost at point (x, y) is given by $h(x, y) = 10 - x - y$ (units are feet). Use a scalar line integral to find the area of one side of the fence.



7. Let $\mathbf{F}$ be the radial vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that if $\mathbf{x}(t)$, $a \le t \le b$, is any path that lies on the sphere $x^2+y^2+z^2 = c^2$, then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = 0$ (Hint: Use the implicit derivative of $[x(t)]^2 + [y(t)]^2 + [z(t)]^2 = c^2$) when computing the line integral).

6.2 Green's Theorem

1. Verify Green's Theorem for the given vector field:

   (a) $\mathbf{F} = -x^2y\mathbf{i} + xy^2\mathbf{j}$, in the disk $D : x^2 + y^2 \le 4$.
   (b) $\mathbf{F} = (x^2 - y)\mathbf{i} + (x + y^2)\mathbf{j}$, in the rectangle D bounded by $x = 0$, $x = 2$, $y = 0$, and $y = 1$.
   (c) $\mathbf{F} = 3y\mathbf{i} - 4x\mathbf{j}$, in the elliptical region $D : x^2 + 2y^2 \le 4$.
   (d) $\mathbf{F} = (x^2y + x)\mathbf{i} + (y^3 - xy^2)\mathbf{j}$, where D is the region inside the circle $x^2 + y^2 = 9$ and outside the circle $x^2 + y^2 = 4$.

   Note:

   - $\sin 2t = 2\sin t \cos t$

- $\cos^2(t) = \frac{1+\cos 2t}{2}$

2. Evaluate

$$\oint_C (x^2 - y^2)dx + (x^2 + y^2)\ dy,$$

where C is the boundary of the square with vertices $(0,0), (1,0), (0,1)$, and $(1,1)$, oriented *clockwise*. Use whatever method of evaluation seems appropriate.

3. Use Green's theorem to find the work done by the vector field

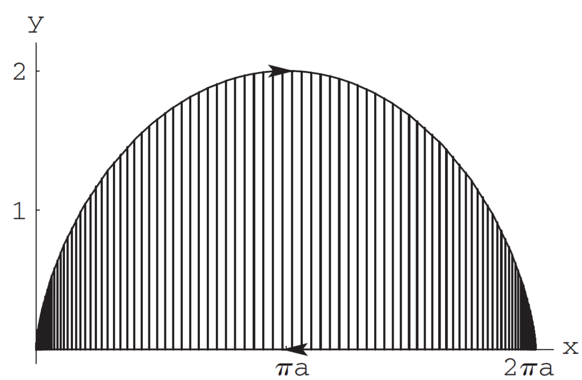
$$\mathbf{F} = (4y - 3x)\mathbf{i} + (x - 4y)\mathbf{j}$$

on a particle as the particle moves counterclockwise once around the ellipse $x^2 + 4y^2 = 4$.

4. OPTIONAL: Let a be a positive constant. Use Green's theorem to calculate the area under one arch of the cycloid

$$\mathbf{x}(t) = \Big[a(t - \sin t), a(1 - \cos t)\Big].$$

One arch of the cycloid is produced from $t = 0$ to $t = 2\pi$.



5. Use Green's theorem to find the area between the ellipse $x^2/9 + y^2/4 = 1$ and the circle $x^2 + y^2 = 25$.

6. Let C be any simple, closed curve in the plane. Show that $\oint_C 3x^2y\ dx + x^3\ dy = 0$.

7. Show that if C is the boundary of any rectangular region in $\mathbb{R}^2$, then

$$\oint_C (x^2y^3 - 3y)\ dx + x^3y^2\ dy$$

depends only on the area of the rectangle, not on its displacement in $\mathbb{R}^2$.

6.3 Conservative Vector Fields

1. Consider the line integral $\int_C z^2 dx + 2y\ dy + xz\ dz$.

   (a) Evaluate this integral, where C is the line segment from (0,0,0) to (1,1,1).

   (b) Evaluate this integral, where C is the path from $(0,0,0)$ to $(1,1,1)$ parametrized by $\mathbf{x}(t) = (t, t^2, t^3),\ 0 \leq t \leq 1$.

   (c) Is the vector field $\mathbf{F} = z^2\mathbf{i} + 2y\mathbf{j} + xz\mathbf{k}$ conservative? Why or why not?

2. Determine whether the given vector field $\mathbf{F}$ is conservative. If it is, find a scalar potential function for $\mathbf{F}$.

   (a) $\mathbf{F} = e^{x+y}\mathbf{i} + e^{xy}\mathbf{j}$.

   (b) $\mathbf{F} = 2x\sin y\mathbf{i} + x^2\cos y\mathbf{j}$.

   (c) $\mathbf{F} = (e^{-y} - y\sin xy)\mathbf{i} - (xe^{-y} + x\sin xy)\mathbf{j}$.

   (d) $\mathbf{F} = (6xy^2 + 2y^3)\mathbf{i} + (6x^2y - xy)\mathbf{j}$.

   (e) $\mathbf{F} = (6xy^2 - 3x^2)\mathbf{i} + (y^2 + 6x^2y)\mathbf{j}$.

3. Of two vector fields

$$\mathbf{F} = xy^2z^3\mathbf{i} + 2x^2y\mathbf{j} + 3x^2y^2z^2\mathbf{k}$$

   and

$$\mathbf{G} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$$

   one is conservative and one is not. Determine which is which, and, for the conservative field, find a scalar potential function.

4. For what values of the constants a and b will the vector field

$$\mathbf{F} = (3x^2 + 3y^2z\sin xz)\mathbf{i} + (ay\cos xz + bz)\mathbf{j} + (3xy^2\sin xz + 5y)\mathbf{k}$$

   be conservative?

5. Let $\mathbf{F} = x^2\mathbf{i} + \cos y\sin z\mathbf{j} + \sin y\cos z\mathbf{k}$.

   (a) Show that $\mathbf{F}$ is conservative and find a scalar potential function f for $\mathbf{F}$.

   (b) Evaluate $\int_x \mathbf{F}\cdot d\mathbf{s}$ along the path $\mathbf{x} : [0,1] \to \mathbb{R}^3,\ \mathbf{x}(t) = (t^2+1, e^t, e^{2t})$.

6. Show that the line integrals are path independent, and evaluate them along the given oriented curve and also by means of *Theorem* 3.3:

   (a) $\displaystyle\int_C (3x - 5y)dx + (7y - 5x)dy$; C is the line segment from (1,3) to (5,2).

   (b) $\displaystyle\int_C \frac{x\ dx + y\ dy}{\sqrt{x^2+y^2}}$; C is the semicircular arc of $x^2 + y^2 = 4$, from (2,0) to (-2,0).

7. Find the work done by the given vector field $\mathbf{F}$ in moving a particle from the point A=$(0,0)$ to the point $B = (2,1)$.

$$\mathbf{F} = (3x^2y - y^2))\mathbf{i} + (x^3 - 2xy)\mathbf{j}$$