Problems

Problem 1.1 Given the real numbers $0 < a < b$ and $c > 0$, prove the inequalities

(a) $a < \sqrt{ab} < \dfrac{a+b}{2} < b$, (b) $\dfrac{a}{b} < \dfrac{a+c}{b+c}$.

Problem 1.2 Prove that $|a+b| = |a| + |b|$ if and only if $ab \geqslant 0$.

Problem 1.3 Prove that

(a) $\max\{x,y\} = \dfrac{x+y+|x-y|}{2}$, (b) $\min\{x,y\} = \dfrac{x+y-|x-y|}{2}$.

Problem 1.4 Find, using the absolute value, a formula to express the function

$$\varphi(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Problem 1.5 Factor out the following expressions of $n \in \mathbb{N}$, so that the corresponding statements become self-evident:

(a) $n^2 - n$ is even,
(b) $n^3 - n$ is a multiple of 6,
(c) $n^2 - 1$ is a multiple of 8 when n is odd.

Problem 1.6 Prove by induction the following statements valid for all $n \in \mathbb{N}$:

(a) $a^n - b^n = (a-b)\displaystyle\sum_{k=1}^{n} a^{n-k} b^{k-1}$,

(b) $n^5 - n$ is a multiple of 5,

(c) $(1+x)^n \geqslant 1 + nx$ if $x \geqslant -1$.

HINT: In (a) make use of the properties of symbolic sums summarised in Appendix A.

Problem 1.7 Prove by induction the following statements valid for all natural numbers $n > 1$:

(a) $n! < \left(\dfrac{n+1}{2}\right)^n$,

(b) $2!4!\cdots(2n)! > \big[(n+1)!\big]^n$,

(c) $1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$.

HINT: In (a) use the inequality $\left(1 + \dfrac{1}{n+1}\right)^{n+1} > 2$, valid for all $n \in \mathbb{N}$. In (b) prove first that $(2n+2)! > (n+2)^n (n+2)!$.

Problem 1.8

(a) Show, with an example, that the sum of two irrational numbers can be rational.
(b) Show, with an example, that the product of two irrational numbers can be rational.
(c) Is it possible to find irrational numbers x and y such that $x^y \in \mathbb{Q}$?

Problem 1.9 Prove that

(a) $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$,
(b) $\sqrt{n} \notin \mathbb{Q}$ if n is not a perfect square (HINT: write $n = k^2 r$, where r does not contain any square factor),
(c) $\sqrt{n-1} + \sqrt{n+1} \notin \mathbb{Q}$ for all $n \in \mathbb{N}$.

Problem 1.10 Prove the identity, valid for all $x \in \mathbb{R}$,

$$\left(\frac{x+|x|}{2}\right)^2 + \left(\frac{x-|x|}{2}\right)^2 = x^2.$$

Problem 1.11 Identify the following sets:

(i) $A = \{x \in \mathbb{R} : |x-3| \leqslant 8\}$,

(ii) $B = \{x \in \mathbb{R} : 0 < |x-2| < 1/2\}$,

(iii) $C = \{x \in \mathbb{R} : x^2 - 5x + 6 \geqslant 0\}$,

(iv) $D = \{x \in \mathbb{R} : x^3(x+3)(x-5) < 0\}$,

(v) $E = \left\{x \in \mathbb{R} : \dfrac{2x+8}{x^2+8x+7} > 0\right\}$,

(vi) $F = \left\{x \in \mathbb{R} : \dfrac{4}{x} < x\right\}$,

(vii) $G = \{x \in \mathbb{R} : 4x < 2x + 1 \leqslant 3x + 2\}$,

(viii) $H = \{x \in \mathbb{R} : |x^2 - 2x| < 1\}$,

(ix) $I = \{x \in \mathbb{R} : |x-1||x+2| = 10\}$,

(x) $J = \{x \in \mathbb{R} : |x-1| + |x+2| > 1\}$.

Problem 1.12 Given real numbers $a < b$ we define, for each $t \in \mathbb{R}$, the real number $x(t) = (1-t)a + tb$. Identify the following sets:

(i) $A = \{x(t) : t = 0, 1, 1/2\}$,

(ii) $B = \{x(t) : t \in (0,1)\}$,

(iii) $C = \{x(t) : t < 0\}$,

(iv) $D = \{x(t) : t > 1\}$.

Problem 1.13 Find supremum and infimum (deciding whether they are maximum and minimum respectively) of the following sets:

(i) $A = \{-1\} \cup [2,3)$,

(ii) $B = \{3\} \cup \{2\} \cup \{-1\} \cup [0,1]$,

(iii) $C = \{2 + 1/n : n \in \mathbb{N}\}$,

(iv) $D = \{(n^2+1)/n : n \in \mathbb{N}\}$,

(v) $E = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$,

(vi) $F = \{x \in \mathbb{R} : (x-a)(x-b)(x-c)(x-d) < 0\}$, with $a < b < c < d$ given real numbers,

(vii) $G = \{2^{-p} + 5^{-q} : p, q \in \mathbb{N}\}$,

(viii) $H = \{(-1)^n + 1/m : n, m \in \mathbb{N}\}$.