## C.6   Continuity

### Problem 6.1

(a) The function $g(x) = |x|$ is a continuous function and $|f(x)| = (g \circ f)(x)$ is continuous because the composition of continuous functions is a continuous function.

As for the reciprocal, take

$$f(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases}$$

It is clearly a discontinuous function, however $|f(x)| = 1$ everywhere, which is continuous. This example illustrates that from the fact that $|f(x)|$ is continuous one cannot conclude that $f(x)$ itself is continuous.

(b) We are talking here about a function $f : \mathbb{R} \mapsto \mathbb{Q}$ that is continuous. One such function would necessarily be constant. Let us see why. Suppose that $f(x_1) = q_1$ and $f(x_2) = q_2 \neq q_1$. Since the function is continuous it must take all intermediate values between $q_1$ and $q_2$ within the interval $[x_1, x_2]$. But between any two rational numbers there are infinitely many irrational numbers, so there must exist $x \in (x_1, x_2)$ such that $f(x)$ is irrational. This is a contradiction and therefore $q_2 \neq q_1$ is not possible.

### Problem 6.2

(a) The information that the function is surjective means that $x_0$ and $x_1$ in $[0, 1]$ such that $f(x_0) = 0$ and $f(x_1) = 1$. Now, consider the interval $[x_0, x_1]$ (or $[x_1, x_0]$, depending on which one is bigger). The function $g(x) = f(x) - x$ is continuous (the sum of two continuous functions) and satisfies

$$g(x_0) = -x_0, \qquad g(x_1) = 1 - x_1.$$

If $x_0 = 0$ then $c = 0$ is the point we are looking for. If $x_1 = 1$ then $c = 1$ is that point. If none of these two things happen then $g(x_0) < 0$ and $g(x_1) > 0$ and we can apply Bolzano's theorem: there must exist $c \in (0, 1)$ such that $g(c) = 0$ —which is equivalent to $f(c) = c$. Whichever the case, we can conclude that there exists $c \in [0, 1]$ such that $f(c) = c$.

(b) Consider the number

$$\mu = \frac{1}{n} \sum_{k=1}^{n} f(x_k).$$

We can obtain a lower bound to $\mu$ by replacing in this expression all the $f(x_k)$ by the smallest one. Thus,

$$\mu \geqslant \min_{k=1,\ldots,n} f(x_k).$$

Likewise, we can obtain an upper bound replacing them by the largest one:

(a) In this case the denominator must never vanish. If $\lambda = 0$ the function $f(x) = 1$ and trivially continuous in $\mathbb{R}$. Consider now $\lambda \neq 0$. Since in this case the denominator is a quadratic polinomial, the requirement that it never vanishes can be rephrase as its two roots being complex. The condition for that is that the discriminant is negative, so

$$4\lambda^2 - 4\lambda < 0 \quad \Leftrightarrow \quad \lambda(\lambda - 1) < 0.$$

This holds if each factor has a different sign, i.e., if $0 < \lambda < 1$. Therefore the function is continuous in $\mathbb{R}$ provided $\lambda \in [0, 1)$.

(b) Any of the values of $\lambda$ found in the previous item make the function continuous in $\mathbb{R}$ —hence also in $[0, 1]$—, so we just have to check what happens if $\lambda < 0$ or if $\lambda \geqslant 1$. In any of these two cases the denominator will have two real roots, so the key point is that none of them lies within the interval $[0, 1]$ where we want $f(x)$ to be continuous.
By solving the quadratic equation we find the two roots as

$$x_1 = \frac{\lambda + \sqrt{\lambda(\lambda - 1)}}{\lambda} = 1 + \sqrt{1 - \lambda^{-1}}, \qquad x_2 = \frac{\lambda - \sqrt{\lambda(\lambda - 1)}}{\lambda} = 1 - \sqrt{1 - \lambda^{-1}}.$$

If $\lambda = 1$ both $x_1 = x_2 = 1$ and so $f$ is not continuous at $x = 1$. Thus $\lambda \neq 1$ is required. In this case $x_1 > 1$, so it will always be outside the interval $[0, 1]$. We can ignore it. On the contrary, $x_2 < 1$, so it will be also ouside the interval provided $x_2 < 0$. This condition implies $\sqrt{1 - \lambda^{-1}} > 1$, which can only hold if $\lambda < 0$.
Summarising, $f(x)$ will be continuous in $[0, 1]$ provided $\lambda < 1$.

**Problem 6.4**

(i) Numerator and denominator are continuous functions in $\mathbb{R}$, so this function will be continuous except when the denominator vanishes. It does when $x^2 - 8x + 12 = (x - 6)(x - 2) = 0$, so $f$ is continuous in $\mathbb{R} - \{2, 6\}$.

(ii) The function is the sum of a plynomial (continuous in $\mathbb{R}$) and the function $e^{3/x}$. The exponential is continuous everywhere and the function $3/x$ too, except for $x = 0$. Besides,

$$\lim_{x \to 0^+} e^{3/x} = \infty,$$

so $f$ is continous in $\mathbb{R} - \{0\}$.

(iii) Polynomials are continuous in $\mathbb{R}$ and so the tangent except when its argument is an odd multiple of $\pi/2$. This means the points

$$3x + 2 = n\pi + \frac{\pi}{2} \quad \Rightarrow x = \frac{n\pi - 2}{3} + \frac{\pi}{6}, \quad n \in \mathbb{Z}.$$

$f$ is continuous except at these infinitely many points.

(iv) The polynomial is continuous in $\mathbb{R}$, so $f$ is continuous wherever the argument of the square root is not negative. This means $x^2 - 5x + 6 = (x - 3)(x - 2) \geqslant 0$, which happens for $x \geqslant 3$

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(viii) The polynomial and the sine function are both continuous everywhere, and so is $1/x$ except at $x = 0$. Function $f$ is defined at $x = 0$ though, so we must check the definition of continuity at this specific point. Since $|x^2 \sin(1/x)| \leqslant x^2$ and $x^2 \to 0$ as $x \to 0$, then

$$\lim_{x \to 0} f(x) = 0 = f(0)$$

and $f$ is continuous in $\mathbb{R}$.

(ix) For $x > 0$ the function is continuous except for $x = (2n-1)\pi/2$, $n \in \mathbb{N}$. For $x < 0$ the function is always continuous. We must compute the two one-sided limits at $x = 0$ to check for continuity at that point. Now,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\tan x}{\sqrt{x}} = \lim_{x \to 0^+} \frac{x}{\sqrt{x}} = \lim_{x \to 0^+} \sqrt{x} = 0.$$

And on the other side,

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0.$$

Thus,

$$\lim_{x \to 0} f(x) = 0 = f(0),$$

so $f$ is continuous in $\mathbb{R} - \{(2n-1)\pi/2 : n \in \mathbb{N}\}$.

(x) As close as we like to a rational number there is always an irrational number. As close as we like to an irrational number there is always a rational number. So, $f$ is discontinuous at every $x \neq 0$. At $x = 0$ function $f(x)$ is continuous though. The reason is that $|f(x)| = |x| \to 0$ as $x \to 0$, so

$$\lim_{x \to 0} f(x) = 0 = f(0).$$

(xi) Each piece of this piecwise function separately is a continuous function, so we just need to check what happens at the joints. Thus,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-1)^3 = 0, \qquad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (|x| - x) = 0,$$

so

$$\lim_{x \to 1} f(x) = 0 = f(1).$$

And

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} (|x| - x) = 2, \qquad \lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} \sin(\pi x) = 0,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-1\}$.

(xii) The two polynomials defining the function for $|x| > 1$ are continuous function. In $(-1, 1)$

And

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^+} (\operatorname{sgn} x + 1) = 0, \qquad \lim_{x\to-1^+} f(x) = \lim_{x\to-1^-} (x+1)^2 = 0,$$

so

$$\lim_{x\to-1} f(x) = 0 = f(-1).$$

Summarising, $f(x)$ is continuous in $\mathbb{R} - \{0\}$.

(xiii) Each of the three pieces of this piecewise function is continuous (a polynomial or the absolute value of a polynomial), so we need to check just the joints. Thus,

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (4x - 5) = 3, \qquad \lim_{x\to2^-} f(x) = \lim_{x\to2^-} |x^2 - 1| = 3,$$

so

$$\lim_{x\to2} f(x) = 3 = f(2).$$

And

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^+} |x^2 - 1| = 3, \qquad \lim_{x\to-2^+} f(x) = \lim_{x\to-2^-} x^2 = 4,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-2\}$.

(xiv) The functions defining $f(x)$ for $|x| > 1$ are both polynomials —hence continuous. Within $|x| \leqslant 1$ it is defined as $g(x) = x - \lfloor x \rfloor$. Now, $g(x) = x + 1$ for all $-1 \leqslant x < 0$, $g(x) = x$ for all $0 \leqslant x < 1$, and $g(1) = 0$. Thus function $f(x)$ can be redefined as

$$f(x) = \begin{cases} (x-1)^2, & x \geqslant 1, \\ x, & 0 \leqslant x < 1, \\ x+1, & x < 0. \end{cases}$$

All three pieces are continuous (polynomials), so we must look at the joints. So,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x-1)^2 = 0, \qquad \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1,$$

and

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x = 0, \qquad \lim_{x\to0^-} f(x) = \lim_{x\to0^-} (x+1) = 1.$$

Therefore the $f(x)$ is continuous in $\mathbb{R} - \{0, 1\}$.

**Problem 6.5**

(i) Denoting $f(x) = x^2 - 18x + 2$, a continuous function in $\mathbb{R}$, we have $f(-1) = 21$, $f(1) = -15$, so Bolzano's theorem guarantees at least one zero in $[-1, 1]$.

(vi) Denoting

$$f(x) = \frac{x^3}{4} - \sin(\pi x) + 3 - \frac{7}{3} = \frac{x^3}{4} - \sin(\pi x) + \frac{2}{3},$$

$f(-2) = -4/3$ and $f(2) = 8/3$, so Bolzano's theorem guarantees at least one zero in $[-2, 2]$.

(vii) Clearly $|\sin x| - \sin x \leqslant 2$, so the equation $|\sin x| - \sin x = 3$ cannot have any solution in $\mathbb{R}$.

**Problem 6.6** If $f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0$ then, as $x \to \pm\infty$ we have $f(x) \sim a_{2n+1}x^{2n+1}$. Therefore the signs of $f(x)$ for large positive $x$ and large negative $x$ are opposite, so we can apply Bolzano and conclude that $f(x)$ must be zero at least at one point in $\mathbb{R}$.