Final Exam

Mathematical Methods of Bioengineering
Ingenería Biomédica (INGLÉS)

14 of May 2019

The maximum time to make the exam is 3 hours. You are allowed to use a calculator and two sheets with annotations.

Problems

1. Consider the function $f(x,y,z) = (e^{xyz}, \tan yz, xy)$ and let $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a differentiable function that verifies $\nabla g(1,1,0) = (1,1,-1)$.

   (a) (**1 point**) If $F = g \circ f$, compute $\nabla F(0, \frac{\pi}{4}, 1)$.

   (b) (**1 point**) Find the tangent plane of $F = 0$ at $(0, \frac{\pi}{4}, 1)$.

SOLUTION

(a) Using the chain rule we are able to calculate the gradient without knowing the explicit expression of g:

$$\nabla F(x,y,z) = \nabla(g \circ f)_{(x,y,z)} = \nabla g[f(x,y,z)] \cdot D_f(x,y,z) =$$

The derivative or Jacobian of f is:

$$D_f(x,y,z) = J_f(x,y,z) = \begin{pmatrix} yz \cdot e^{xyz} & xz \cdot e^{xyz} & xy \cdot e^{xyz} \\ 0 & z(1+\tan^2(yz)) & y(1+\tan^2(yz)) \\ y & x & 0 \end{pmatrix}$$

Evaluating the expression at the given point we get:

$$D_f(0, \pi/4, 1) = \begin{pmatrix} \frac{\pi}{4} & 0 & 0 \\ 0 & 2 & \frac{\pi}{2} \\ \frac{\pi}{4} & 0 & 0 \end{pmatrix}$$

On the other hand, $f(0, \pi/4, 1)$=(1,1,0). Then, $\nabla g[f(x,y,z)] = \nabla g[(1,1,0)] = (1,1,-1)$ by the statement of the exercise. So finally plugging-in both result we conclude that:

$$\nabla F(0, \pi/4, 1) = \nabla g[f(0, \pi/4, 1)] \cdot D_f(0, \pi/4, 1) = (1,1,-1) \begin{pmatrix} \frac{\pi}{4} & 0 & 0 \\ 0 & 2 & \frac{\pi}{2} \\ \frac{\pi}{4} & 0 & 0 \end{pmatrix} = (0, 2, \pi/2)$$

(b) The implicit formula for the tangent plane of surface with equation $F(x, y, z) = 0$ at a point $P = (x_0, y_0, z_0)$ is

$$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Substituting for $P = (0, \pi/4, 1)$ we get

$$(0, 2, \pi/2) \cdot (x - 0, y - \pi/4, z - 1) = 0$$

Computing the dot product and simplifying, results in:

$$2y + \frac{\pi}{2} z = \pi$$

2. A laboratory that designs **nasogastric tubes** decides to model a tube of a *Levine catheter*. Assume that the thickness of the tube is one and that can be modelled with the parametric equations:

$$\mathbf{x}(t) = \left(\sqrt{2} \cdot t, e^t, e^{-t}\right), \quad 0 \le t \le 1 \ .$$

(a) (**1 point**) Compute the length of the catheter tube.

(b) (**1 point**) Suppose that through the tube is going to be administrated a medicament with density:

$$f(x, y, z) = \frac{1}{y^2 z^2}$$

Compute the total quantity of medicament that can be accumulated inside the catheter tube.

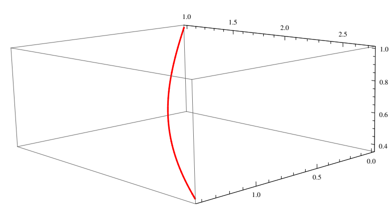

Figure 1: Catheter tube.

Note: consider $\sinh t := \frac{e^t - e^{-t}}{2}$, $\cosh t := \frac{e^t + e^{-t}}{2}$. *It may help you to know that* $\cosh^2 t = \frac{e^{2t} + e^{-2t} + 2}{4}$ and that $\frac{d(\sinh t)}{dt} = \cosh t$.

SOLUTION

(a) The length of the catheter can be computed using the formula for the length of a curve.

$$\begin{aligned}
\mathbf{x}(t) &= \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \\
\mathbf{x}'(t) &= \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \\
\|\mathbf{x}'(t)\| &= \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = 2\sqrt{\cosh^2 t} = 2\cosh t \\
L(\mathbf{x}) &= \int_0^1 \|\mathbf{x}'(t)\| dt = \int_0^1 2\cosh t \ dt = 2\sinh t|_0^1 = e^t - e^{-t}|_0^1 = e - \frac{1}{e} = 2.3504
\end{aligned}$$

(b) Integrating the density function over the catheter path we get the total mass of the medicament on the tube. Then,

$$\begin{aligned}
f(\mathbf{x}(t)) &= \frac{1}{(e^t)^2(e^{-t})^2} = 1 \\
\|\mathbf{x}'(t)\| &= \sqrt{2 + e^{2t} + e^{-2t}} = 2\cosh t \\
\int_{\mathbf{x}} f ds &= \int_0^1 f(\mathbf{x}(t))\|\mathbf{x}'(t)\| \; dt = \int_0^1 1 \cdot 2\cosh t dt = 2.3504
\end{aligned}$$

3. Consider the surface shown in figure 2 and the region $D_{r_0} = \{(x,y) : x^2 + y^2 \leq r_0^2\}$ where r_0 is a constant value.

   (a) (**1 point**) Find the volume under the surface over the region D_{r_0}.

   (b) (**1 point**) Find the volume under the surface over all $\mathbb{R}^2$.

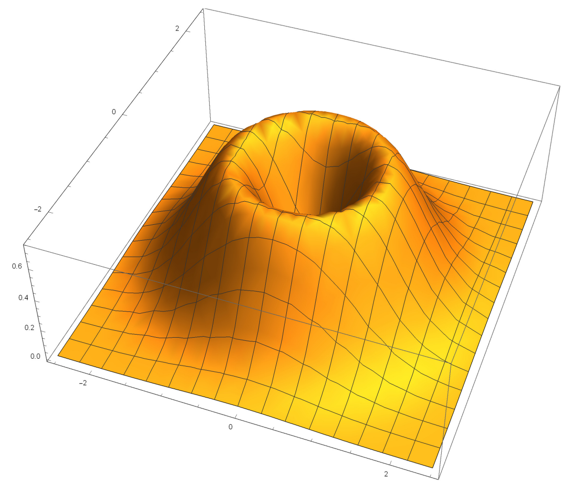

Figure 2: $f(x,y) = 2(x^2+y^2)e^{-x^2-y^2}$.

Observation: $\lim_{r_0 \to \infty} D_{r_0} = \mathbb{R}^2$.

Integration by parts: $\int u \cdot v' dx = u \cdot v - \int u' \cdot v \; dx$

(a) The volume under the surface is obtained integrating the surface:

$$V = \int\int_{D_{r_0}} f(x,y) dA = \int\int_{D_{r_0}} 2(x^2+y^2)e^{-x^2-y^2} dxdy$$

We observe that a change of variables to polar coordinates is necessary. Because D_{r_0} is a disk of radius r_0 can be described on polar coordinates as:

$$D_{r_0} = \{(r,\theta) : 0 \leq r \leq r_0, \; 0 \leq \theta \leq 2\pi\}$$

Using the change of variables theorem for double integrals,

$$\begin{aligned}
V &= \int\int_{D_{r_0}} f(x,y) dA = \int\int_{D_{r_0}} f(r,\theta) \cdot r \; dA = \int\int_{D_{r_0}} 2(r^2)e^{-r^2} r \; dr d\theta \\
&= \int_0^{2\pi}\int_0^{r_0} 2r^3 e^{-r^2} \; dr d\theta
\end{aligned}$$

To compute a integral of the type "$\int xe^x\ dx$" we integrate by parts,

$$\begin{array}{ll} u = r^2 & v = e^{-r^2} \\ du = 2r\ dr & dv = (-2r)e^{-r^2}dr \end{array}$$

$$\begin{aligned} V &= \int_0^{2\pi} \left(\int u \cdot v' dr \right) d\theta = \int_0^{2\pi} \left(u \cdot v - \int u' \cdot v\ dr \right) d\theta = \\ &= \int_0^{2\pi} \left(- [r^2 e^{-r^2}|_0^{r_0} - \int_0^{r_0} 2re^{r_0}\ dr] \right) d\theta = -2\pi \cdot [r_0^2 e^{-r_0^2} + e^{-r^2}|_0^{r_0}] = \\ &= -2\pi \cdot [r_0^2 e^{-r_0^2} + e^{-r_0^2} - 1] = 2\pi \cdot [1 - r_0^2 e^{-r_0^2} - e^{-r_0^2}] \end{aligned}$$

(b) When we tend to infinite the radius of a disk we cover all $\mathbb{R}^2$, $\lim_{r_0 \to \infty} D_{r_0} = \mathbb{R}^2$. So,

$$\int\int_{\mathbb{R}^2} f(x,y)dA = \lim_{r_0 \to \infty} \int\int_{D_{r_0}} f(x,y)dA = \lim_{r_0 \to \infty} 2\pi \cdot [1 - r_0^2 e^{-r_0^2} - e^{-r_0^2}] = 2\pi$$

4. Suppose that you go every day to work by subway. You walk to the same subway station, which is served by two subway lines, both stopping near where you work. Each subway line sends trains to arrive at the stop every 6 minutes, but the dispatchers (train drivers) begin the schedules at random times.

   (a) (**0.5 points**) Given the arrival time x minutes for the first line train, and y minutes for the second line train, which is the function $T(x,y)$ that give the waited time?

   (b) (**1 point**) What is the average time you expect to wait for a subway train?

   Note: You could model the waiting time for the two subway lines by using a point (x,y) in the square $[0,\ 6] \times [0,\ 6]$.

   Note: The average of function T over a region $D \subset \mathbb{R}^2$ is $T_{avg} = \dfrac{1}{A(D)} \int\int_D T(x,y)\ dxdy$.

SOLUTION

(a) We catch the first train to arrive. If the **first** line train arrives first, we wait x and if the **second** line train arrive first, we wait y. So the function T is:

$$T(x,y) = \min(x,y) = \begin{cases} x & if \quad x \leq y \\ \\ y & if \quad x > y \end{cases}$$

A representation of the function is shown in 3.

(b) In order to find the average time we use the formula for the average of a function, $T_{avg} = \dfrac{1}{V(D)} \int\int_D T(x,y)\ dxdy$. In our case, D is the square region $[0,\ 6] \times [0,\ 6]$. Then, the area is $A(D) = 6 \cdot 6$. Finally,

$$T_{avg} = \frac{1}{36} \int_0^6 \int_0^6 min(x,y)\ dxdy = \frac{1}{36} \int_0^6 \int_x^6 x\ dydx + \frac{1}{36} \int_0^6 \int_0^x y\ dydx = ... = 2$$

So the expected waited time is 2 minutes.

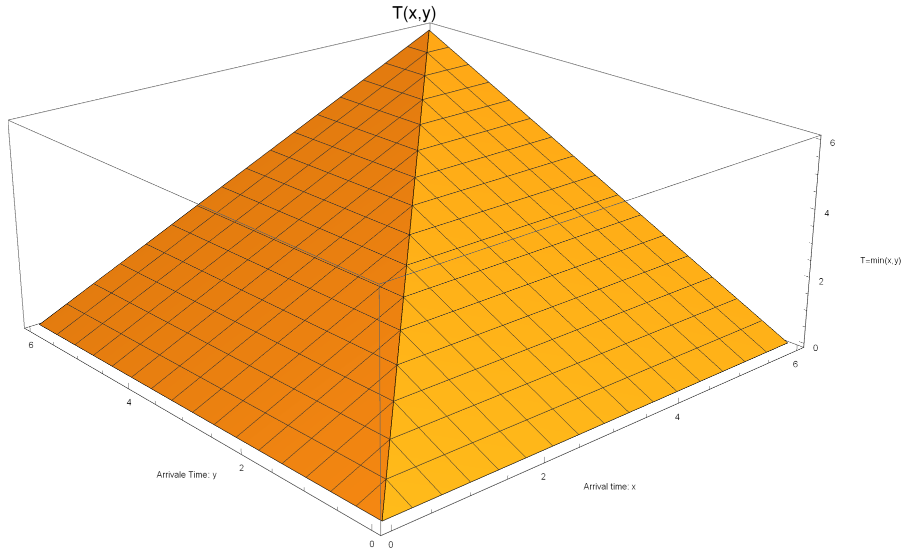

Figure 3: $T(x, y)$.

5. Consider the force field given by $\mathbf{F}(x, y) = (x + y, x - y)$.

   (a) (**1.5 points**) Prove that $\mathbf{F}$ is conservative and compute its potential function $f(x, y)$.

   (b) (**1 points**) Compute the line integral of $\mathbf{F}$ over the parabola $y = x^2$ with $x \in [0, 1]$.

   (c) (**0.5 points**) Compute the line integral of $\mathbf{F}$ over the path delimited by the parabola $y = x^2$ and the line $y = 1$ with counterclockwise orientation.

SOLUTION

(a) $\mathbf{F}$ is a vector field of class C^1 and $\mathbb{R}^2$ is a simply connected region. So, $\mathbf{F} = \nabla f \iff \nabla \times \mathbf{F} = 0$.

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$
$$1 = 1$$

Now we compute $f(x, y)$.

$$(F_1, F_2) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x} = x + y \longrightarrow f(x, y) = \int x + y \, dx = \frac{x^2}{2} + yx + h(y)$$

Taking the partial of $\left(-\frac{x^2}{2} + yx + h(y)\right)$ respect of y we get the equation

$$F_2 = \frac{\partial\left(\frac{x^2}{2} + yx + h(y)\right)}{\partial y}$$

$$x - y = x + h'(y) \longrightarrow h(y) = \int -y\ dy = -\frac{y^2}{2} + C$$

Then, $f(x,y) = \frac{x^2}{2} - \frac{y^2}{2} + yx + C$, C constant.

(b) Because $\mathbf{F}$ is conservative $\int_c \mathbf{F}\ d\mathbf{s} = f(B) - f(A)$. Using that, $A = (0, 0^2),\ B = (1, 1^2)$ we get that $\int_c \mathbf{F}\ d\mathbf{s} = f(1,1) - f(0,0) = {}^{1^2}/_2 - {}^{1^2}/_2 + 1 \cdot 1 + C - (0 - 0 + 0 + C) = 1$

(c) The path is closed as we can see in figure 4. We know that $\oint_c \mathbf{F}\ d\mathbf{s} = 0$ for any closed path.

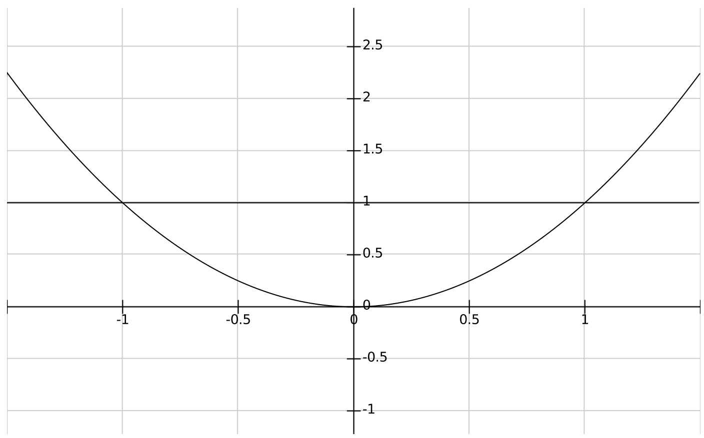

Figure 4: Path from c).