Partial Exam

Mathematical Methods of Bioengineering
Ingenería Biomédica

20 of March 2019

The maximum time to make the exam is 2 hours. You are allowed to use a calculator and two sheets with annotations.

Problems

1. (**2 points**) Find the equation of a plane that contains the line $l(t) = (-1, 1, 2) + t(3, 2, 4)$ and is perpendicular to the plane $2x + y - 3z + 4 = 0$.

   Note: Two planes are perpendicular when their normal vector are.

   SOLUTION

   Let π be our plane. Because π is perpendicular to the plane $2x + y - 3z + 4 = 0$, his normal $(2, 1, -3)$ must lie on π. Because the line $l(t)$ is on the plane his director vector must lie on π. Then we have that $(2, 1, -3)$ and $(3, 2, 4)$ lies on the plane so $n = (2, 1, -3) \times (3, 2, 4) = (10, -17, 1)$ is a normal vector of the plane π. Looking at $l(t)$ a point on the plane is $(-1, 1, 2)$. So π is

   $$10(x + 1) - 17(y - 1) + (z - 2) = 0$$

   or

   $$10x - 17y + z + 25 = 0$$

2. The three-dimensional **heat equation** is the partial differential equation

   $$k\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\Big) = \frac{\partial T}{\partial t}$$

   (a) (**1 point**) First we examine a simplified version of the heat equation. Consider a straight wire modelled by x. Then the temperature $T(x, t)$ at time t and position x along the wire is modelled by the one-dimensional heat equation

   $$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

   Show that the function $T(x, t) = e^{-kt}\cos x$ satisfies this equation. What happens to the temperature of the wire after a long period of time?

   (b) (**1 point**) Now show that $T(x, y, z, t) = e^{-kt}(\cos x + \cos y + \cos z)$ satisfies the three-dimensional heat equation.

   SOLUTION

(a)
- $\frac{\partial T}{\partial x} = -e^{-kt}\sin x \Longrightarrow \frac{\partial^2 T}{\partial x^2} = -e^{-kt}\cos x$
- $\frac{\partial T}{\partial t} = -ke^{-kt}\cos x$

Then, $k\frac{\partial^2 T}{\partial x^2} = k(-e^{-kt}\cos x) = \frac{\partial T}{\partial t}$.
After a long period of time, the temperatures goes to zero because the negative exponential function of the time.

(b)
- $\frac{\partial^2 T}{\partial x^2} = -e^{-kt}\cos x$
- $\frac{\partial^2 T}{\partial y^2} = -e^{-kt}\cos y$
- $\frac{\partial^2 T}{\partial z^2} = -e^{-kt}\cos z$
- $\frac{\partial T}{\partial t} = -ke^{-kt}(\cos x + \cos y + \cos z)$

Then, $k(-e^{-kt}\cos x - e^{-kt}\cos y - e^{-kt}\cos z) = -ke^{-kt}(\cos x + \cos y + \cos z)$

3. A bioinvestigation laboratory works with cells whose surface are represented in the next figure.

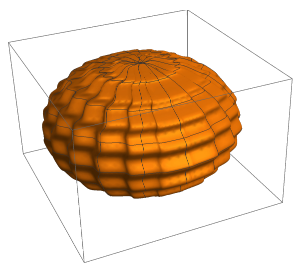

Figure 1: Cell.

While proceeding with the experiment, an unexpected incident disturbs the pressure in the essay area. The pressure at each point of the space is now given by the function

$$T(x, y, z) = xy + xz + yz$$

(a) (**2 points**) Suppose you can model the surface of each of the cells with the following equation:

$$\mathbf{x}(s,t) \equiv \begin{cases} x(s,t) = 1.5\sin s\sin t + 0.05\cos 20t \\ y(s,t) = 1.5\cos s\sin t + 0.05\cos 20s, \\ z(s,t) = \cos t \end{cases} \qquad t, s \in [-\pi, \pi]$$

Compute the variation of the pressure on the surface when $s = \frac{\pi}{2}$ and $t = \frac{\pi}{2}$.

(b) (**1 point**) Suppose a microorganism is at the point $(-1, 0, 0)$. In which direction should the cell move in order to keep pressure constant? Explain your answer.

SOLUTION

(a) Using the chain rule

- On one hand,

$$\frac{\partial T}{\partial s} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial s}$$
$$= (y+z)\cdot 1.5\cos s\sin t + (x+z)\cdot(-1.5\sin s\sin t - \sin 20s) + (x+y)\cdot 0$$
$$= (1.5\cos s\sin t + 0.05\cos 20s + \cos t)\cdot(1.5\cos s\sin t) +$$
$$+(1.5\sin s\sin t + 0.05\cos 20t + \cos t)\cdot(-1.5\sin s\sin t - \sin 20s)$$

- On the other hand,

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial t}$$
$$= (y+z)\cdot(1.5\cos t\sin s - \sin 20t) + (x+z)\cdot(1.5\cos s\cos t) + (x+y)\cdot(-\sin t)$$
$$= (1.5\cos s\sin t + 0.05\cos 20s + \cos t)\cdot(1.5\cos t\sin s - \sin 20t)$$
$$+(1.5\sin s\sin t + 0.05\cos 20t + \cos t)\cdot(1.5\cos s\cos t)$$
$$+(1.5\sin s\sin t + 0.05\cos 20t + 1.5\cos s\sin t + 0.05\cos 20s)\cdot(-\sin t)$$

Using that $\cos\pi/2 = \sin(20\pi/2) = 0$, $\sin\pi/2 = \cos(20\pi/2) = 1$ we have,

$$\left.\frac{\partial T}{\partial s}\right|_{s=\pi/2,t=\pi/2} = -2.325$$
$$\left.\frac{\partial T}{\partial t}\right|_{s=\pi/2,t=\pi/2} = -1.6$$

(b) We compute the gradient of the temperature function at the point

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = (y+z, x+z, x+y)$$
$$\nabla T(-1,0,0) = (0,-1,-1)$$

Then we are looking for a direction $\mathbf{u}$ such that

$$D_{\mathbf{u}}T(-1,0,0) = \nabla T(-1,0,0)\cdot\mathbf{u} = (0,-1,-1)\cdot\mathbf{u} = 0$$

This defines a plane of directions $\pi : \{(u_1,u_2,u_3) \in \mathbb{R}^3 : u_2 + u_3 = 0\}$. Then any direction contained in the plain works, for example, $(0,-1,1)$ or $(0,1,-1)$.

4. A laboratory is working in a **nanotechnology** experiment that is trying to model a new prototype of carbon nanotube as shown in figure 2. The surface in nanometers (nm) is given by the equation

$$z = 2(x^2+y^2)e^{-x^2-y^2}$$

(a) (**1 point**) Find the critical points of the nanotube.

(b) (**1 point**) Is the origin a minimum/maximum? Explain your answer.

(c) (**1 point**) Write the equation in cylindrical coordinates. Which variables appear on the equation? Does the equation represent the same figure when $\theta = 0$ and $\theta = \pi/2$?

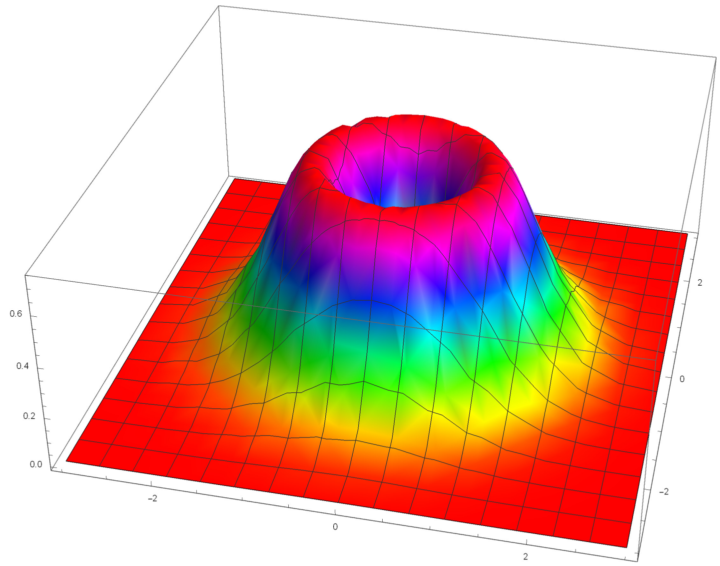

Figure 2: Representation of the prototype.

SOLUTION

(a) We need to find the points where $\nabla f(x,y) = (0,0)$. So,

$$\nabla f(x,y) = 2\Big(2xe^{-x^2-y^2} - 2xe^{-x^2-y^2}(x^2+y^2), 2ye^{-x^2-y^2} - 2ye^{-x^2-y^2}(x^2+y^2)\Big)$$
$$\nabla f(x,y) = 2e^{-x^2-y^2}\Big(2x - 2x(x^2+y^2), 2y - 2y(x^2+y^2)\Big)$$
$$\nabla f(x,y) = 4e^{-x^2-y^2}\Big(x(1-x^2-y^2), y(1-x^2-y^2)\Big)$$

Then, $\nabla f(x,y) = (0,0) \iff (x,y) = (0,0)$ or $x^2+y^2 = 1$. In set notation,

$$C = \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$$

(b) We compute the Hessian matrix in order to see if the origin is an extrema.

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} =$$
$$4e^{-x^2-y^2}\begin{bmatrix} -2x^2(1-x^2-y^2) + (1-3x^2-y^2) & -2xy(2-x^2-y^2) \\ -2xy(2-x^2-y^2) & -2y^2(1-x^2-y^2) + (1-x^2-3y^2) \end{bmatrix}$$

At the origin we get,

$$H_f(0,0) = 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq 0$$

The matrix es positive definite so the origin is a **minimum**.

(c) Using that $r^2 = x^2+y^2$, tthe equation is written as

$$z = 2r^2e^{-r^2}$$

The variables who appear in the equation are z and r. The equation doesn't depend on θ so then for any fixed angle the figure will be the same.