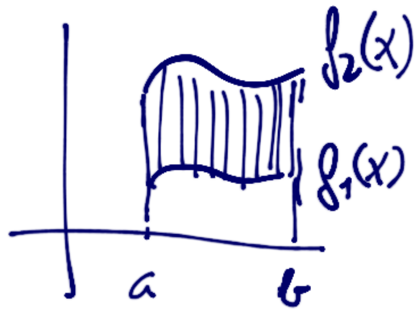
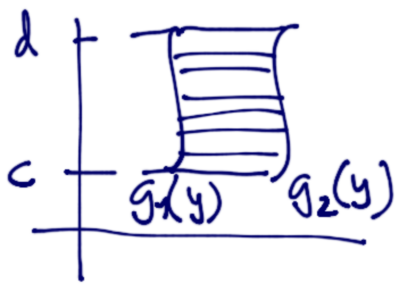
# Integration in non-rectangular regions

In 2D

$\boxed{\text{Type I}}$



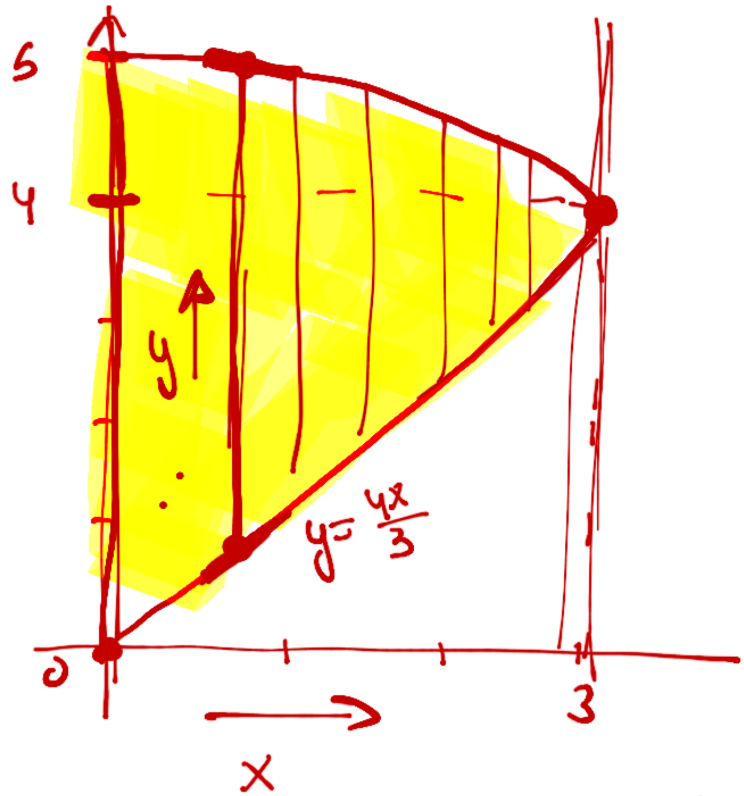

$\boxed{\text{Type II}}$




$\boxed{\text{Type 3}}$ Both of them.

Problem 5 – 3.1

i) $$\int_0^3 \int^{\sqrt{25-x^2}} f(x,y)\,dy\,dx$$



$$y = \frac{4x}{3}$$

$$y = \sqrt{25-x^2}$$

If $x=3$ $y=4$

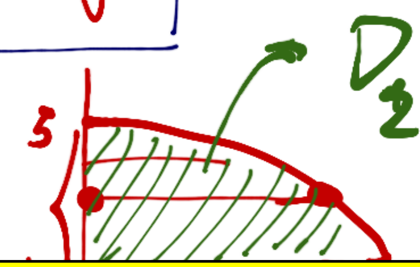
If $x=0$, $y=5$

Change the order of integration $\left[ \frac{4x}{3} \le y \le \sqrt{25-x^2} \right]$

$$\frac{4x}{3} \le y \Rightarrow \boxed{x \le \frac{3y}{4}}$$

$$0 \le y \le \sqrt{25-x^2} \Rightarrow \boxed{x \le \sqrt{25-y^2}}$$

$x \ge 0$



$0 \le x \le \sqrt{25-y^2}$

New region of integration.

$$D = D_1 \cup D_2 = \underbrace{\{(x,y) \in \mathbb{R}^2,\ 0 \le y \le 4,\ 0 \le x \le \tfrac{3y}{4}\}}_{D_1}$$

$$\cup \underbrace{\{(x,y) \in \mathbb{R}^2,\ 4 \le y \le 5,\ 0 \le x \le \sqrt{25-y^2}\}}_{D_2}$$

$$\int_D f(x,y)\,dy\,dx = \int_{D_1} f(x,y)\,dx\,dy + \int_{D_2} f(x,y)\,dx\,dy.$$

## Triple Integrals

$\boxed{\text{Type 1}}$ $\quad a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)$

and $\phi_1(x,y) \le z \le \phi_2(x,y)$



and $\phi_1(x,y) \le z \le \phi_2(x,y)$

$\boxed{\text{Type 2}}$ any other combination.

## Change of variables in multiple integration

In linear Algebra you have several transformations

rotations, $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \to T(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix} = A\bar{x}$$

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

reflections, projections, ...

We use those transformations to simplify the

Remember what happens in 1D

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} \widehat{f(g(t))} \cdot \widehat{g'(t)}\,dt$$

$$\boxed{x = g(t)}$$

$$\boxed{dx = g'(t)\,dt}$$

If $x = a = g(t)$

$\Downarrow$

$t_0 = g^{-1}(a)$

$t_1 = g^{-1}(b)$

The composition in several variables is similar

$f(g(t)) \sim f(x,y)$ with $x = g_1(s,t)$

$y = g_2(s,t)$

$f(g_1(s,t), g_2(s,t))$

Definition

Let $D^*$ be a subset in $\mathbb{R}^N$ and

$T : D^* \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ a transformation of class $C^1$

We define the Jacobian of the transformation $T$ as

$$JT = \det(DT) = \det \begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \cdots & \frac{\partial T_1}{\partial x_N} \\ - & - & - & - \\ \frac{\partial T_N}{\partial x_1} & \frac{\partial T_N}{\partial x_2} & \cdots & \frac{\partial T_N}{\partial x_N} \end{pmatrix}$$

In particular in $\mathbb{R}^2$

$T : D^* \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T \in C^1$



$\frac{\partial}{\partial u}$ $\frac{\partial}{\partial v}$

## Important transformations

In $\mathbb{R}^2$ - Polar coordinates.

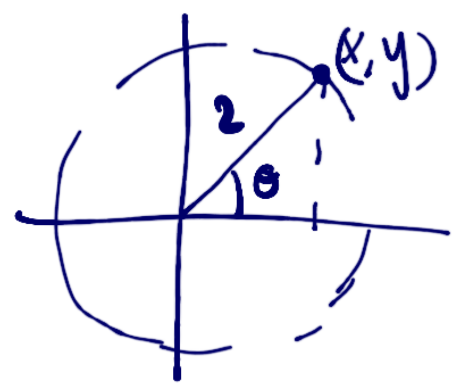
Assume a transformation

$$T : D^* \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(r,\theta) \longrightarrow T(r,\theta) = (x,y)$$

$$= (r\cos\theta, r\sin\theta)$$

$$\boxed{\left.\begin{array}{l} x = \overbrace{r\cos\theta}^{x(r,\theta)} \\ y = \underbrace{r\sin\theta}_{y(r,\theta)} \end{array}\right\} : T}$$




Inverse transformation $T^{-1}$

$$JT = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r > 0$$

if $r \neq 0$

and

$$D^* = \{(r,\theta) \in \mathbb{R}^2 \,,\; r \geq 0 \,,\; 0 \leq \theta \leq 2\pi\}$$

Since $JT \neq 0$ we can guarantee the existence of the inverse transformation

In $\mathbb{R}^3$ – Cylindrical coordinates.

$(x,y,z) \in \mathbb{R}^3$ $\boxed{x = r\cos\theta\,,}$

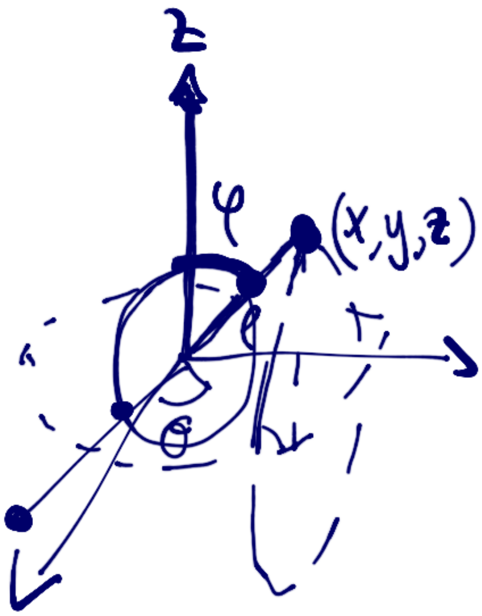
$$D^* = \{(r,\theta,z) \in \mathbb{R}^3,\ r \geq 0,\ z \in \mathbb{R},\ 0 \leq \theta \leq 2\pi\}$$

$$JT = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix}$$

$$= \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r > 0$$

# Spherical coordinates.

$(x,y,z)\in\mathbb{R}^3$




$$\left.\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}\right\} : T$$

$$JT = -\rho^2\sin\varphi \neq 0$$

we cannot duplicate points.

$$D^* = \{(\rho,\theta,\varphi)\ ;\ \rho\geq 0,\ 0\leq\theta\leq 2\pi,\ 0\leq\varphi\leq\pi\}$$

## Theorem of change of variables

Let $D^*$ and $D$ be two regions in $\mathbb{R}^N$

and $T: D^* \to D$ a transformation.

$$\begin{cases} \text{bijective. and of class } C^1 \\ \text{so that } JT \neq 0 \end{cases}$$

Then for any integrable function $f$

$$\int_D f(x_1, x_2, \dots x_N)\, dx_1 \dots dx_N = \int_{D^*} \overbrace{f(T(u_1 \dots u_N))}^{(f\circ T)(u_1 \dots u_N)} \underbrace{|JT|}_{\substack{\text{absolute}\\ \text{value of}\\ \text{the } JT}} du_1 \dots du_N$$

Remark

$T^{-1}$ denotes the inverse transformation of $T$

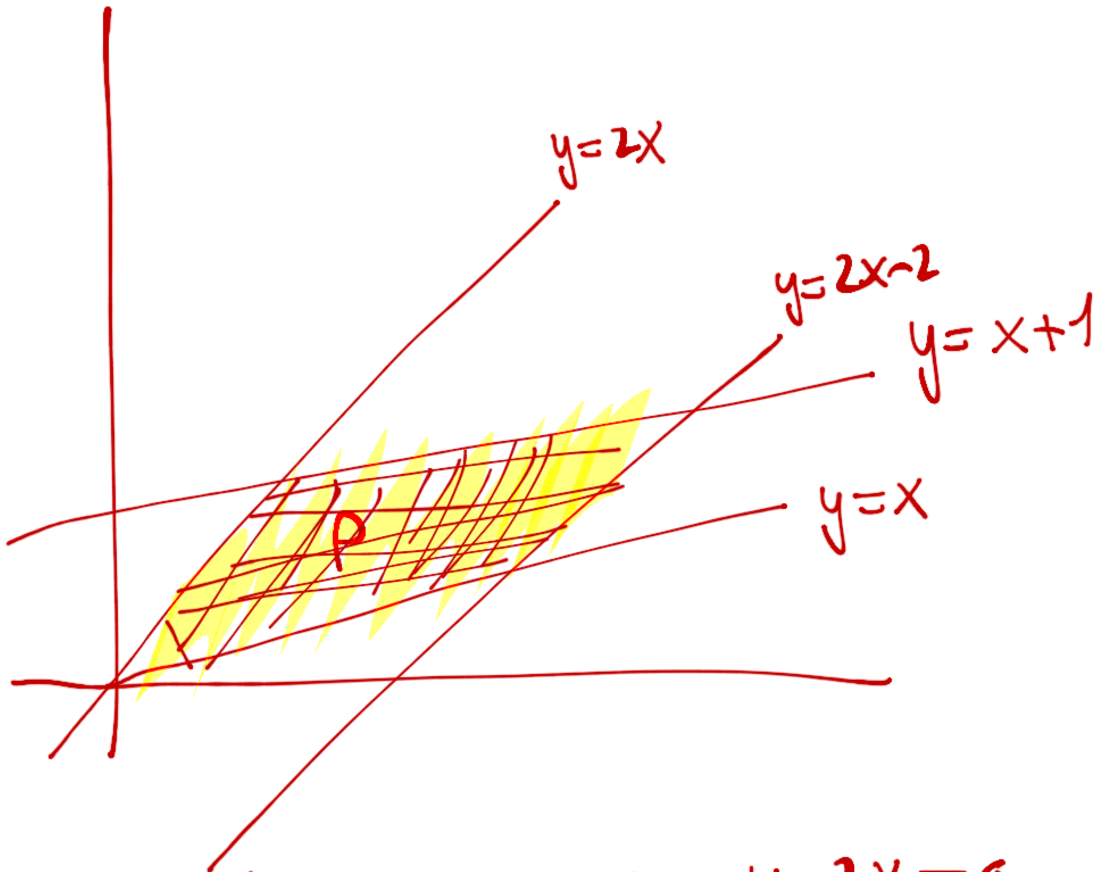
$T$ is onto and one-to-one, and of class $C^1$

$$JT \neq 0$$

Then $$(JT)\cdot(JT^{-1}) = 1$$

Example: Consider the parallelogram bounded by

$$P : \begin{cases} y = 2x & y = 2x-2 \\ y = x & y = x+1 \end{cases}$$




$$\begin{cases} y=2x \Rightarrow \underline{y-2x}=0 \\ y=2x-2 \Rightarrow \underline{y-2x}=-2 \end{cases}$$

$$\begin{cases} y=x \Rightarrow \underline{y-x}=0 \\ y=x+1 \Rightarrow \underline{y-x}=1 \end{cases}$$



$u = y-x$ $\quad 0 \leq u \leq 1$

$$u = y - 2x \qquad v = y - x \qquad \left\{ \begin{array}{l} y = v + x \\ u = v + x - 2x \end{array} \right.$$

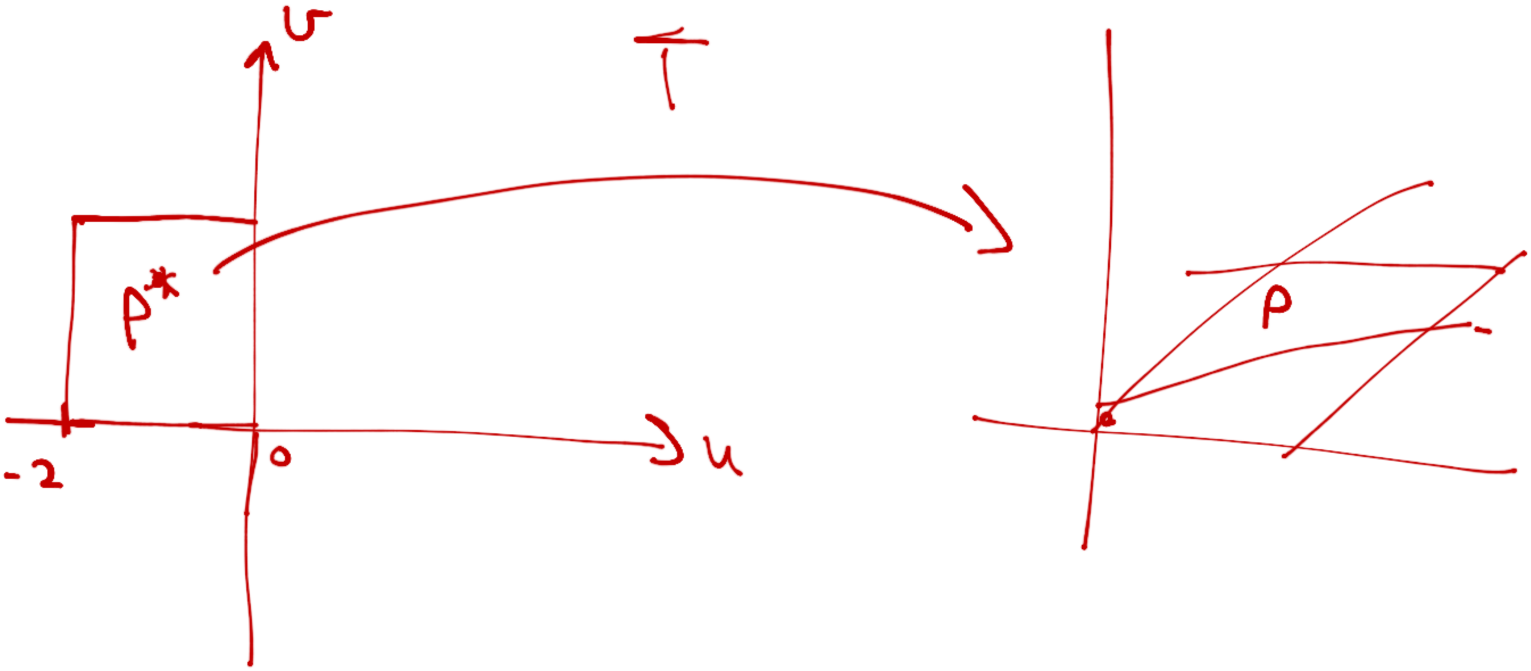
$$u - v = -x$$

$$\left\{ \begin{array}{l} \boxed{x = v - u} \\ \boxed{y = 2v - u} \end{array} \right. : T$$

$$JT = \det \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = -2 + 1 = -1$$

So that

$$\int_P xy \, dx\,dy = \int_{P^*} (v-u)(2v-u) \cdot \overbrace{|JT|}^{=1} \, du\,dv$$




$$\int_P xy\,dx\,dy = \int_0^1 \int_{-2}^{0} (2v^2 + u^2 - 3uv)\,du\,dv$$

$$= \int_0^1 \left[\left(2v^2 u + \frac{u^3}{3} - 3\frac{u^2}{2} v\right)\right]_{\boxed{-2}}^{0} dv$$

# Applications of the integral

- **Area of a region $D$**

$$A = \iint_D dx\,dy$$

- **Volume of $Q$**

$$V = \iiint_Q dx\,dy\,dz$$

in general we call the measure of a region $D$

$$|D| = \int_D dx_1 \cdots dx_n$$

- **Mean value for a bounded region $D$ and $f$**

The mean value is an extension of what happens for the discrete case.

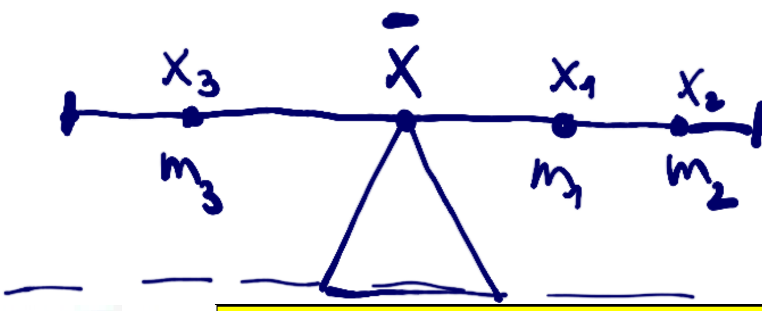
Assume a number values.

$x_1 \dots x_n$

then the mean value will be

$$\frac{x_1 + \dots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

If we add a mass at each point $x_i$




Balance of a rod.

$\bar{x}$ = mean value.

Newton's principle of balance

$$\sum_{i=1}^{n} m_i (x_i - \bar{x}) = 0$$

$$\Downarrow$$

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad \underline{\text{centre of mass}}$$

In $\mathbb{R}^3$ and with a continuous density. $\rho(x,y,z)$

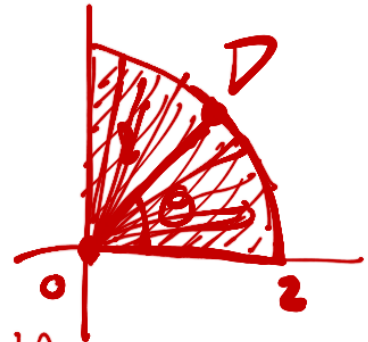
centre of

$$\bar{x} = \frac{1}{M} \iiint_D x \, \rho(x,y,z) \, dx\,dy\,dz$$

Total mass,

$$M \quad \iiint_D$$

## Problem 6 - 3.3




Mass for $D: \{x^2+y^2 \leq 4\}$ $x,y \geq 0$

density proportional to distance to the origin

$$\rho(x,y) = K r = K\sqrt{x^2+y^2} \quad K \equiv \text{constant.}$$

$$M = \iint_D \rho(x,y)\,dx\,dy = \iint_D K\sqrt{x^2+y^2}\,dx\,dy$$

use polar coordinates change of variable

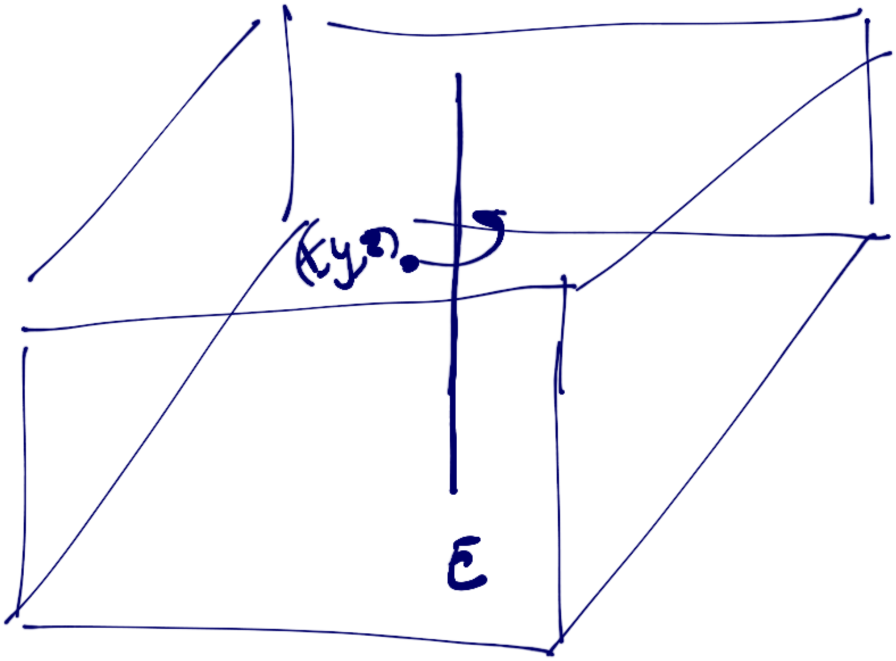
$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} ; T \qquad J_T = r$$

$$M = \iint_D \rho(x,y)\,dx\,dy = \int_0^2 \int_0^{\pi/2} \underbrace{kr}_{\sqrt{x^2+y^2}=r \;\; k\sqrt{x^2+y^2}} \cdot \underbrace{r}_{|J T|}\, d\theta\, dr$$

$$= k\left(\int_0^{\pi/2} d\theta\right)\left(\int_0^2 r^2\,dr\right) = \frac{k\pi}{2}\left.\frac{r^3}{3}\right]_0^2 = \frac{k\pi 8}{6} = \frac{4k\pi}{3}$$

## Moment of inertia

It measures how a body reacts when turning around an axis. (in 3D)




dist((x,y,z), E) ≡ distance from a point $(x,y,z)$ in D to the axis E

If $E = X$ dist((x,y,z), X) = $y^2+z^2$

If $E = Y$ dist((x,y,z) Y) = $x^2+z^2$

If $E = Z$ dist((x,y,z) Z) = $x^2+y^2$