Constrained extrema

we look for max and min of a function

$$f:\mathbb{R}^N \longrightarrow \mathbb{R}.$$

in particular $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) \quad \text{or} \quad f(x_1, \dots x_N)$$

constrained to $g(x,y)=0$

$$f\big|_{\{g(x,y)=0\}}$$

Graphically highest and lowest of the surface

We might use two methods:

a) Intersection with vertical planes

b) Intersection with horizontal planes (level curves)

Analytically, this equivalent to

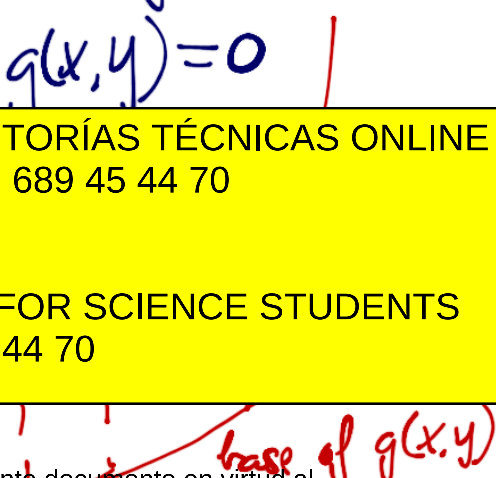
a) Reduce the problem to one variable.

b) Lagrange Multipliers method.

Vertical planes:

Surface $z=f(x,y)$ we cut it by a plane

cylinder with base $g(x,y)=0$



Example: $f(x,y) = x^2 + y^2$

Find the minimum of f constrained to

$$g(x,y) = x + y - 1 = 0$$

Min of $f\big|_{g(x,y)=0}$

We reduce the problem to one variable

and $g(x,y) = \underbrace{x+y-1}_{\text{line in } \mathbb{R}^2} = 0 \implies y = 1 - x$

Substituting into $f(x,y)$

$$h'(x) = 4x - 2 = 0 \Rightarrow x = \frac{1}{2}$$

$$y = 1 - x \Rightarrow y = \frac{1}{2}$$

$h''(x) = 4 > 0 \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ minimum

for $f(x,y)$ constrained to $g(x,y)=0$

and the value is

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Remark

We might have problems finding an explicit

Intersection with level curves

Looking for maximum and minimum using level curves

$$f(x,y)=c\ ,\quad c\in\mathbb{R}.$$

we will move such level curves along the z-axis.

Since we are constrained to

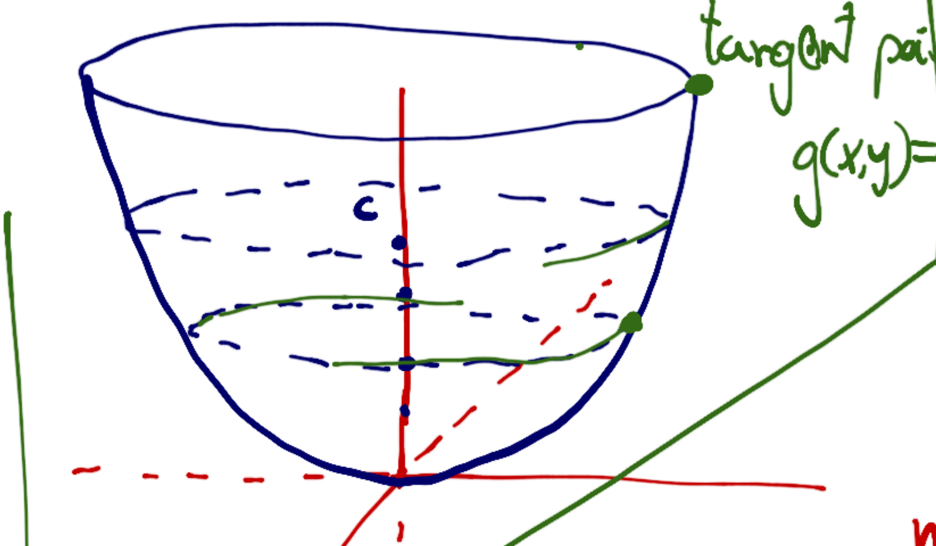
$$g(x,y)=0$$

we must find the intersection of $g(x,y)$ with

Example: $f(x,y) = x^2+y^2$
$g(x,y) = x+y-1=0$ } Max and min.

Level curves of $f(x,y)$

$$f(x,y) = \underbrace{x^2+y^2}_{\text{circumferences of radius } \sqrt{c} \text{ and centered at the origin}} = c$$



tangent point between f and g.
$g(x,y)=0$

min point of f

Lagrange Multiplier method is base on level curves.

Indeed, using level curves we are looking for the points (x,y) where the level curves of the surface

$$z = f(x,y)$$

are tangent to the constrained

$$g(x,y) = 0$$

We know that two curves are tangent if the orthogonal vectors are parallel.

$\lambda \in \mathbb{R}$, λ = Lagrange multiplier.

Theorem - Lagrange Multiplier

Let $A \subset \mathbb{R}^n$ be open set in $\mathbb{R}^n$ and

$$f, g : A \subset \mathbb{R}^n \to \mathbb{R}$$

two scalar functions of class C^1 in A

Let $x_0 \in A$ and $g(x_0) = c$

$S \equiv$ level curve of g at the value c

$$S = \{ x \in \mathbb{R}^n ; \; g(x) = c \}$$

If we consider f constrained to S

f

$f|_S$

Then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

and x_0 is a critical point.

Remark

Lagrangian $\equiv$ $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

look for local extrema there.

Example: $f(x,y) = x^2 + y^2$

$\{ g(x,y) = x + y - 1 = 0 \} = S$

Applying Lagrange Multiplier, we must solve the following system:

$$\left.\begin{array}{l}\nabla f(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=0\end{array}\right\}$$

$$\Downarrow$$

$$\underbrace{\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)}_{\nabla f(x,y)}=\lambda\underbrace{\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right)}_{\nabla g(x,y)}$$

$$(2x,2y)=\lambda(1,1)\Big(\Rightarrow\left.\begin{array}{l}2x=\lambda\\ 2y=\lambda\end{array}\right\}$$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \end{aligned}\right\} \Rightarrow x = y \quad \text{then} \quad x+y-1=0$$

$$\Downarrow$$

$$2x-1=0$$

$$x = \frac{1}{2} = y.$$

we arrive at only one critical point

$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$ minimum.

Global extrema

We find extrema in:

- The whole space $\mathbb{R}^n$

To guarantee the existence of extrema we need the following result

Theorem of Weierstrass

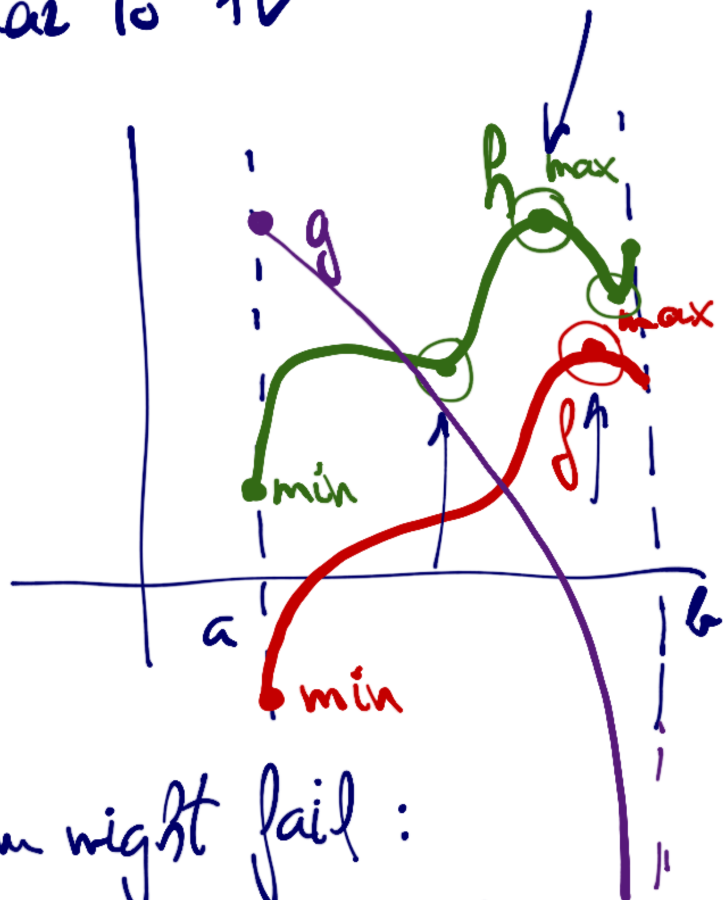
Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous scalar function in $K \subset \mathbb{R}^N$, K compact set.

Then there exist x_m and x_M in K such that for any $x \in K$

$$f(x_m) \leq f(x) \leq f(x_M)$$

Remark

Similar to 1D



$[a,b] = K$ compact set

g has a max in $[a,b]$ but not a min because g is not cont. in $[a,b]$.

Theorem might fail:

- If the function is not cont.
- If K is not compact.

mun.

Process to find extrema

a) Find critical points in the interior of A

$$\nabla f(x_1, \ldots x_N) = 0$$

b) Find critical points on the boundary

$$f\big|_{\partial A} \Rightarrow \text{Lagrange multiplier / vertical planes.}$$

c) Points where the function is not diff.

min (global)

Example: Find the extrema of the function

$$f(x,y,z) = x+y+z$$

in the ellipsoid $\underbrace{x^2+2y^2+3z^2}_{g(x,y,z)} \leq 1$

First, we observe that

- f is continuous

  and

- $V = \{(x,y,z) \in \mathbb{R}^3,\ x^2+2y^2+3z^2 \leq 1\}$
  is a compact set.

Thanks to Weierstrass Th. f will have a

In the interior:

$$\nabla f(x,y,z) = (1,1,1) \neq (0,0,0)$$

There is no critical point in the interior.

So that, there are no extrema in the interior.

On the boundary:

We apply Lagrange Multiplier.

$$\left.\begin{array}{l} \nabla f(x,y,z) = \lambda\, g(x,y,z) \\ g(x,y,z) = 0 \end{array}\right\}$$

Non-linear system

$$\left.\begin{array}{l} 1 = 2\lambda x \\ 1 = 4\lambda y \\ 1 = 6\lambda z \\ x^2 + 2y^2 + 3z^2 = 1 \end{array}\right\}$$

Obviously $\lambda \neq 0$. so that

$$x = \frac{1}{2\lambda}, \quad y = \frac{1}{4\lambda}, \quad z = \frac{1}{6\lambda}$$

substituting into the last equation.

$$\frac{1}{\ldots} + \frac{2}{\ldots} + \frac{3}{\ldots} = 1$$

We find two points:

$$M_\pm = \left(\pm\frac{1}{2}\sqrt{\frac{24}{11}}\,,\ \pm\frac{1}{4}\sqrt{\frac{24}{11}}\,,\ \pm\frac{1}{6}\sqrt{\frac{24}{11}}\right)$$

$$\underline{f(M_\pm) = \pm\frac{11}{12}\sqrt{\frac{24}{11}}} \quad \begin{cases} M_+ \ \text{max} \\ M_- \ \text{min.} \end{cases}$$

f is also differentiable so that

$$\boxed{\begin{array}{l} M_+ \ \text{global max} \\ M_- \ \text{global min} \end{array} \quad \text{for } f|_V}$$