Derivatives: Partial derivatives

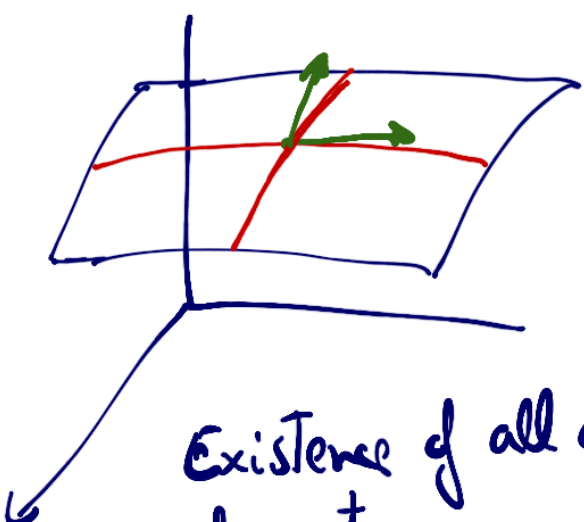
If $f(x,y)$, $\underbrace{\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}}$

ratio of change on the direction of the axis.

Directional derivatives:

$D_v f(x,y)$, v = vector.

ratio of change on the direction v.

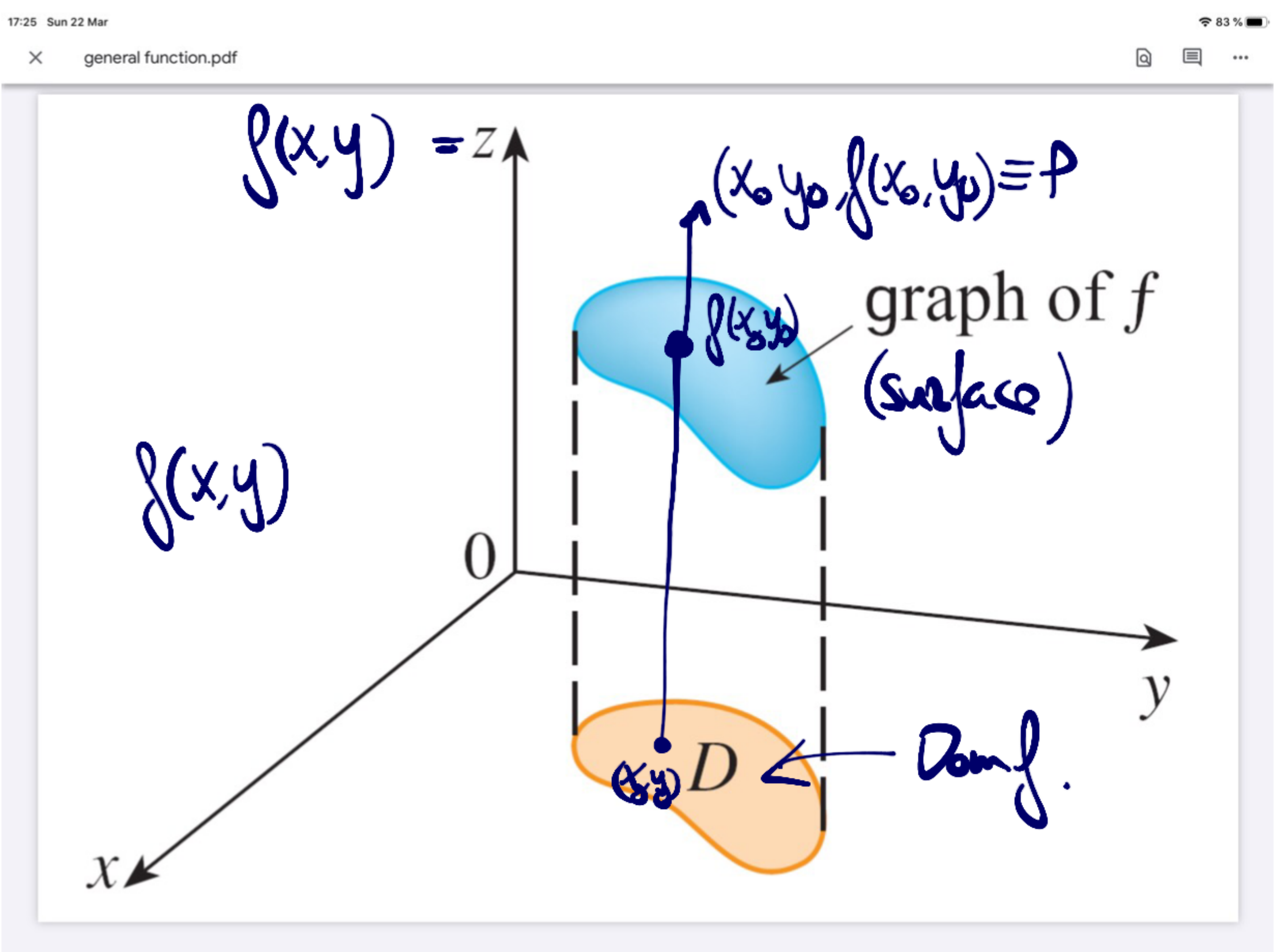


Existence of all directional derivatives $\not\Rightarrow$ differentiability

$\nexists$ tangent plane

Differentiability

We have $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ a scalar function and we would like to see if f is differentiable at a point P.



At a point $(x_0, y_0, f(x_0,y_0)) \in \mathbb{R}^3$ will have a plane going through that point with equation

any plane. $\{ z = f(x_0,y_0) + A(x-x_0) + B(y-y_0)$

$A, B \in \mathbb{R}$ coefficients for the plane.

But, if we want to have the tangent plane to the surface $z = f(x,y)$ at $P = (x_0, y_0, f(x_0,y_0))$ we need that

$$f\big((x_0,y_0)+(\underbrace{h}_{(x-x_0)},\underbrace{k}_{(y-y_0)})\big) = f(x_0,y_0) + Ah + Bk + \underline{\underline{2(h,k)}}$$

$$\left[f(x_0+h) = f(x_0) + f'(x_0)(x-x_0) + 2(h) \right]$$

$$\lim_{h \to 0} \frac{2(h)}{h} = 0$$

That would be the tangent plane if

$$\boxed{\lim_{(h,k)\to(0,0)} \frac{\varepsilon(h,k)}{\|(h,k)\|}=0} = \lim_{(x,y)\to(x_0,y_0)} \frac{\varepsilon((x,y)-(x_0,y_0))}{\|(x,y)-(x_0,y_0)\|}$$

<u>Definition - Differentiability</u>

Let $A\subset\mathbb{R}^2$ be a set in $\mathbb{R}^2$ such that $(x_0,y_0)\in A$

$f: A\subset\mathbb{R}^2 \to \mathbb{R}$ a scalar function

Then, f is differentiable at (x_0,y_0) if

a) $\dfrac{\partial f(x_0,y_0)}{\partial x}$, $\dfrac{\partial f(x_0,y_0)}{\partial y}$ exist

b) $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-f(x_0,y_0)-\frac{\partial f(x_0,y_0)}{\partial x}(x-x_0)-\frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\|(x,y)-(x_0,y_0)\|}=0$$

We might write that limit as

$$\left\{ \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-f(x_0,y_0)-\nabla f(x_0,y_0)\cdot\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix}}{\|(x,y)-(x_0,y_0)\|} = 0 \right.$$

If that is the case we write the tangent plane to the graph of f at (x_0,y_0) as

$$z = f(x_0,y_0) + \underbrace{\frac{\partial f(x_0,y_0)}{\partial x}}_{=A}(x-x_0) + \underbrace{\frac{\partial f(x_0,y_0)}{\partial y}}_{=B}(y-y_0)$$

These two particular coefficients define the tangent plane to $z=f(x,y)$ (slopes in the directions of the axis)

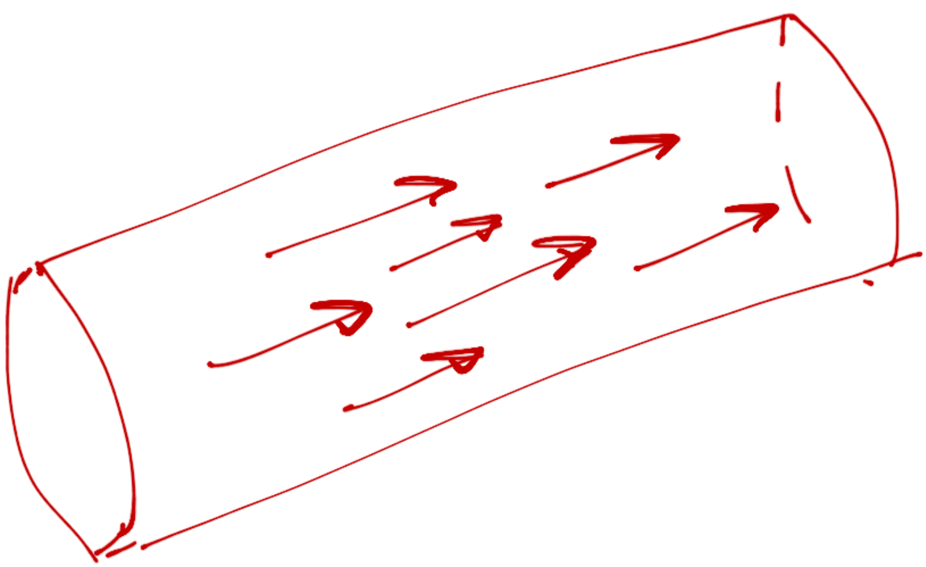
Definition

$A \subset \mathbb{R}^N$, $x_0 \in A$, $f : A \to \mathbb{R}^M$

f is differentiable at $x_0 \in \mathbb{R}^N$ if

a) all partial derivatives exist at x_0

b) $$\lim_{x \to x_0} \frac{\left\| f(x) - f(x_0) - \overbrace{Jf(x_0)}^{\text{Jacobian Matrix}} \overbrace{(x - x_0)}^{\text{vector}} \right\|}{\|x - x_0\|} = 0$$



Problem 5 of set 1.2

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

i) f differentiable at $(0,0)$?

we might study the continuity first at $(0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2\lambda x^2}{x^2(1+\lambda^2)}$$

$$y = \lambda x$$

$$= \lim_{x\to 0} \frac{2\lambda}{1+\lambda^2} = \frac{2\lambda}{1+\lambda^2}$$ The limit depends on the direction so it does not exist.

f is not continuous at $(0,0) \Rightarrow f$ is not diff.

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{r\to 0} \frac{2r^2\cos\theta\sin\theta}{r^2} = \underline{2\cos\theta\sin\theta}$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$

The limit depends on the direction θ so it does not exist.

$$\underbrace{\text{Existence of all directional derivatives}}_{\not\Downarrow \ \text{differentiability.}} \not\Longrightarrow \text{continuity}$$

ii) Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at $(0,0)$

Show that $\frac{\partial f}{\partial x}$ is not continuous.}

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 = \frac{\partial f(0,0)}{\partial y}$$

$$\lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$f(x,y) = \frac{2xy}{x^2+y^2} \Rightarrow f(h,0) = \frac{2 \cdot 0 \cdot h}{h^2+0} = 0$$

$$f(0,0) = 0$$

$$f(0,k) = \frac{2 \cdot 0 \cdot k}{0+k^2} = 0$$

Partial derivative with respect to x

If $(x,y) \neq (0,0)$ (rules of derivatives)

$$\frac{\partial f(x,y)}{\partial x} = \frac{2y(x^2+y^2) - \overbrace{2xy\,2x}^{4x^2y}}{(x^2+y^2)^2}$$

$$= \frac{-2x^2y + 2y^3}{(x^2+y^2)^2}$$

At $(0,0)$ $\frac{\partial f(0,0)}{\partial x} = 0$

$$\frac{\partial f(x,y)}{\partial x} = \begin{cases} \dfrac{-2x^2y+2y^3}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

Continuous at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} \frac{-2x^2y+2y^3}{(x^2+y^2)^2} = 0 = \frac{\partial f(0,0)}{\partial x}$$

Using polar coordinates. $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$

$$\lim_{(x,y)\to(0,0)} \frac{-2x^2y+2y^3}{(x^2+y^2)^2} = \lim_{r\to 0} \frac{-2r^3\cos^2\theta\sin\theta + 2r^3\sin^3\theta}{r^4}$$

$$= 2 \lim_{r\to 0} \underset{\downarrow\,\infty}{\frac{1}{r}} \underbrace{\left(\sin^3\theta - \cos^2\theta\sin\theta\right)}_{\text{bounded.}}$$

$$= \infty$$

Problem 10 i)

$$f(x,y) = x - y + 2$$
$$(x_0 y_0) = (1,3)$$

Find the tangent plane.

f linear so that tangent plane $\equiv f$.

$$z = f(x_0 y_0) + \frac{\partial f(x_0 y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0 y_0)}{\partial y}(y - y_0)$$

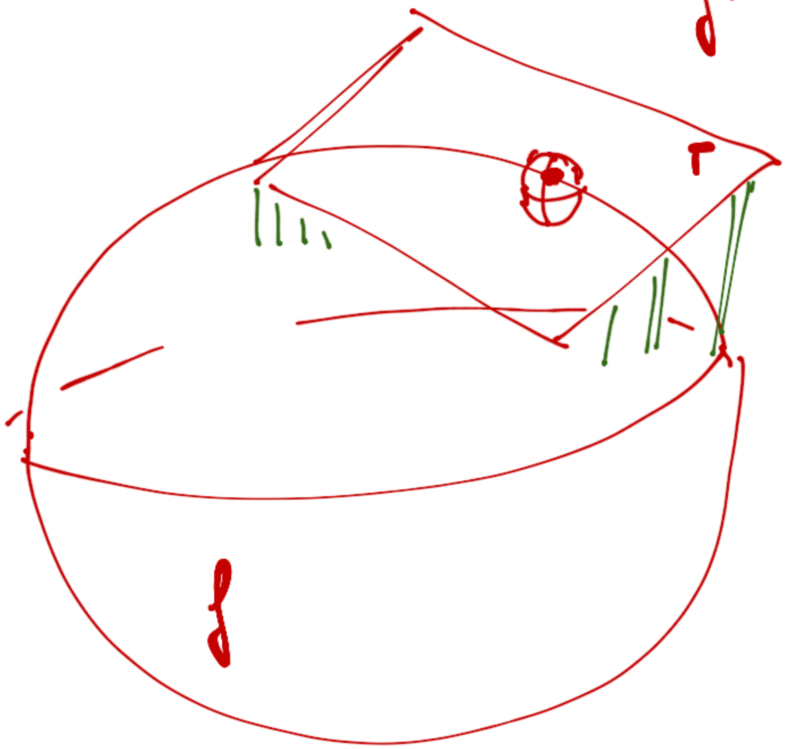
$$f(x_0 y_0) = 1 - 3 + 2 = 0$$

$$\frac{\partial f(x_0 y_0)}{\partial x} = 1 \qquad \frac{\partial f(x_0 y_0)}{\partial y} = -1$$

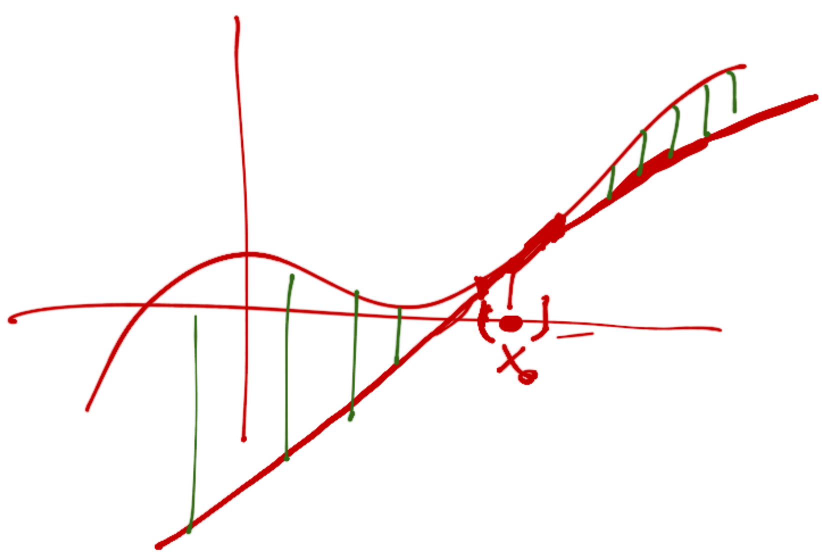
$$z = (x-1) - (y-3) = x - y + 2$$

$\boxed{z = x - y + 2}$ Tangent plane.

$f \sim T$ in the ball



But f is away from T if we are outside of the ball



Proposition

$A \subset \mathbb{R}^N$, $x_0 \in A$ and $\boxed{f : A \to \mathbb{R} \text{ differentiable}}$ at x_0 and $v \in \mathbb{R}^N \setminus \{0\}$ a vector.

Then,

$$\nabla_v f(x_0) = \sum_{i=1}^{N} \frac{\partial f(x_0)}{\partial x_i} \cdot v_i = \langle \nabla f(x_0), v \rangle$$

v is normalised vector

$\|v\| = 1$

- $\langle \nabla f(x_0), (\alpha v) \rangle = \alpha \langle \nabla f(x_0), v \rangle$

$\not\parallel$ if $\alpha \neq 1$

$\langle \nabla f(x_0), v \rangle$

We must have

$\|v\| = 1$

Example: Set $\underline{1.3}$ problem 1 i)

$f(x,y) = x^2 + y^2$ at $(1,1)$ along the direction $(1,-1)$

$$D_v f(1,1) = \lim_{(x,y)\to(1,1)} \frac{f((1,1)+t(1,-1)) - f(1,1)}{t\,\|(1,-1)\|}$$

f is differentiable everywhere (it is a polynomial)

$$D_v f(1,1) = \left\langle \nabla f(1,1), \frac{(1,-1)}{\|(1,-1)\|} \right\rangle$$

$$\nabla f(x,y) = (2x, 2y) \Rightarrow \nabla f(1,1) = (2,2)$$

$$\|(1,-1)\| = \sqrt{1+1} = \sqrt{2}$$

$$D_v f(1,1) = \left\langle (2,2), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\rangle = \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 0$$

Remark

$$-\|\nabla f(x_0)\| \leq \underline{D_v f(x_0)} = \langle \nabla f(x_0), v \rangle = \|\nabla f(x_0)\| \underbrace{\|v\|}_{=1} \cos(\nabla f, v)$$

$$= \|\nabla f(x_0)\| \cos(\nabla f, v) \leq \|\nabla f(x_0)\|$$

We obtain that the directional derivative of f at x_0 in the direction of v is maximal in the direction of ∇f

$$\cos(\nabla f, v) = 1 \Rightarrow \text{angle}(\nabla f, v) = 0$$

or

$$\nabla f \parallel v$$

Also,

$-\nabla f(x_0)$ maximal decreasing for f.

- Problem 5, set 1.3

Temperature of a metal plate

$$T(x,y) = e^x \cos y + e^y \cos x$$

a) Direction of maximal increasing for T at (0,0)

$$\nabla T(x,y) = \Big(\underbrace{e^x \cos y - e^y \sin x}_{\frac{\partial T}{\partial x}}, \underbrace{-e^x \sin y + e^y \cos x}_{\frac{\partial T}{\partial y}}\Big)$$

$$\boxed{\nabla T(0,0) = (1, 1)}$$

b) T decreasing the fastest

$$-\nabla T(0,0) = (-1, -1)$$

Proposition

$A \subset \mathbb{R}^n$, $x_0 \in A$, $f : A \to \mathbb{R}$ differentiable at x_0 with $\nabla f(x_0) \neq 0$ then

$\nabla f(x_0) \perp$ to the level curve of f at $f(x_0)$

For example in $\mathbb{R}^3$

$$\nabla f(x,y,z) \neq (0,0,0)$$

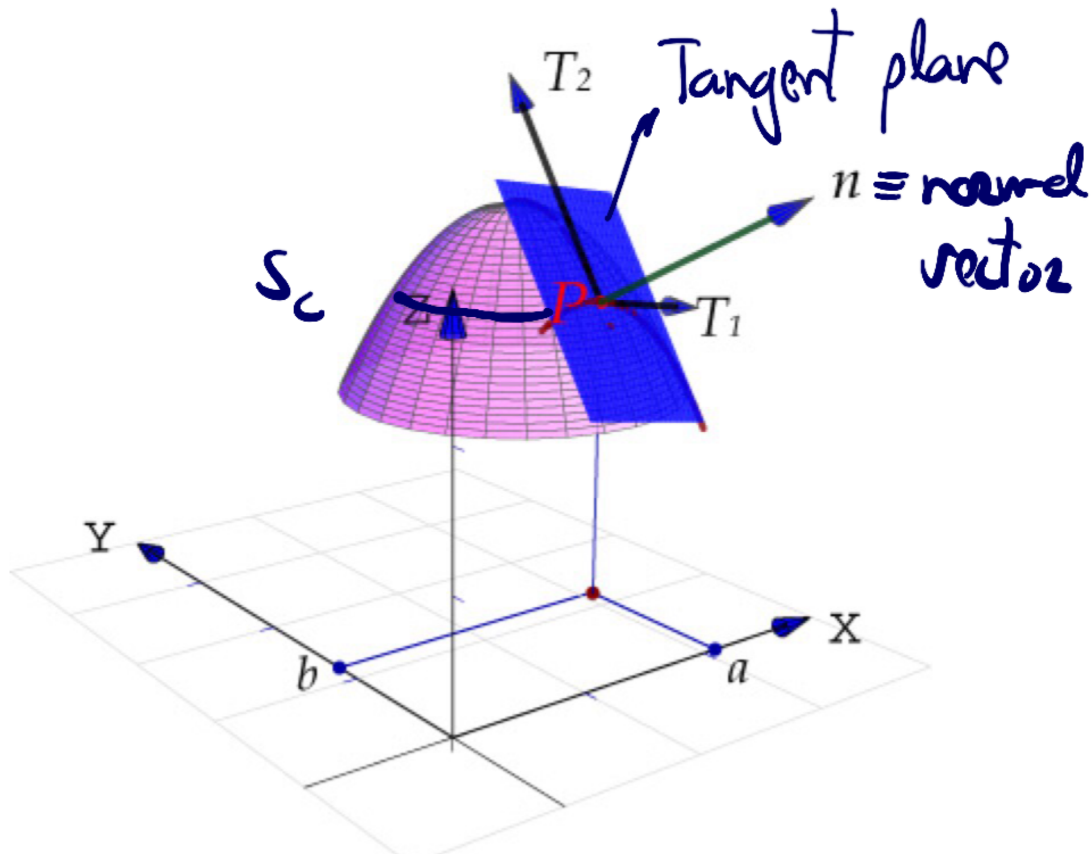
$(x,y,z) \in \Omega$, Ω domain in $\mathbb{R}^3$

Assume a level curve

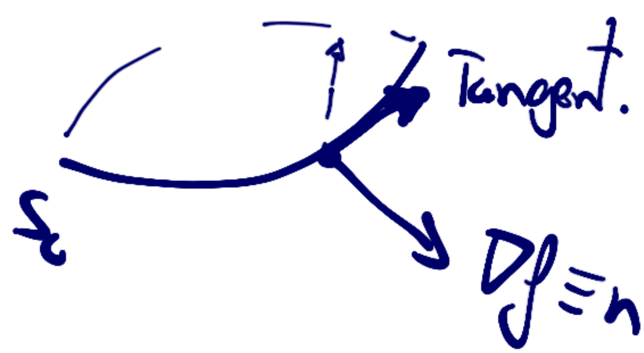
$f(x,y,z) = c \in \mathbb{R}$

S_c = specific level curve at c.

Since f is diff. the tangent plane will be perpendicular to the level curve.



So that ∇f is the normal vector to the level curve.



In other words, take two points on the tangent plane.

$$P^* = (P_1, P_2, P_3) \quad \text{and} \quad P = (x, y, z)$$

with $n = (n_x, n_y, n_z)$ as the normal vector to the tangent plane.

$$(P - P^*) \cdot n = 0 \Rightarrow (x - P_1)n_x + (y - P_2)n_y + (z - P_3)n_z = 0$$

There is only one normal vector !!

$$\boxed{\nabla f(x, y, z) \cdot (P - P^*) = 0}$$

<u>Definition</u>

Let S be a surface in $\mathbb{R}^3$, then the tangent plane to S at $(x_0, y_0, z_0) \in S$ is given by $\nabla f(x_0, y_0, z_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$

Problem 11. Set $\underline{\underline{1.2}}$ i)

Sphere.

Surface: $x^2+y^2+z^2=3$ at $(x_0\,y_0\,z_0)=(1,1,1)$

$$\nabla f(x,y,z)=(2x,\,2y,\,2z)$$

$$\nabla f(1,1,1)=(2,2,2)$$

Tangent plane:

$$\nabla f(1,1,1)\cdot\begin{pmatrix}x-1\\y-1\\z-1\end{pmatrix}=0$$

$$2x-2+2y-2+2z-2=0$$

$$\boxed{x+y+z=3}$$