Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 14

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Outline

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Definition 3.1

A vector field on $\mathbb{R}^n$ is an application or mapping of the form $\mathbf{F} : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ who allocate each **point x** on A a **vector** $\mathbf{F}(\mathbf{x})$. For $n = 2$ we call it vector field on the plane and for $n = 3$ we call it vector field on the space.

Remarks

- We are concerned primarily with vector fields on $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Mainly the vector fields represents a physical measure as force or velocity associated with the point $\mathbf{x}$.
- This vector $\mathbf{F}(\mathbf{x})$ is represented by an arrow whose tail is at the point $\mathbf{x}$.

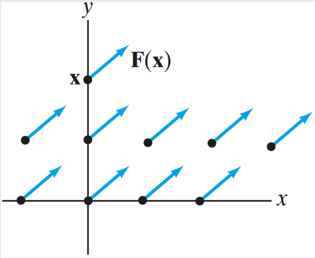
This perspective allows us to
visualise vector fields in a reasonable way

Example 1

- Suppose $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{F}(\mathbf{x}) = \mathbf{a}, \quad \text{where } \mathbf{a} \text{ is a constant vector.}$$

- Then, $\mathbf{F}$ assigns $\mathbf{a}$ to each point of $\mathbb{R}^2$.
- We can picture $\mathbf{F}$ by drawing the same vector (parallel translated) emanating from each point in the plane.



$$\mathbf{F}(\mathbf{x}) = \mathbf{i} + \mathbf{j}$$

Example 2

- Let us depict $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\mathbf{G}(\mathbf{x}) = \mathbf{G}(x, y) = y\mathbf{i} - x\mathbf{j}$$

- We can begin calculating some specific values of $\mathbf{G}$

(x, y)	$\mathbf{G}(x, y)$
$(0, 0)$	$\mathbf{0}$
$(1, 0)$	$-\mathbf{j}$
$(0, 1)$	$\mathbf{i}$
$(1, 1)$	$\mathbf{i} - \mathbf{j}$

- To understand $\mathbf{G}$ somewhat better, note that,

$$\|\mathbf{G}(x, y)\| = \|y\mathbf{i} - x\mathbf{j}\| = \sqrt{y^2 + x^2} = \|\mathbf{r}\|$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is the position vector of the point (x, y).

Example 2

- Let us depict $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ where,

$$\begin{aligned} \mathbf{G}(\mathbf{x}) &= \mathbf{G}(x, y) = y\mathbf{i} - x\mathbf{j} \\ \|\mathbf{G}(x, y)\| &= \|y\mathbf{i} - x\mathbf{j}\| = \sqrt{y^2 + x^2} = \|\mathbf{r}\| \end{aligned}$$

  where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is the position vector of the point (x, y)
- It follows that $\mathbf{G}$ has constant length a on the circle

$$x^2 + y^2 = a^2$$

- In addition, we have,

$$\mathbf{r} \cdot \mathbf{G}(x, y) = (x\mathbf{i} + y\mathbf{j}) \cdot (y\mathbf{i} - x\mathbf{j}) = 0$$

- Hence, $\mathbf{G}(x, y)$ is always perpendicular to the position vector of the point (x, y).

Example 2

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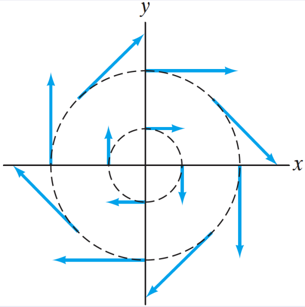

Scalar Fields

Usually, one thinks of a vector field on $\mathbb{R}^n$ as attaching vector information to each point. But is also interesting giving *scalar information* to each point:

- A scalar-valued function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is called a scalar field.
- So, one can think of a scalar field on $\mathbb{R}^n$ as attaching real number information to each point, e.g.,

Temperature or Pressure

Outline

Example 3: Inverse Square Vector Field

- Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- Consider the so-called inverse square vector field in $\mathbb{R}^n$
- It is a function $\mathbf{F} : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}^3$ given by:

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}, \quad \text{where } c \text{ is any (nonzero) constant.}$$

- If we set $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ then $\mathbf{F}$ is given by:

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r} = \frac{c}{\|\mathbf{r}\|^2}\mathbf{u} \quad \text{(inverse square)}$$

- The direction of $\mathbf{F}$ at the point $P(x, y, z) \neq (0, 0, 0)$ is parallel to the vector from the origin to P.
- The magnitude of $\mathbf{F}$ is inversely proportional to the square of the distance from the origin to P.

Example 3: Inverse Square Vector Field

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- $\mathbf{F}$ points away from the origin if c is positive.
- $\mathbf{F}$ points toward the origin if c is negative.

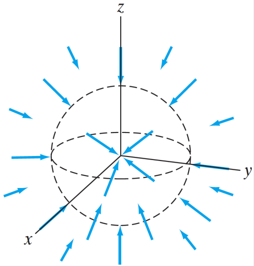
Example 3: Inverse Square Vector Field

- Consider the so-called inverse square vector field in $\mathbb{R}^n$

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r} = \frac{c}{\|\mathbf{r}\|^2}\mathbf{u} \quad \text{(inverse square)}$$

Example: Newtonian gravitational field between two bodies

$$\mathbf{F} = -\frac{GMm}{\|\mathbf{r}\|^2}\mathbf{u}$$



Gradient Fields and Potentials

Inverse square fields are examples of gradient fields

- A gradient field on $\mathbb{R}^n$ is a vector field $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{F}$ is the gradient of some (differentiable) scalar-valued function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x}), \quad \text{at all } \mathbf{x} \text{ in } X$$

- The function f is called a (scalar) potential function for the vector field $\mathbf{F}$.

Gradient Fields and Potentials

- In the case of the inverse square field, we write out the components of $\mathbf{F}$ explicitly.
- Assume $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \frac{c}{\|\mathbf{r}\|^2}\mathbf{u} = \left(\frac{c}{x^2+y^2+z^2}\right)\left(\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}\right)\\ &= \frac{cx}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{cy}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{cz}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}\end{aligned}$$

- Then, it can be shown that, $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$ where f is given by:

$$f(x,y,z) = -\frac{c}{\sqrt{x^2+y^2+z^2}} = -\frac{c}{\|\mathbf{r}\|}$$

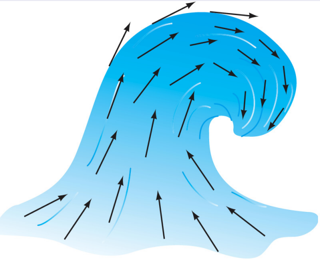
$$f : \mathbb{R}^3 - \{\mathbf{0}\} \to \mathbb{R}$$

Outline

1. Vector Fields
   - Introduction
   - Potentials
   - Flow lines

Flow Lines

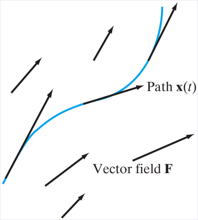
- Suppose you are drawing a sketch of a vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$
- It helps to imagine that the arrows represent the velocity of some fluid moving through space



Analytically we are drawing paths whose velocity vectors coincide with those of the vector field

Definition 3.2

- Let $\mathbf{F}$ be a vector field
$$\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$$
- A flow line of the vector field $\mathbf{F}$ is a differentiable path $\mathbf{x} : I \to \mathbb{R}^n$ such that
$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$$
- That is, the velocity vector of $\mathbf{x}$ at time t is given by the value of the vector field $\mathbf{F}$ at the point on $\mathbf{x}$ at time t.

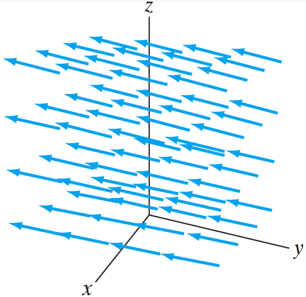


Example 4

- We calculate the flow lines of the constant vector field

$$\mathbf{F}(x, y, z) = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

- A picture of this vector field: makes it easy to believe that the flow lines are straight-line paths.



Example 4

- We calculate the flow lines of the constant vector field:

$$\mathbf{F}(x, y, z) = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

- If $\mathbf{x}(t) = (x(t), y(t), z(t))$ is a flow line, then, by Definition 3.2, we must have:

$$\mathbf{F}(x, y, z) = \mathbf{x}'(t) = (x'(t), y'(t), z'(t)) = (2, -3, 1) = \mathbf{F}(\mathbf{x}(t))$$

- Equating components, we see

$$\begin{cases} x'(t) = 2 \\ y'(t) = -3 \\ z'(t) = 1 \end{cases} \Rightarrow \begin{cases} x(t) = 2t + x_0 \\ y(t) = -3t + y_0 \\ z(t) = t + z_0 \end{cases}, \quad x_0, y_0, z_0 \text{ constants}$$

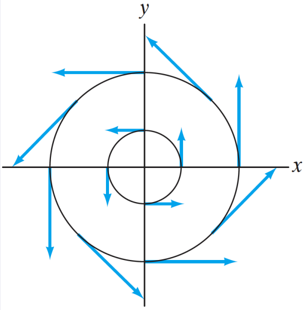
- These differential equations have been solved by direct integration.

Example 5

- Consider the vector field

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

- Intuition should lead to suspect that a flow line of the vector field **F** should be circular:



Example 5

- Consider the vector field

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

- Suppose $\mathbf{x} : [0, 2\pi) \to \mathbb{R}^2$ is given by

$$\mathbf{x}(t) = (a\cos t, a\sin t), \quad \text{where } a \text{ is constant}$$

- Then

$$\mathbf{x}'(t) = \ -a\sin t\mathbf{i} + a\cos t\mathbf{j} = \mathbf{F}(a\cos t, a\sin t)$$

- So such paths are indeed flow lines.
- Finding all possible flow lines of $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ is a more involved task.

Example 5

$$\begin{aligned}\mathbf{F}(x,y) &= -y\mathbf{i} + x\mathbf{j}\\ \mathbf{x}(t) &= (a\cos t, a\sin t), \quad \text{where } a \text{ is constant}\\ \mathbf{x}'(t) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j} = \mathbf{F}(a\cos t, a\sin t)\end{aligned}$$

- If $\mathbf{x}(t) = (x(t), y(t))$ is a flow line, then, by Definition 3.2

$$\mathbf{x}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} = -y(t)\mathbf{i} + x(t)\mathbf{j} = \mathbf{F}(\mathbf{x}(t))$$

- Equating components:

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

- This is an example of a first-order system of differential equations.

Example 5

$$\begin{aligned}
\mathbf{F}(x,y) &= -y\mathbf{i} + x\mathbf{j} \\
\mathbf{x}(t) &= (a\cos t, a\sin t), \quad \text{where } a \text{ is constant} \\
\mathbf{x}'(t) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j} = \mathbf{F}(a\cos t, a\sin t) \\
\mathbf{x}'(t) &= x'(t)\mathbf{i} + y'(t)\mathbf{j} = -y(t)\mathbf{i} + x(t)\mathbf{j} = \mathbf{F}(\mathbf{x}(t))
\end{aligned}$$

- Equating components:

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

- All solutions to this system are of the form:

  $x(t) = (a\cos t - b\sin t, a\sin t + b\cos t)$, a and b constants

- It's not difficult to see that such paths trace circles when at least one of a or b is nonzero.

First-order systems of differential equations

- We consider a general case
- Assume **F** is a vector field on $\mathbb{R}^n$
- Finding the flow lines of **F** is equivalent to solving the first-order system of differential equations:

$$\begin{cases} x_1'(t) = F_1(x_1(t), x_2(t), \ldots, x_n(t)) \\ x_2'(t) = F_2(x_1(t), x_2(t), \ldots, x_n(t)) \\ \qquad \vdots \\ x_n'(t) = F_n(x_1(t), x_2(t), \ldots, x_n(t)) \end{cases}$$

- The functions $x_1(t), \ldots, x_n(t)$ are the components of the flow line $\mathbf{x}$.
- The function F_i is just the ith component function of the vector field **F**.