

## Ordering

The set of real numbers is equipped with an order relation, denoted by  $<$ , such that for any given two real numbers  $a$  and  $b$ , there are three mutually exclusive possibilities:

(i)  $a < b$  (a is less than b)

(ii)  $a = b$  (a equals b)

(iii)  $b < a$  (b is less than a)

$$b > a$$

We define

(iv)  $a \leq b$  (a is less than or equal to b)

(v)  $a \geq b$  (a is greater than or equal to b).

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For every  $x, y, z \in \mathbb{R}$ , the following properties hold:

**Transitivity:**

If  $x < y$  and  $y < z$ , then  $x < z$ .

**Compatibility with addition:**

If  $x < y$ , then  $x + z < y + z$ .

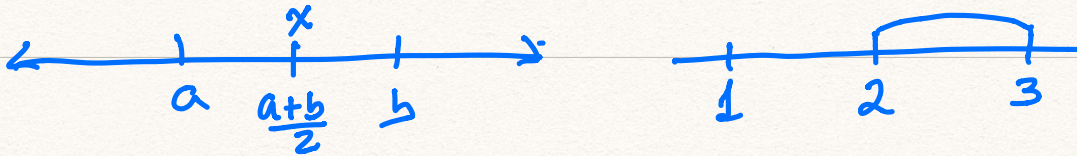
**Multiplication by a positive factor:**

If  $x < y$  and  $0 < z$ , then  $xz < yz$ .

**Multiplication by a negative factor:**

If  $x < y$  and  $z < 0$ , then  $xz > yz$ .

Example: Suppose that  $a < b$ . There exists a real number  $x$  satisfying  $a < x < b$ .



We will prove that  $x = \frac{a+b}{2}$  satisfies  $a < x < b$ .

$$a < b \Rightarrow 2a < a+b \Rightarrow a < \frac{a+b}{2} \quad (1)$$

$\uparrow$  add a to both sides                       $\uparrow$  divide by 2

$$a < b \Rightarrow a+b < 2b \Rightarrow a+b < b \quad (2)$$



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Example: If  $b > 0$  and  $B > 0$  and

$$\frac{a}{b} < \frac{A}{B} \implies \frac{a \cdot B}{b} < \frac{A \cdot b}{B} \implies aB < Ab$$

mult. both sides by  $bB$

then  $aB < bA$ . Deduce that

$$\frac{a}{b} < \frac{a+A}{b+B} < \frac{A}{B}$$

•  $\frac{a}{b} < \frac{a+A}{b+B}$  :

~~$$\begin{aligned} a(b+B) &< (a+A)b \\ ab + aB &< ab + Ab \\ aB &< Ab \end{aligned}$$~~

Adding  $ab$  to both sides of  $aB < Ab$  we get

$$ab + aB < Ab + ab \implies a(b+B) < (A+a)b \implies$$

$\downarrow$   
 $b+B > 0$   
 $b > 0$

$$\frac{a(b+B)}{b(b+B)} < \frac{(A+a)b}{b(b+B)} \implies \frac{a}{b} < \frac{A+a}{b+B}$$

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Example: If  $\underbrace{a > 0}_{\text{Always true}}$ , then  $\underbrace{a^{-1} > 0}_{\text{Prove this is true.}}$ .

Assume that  $a > 0$  and  $a^{-1} \leq 0$ .

$a^{-1} < 0$  or  $\cancel{a^{-1} = 0}$ .  
Case II Case I

Case I:

Suppose that  $a^{-1} = 0$ . We know that  $a \cdot a^{-1} = 1$ .

But  $a^{-1} = 0$  so  $a \cdot a^{-1} = 0 \Rightarrow 0 = 1$  Contradiction!

So  $a^{-1} \neq 0$ .

$\rightarrow$  state 1  
false  $\rightarrow$  state 2 } Contradiction

Case II:

Now suppose that  $a^{-1} < 0$ . Multiplying both

sides by  $a > 0$ , we get  $a \cdot a^{-1} < a \cdot 0$  or

$$1 = a \cdot a^{-1} < a \cdot 0 = 0 \Rightarrow 1 < 0.$$

So  $\bar{a}^{-1} > 0$ .

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Example: If  $x$  and  $y$  are positive, then  $x < y$  if and only if  $x^2 < y^2$ .

• If  $x < y$  then  $x^2 < y^2 \implies$

• If  $x^2 < y^2$  then  $x < y \Leftarrow$

• If  $x < y$  then  $x^2 < y^2$

Multiply both sides of  $x < y$  times  $x$  we get  $x^2 < xy$ .

Multiply both sides of  $x < y$  times  $y$  we get  $xy < y^2$

By transitivity  $x^2 < y^2$ .

• If  $x^2 < y^2$  then  $x < y$

$$\begin{array}{l} x < y \quad x+y > 0 \\ x-y < 0 \end{array}$$

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example,  $(x+y)^{-1} > 0$ . Multiplying  
both sides of (1) by  $(x+y)^{-1}$  we get

$$x-y < 0 \Rightarrow x < y.$$

add  
y to both  
sides

$$\cancel{(x+y)^{-1}} \cancel{(x+y)} (x-y) < 0 \quad \cancel{(x+y)^{-1}} = 0$$

$$x-y < 0$$

$$a = x+y > 0$$

$$a^{-1} = (x+y)^{-1} > 0$$

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