Ordering

The set of real numbers is equiped with an order relation, denoted by $<$, such that for any given two real numbers a and b, there are three mutually exclusive posibilities:

(i) $a < b$ (a is less than b) $\boxed{b > a}$

(ii) $a = b$ (a equals b)

(iii) $b < a$ (b is less than a)

We define

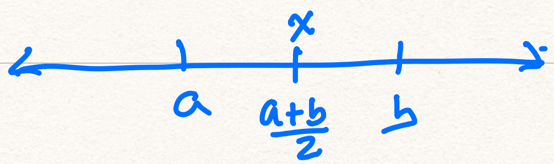
(iv) $a \leq b$ (a is less than or equal to b)

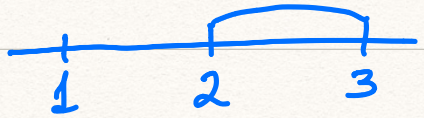
(v) $a \geq b$ (a is greater than or equal to b).

For every $x, y, z \in \mathbb{R}$, the following properties hold:

Transitivity:	If $x < y$ and $y < z$, then $x < z$.
Compatibility with addition:	If $x < y$, then $x + z < y + z$.
Multiplication by a positive factor:	If $x < y$ and $0 < z$, then $x\,z < y\,z$.
Multiplication by a negative factor:	If $x < y$ and $z < 0$, then $x\,z > y\,z$.

Example: Suppose that $\boxed{a<b}$. There exists a real number x satisfying $a<x<b$.





We will prove that $x = \frac{a+b}{2}$ satisfies $a<x<b$.

$$\boxed{a < b} \underset{\substack{\uparrow \\ \text{add } a \\ \text{to both} \\ \text{sides}}}{\Rightarrow} 2a < a+b \underset{\substack{\uparrow \\ \text{divide} \\ \text{by } 2}}{\Rightarrow} a < \frac{a+b}{2} \qquad (1)$$

$$a < b \Rightarrow a+b < 2b \Rightarrow a+b < b \qquad (2)$$

<u>Example</u>: If $b>0$ and $B>0$ and

$\frac{a}{b} < \frac{A}{B}$ $\xrightarrow[\text{by } bB]{\text{mult. both sides}}$ $\frac{a}{\cancel{b}} \cdot B\cancel{b} < \frac{A}{\cancel{B}} \cdot b\cancel{B} \Rightarrow aB < Ab$

then $aB < bA$. Deduce that

$$\frac{a}{b} < \frac{a+A}{b+B} < \frac{A}{B}.$$

~~$a(b+B) < (a+A)b$~~
~~$a\cancel{b} + aB < a\cancel{b} + Ab$~~
~~$aB < Ab$~~

- $\frac{a}{b} < \frac{a+A}{b+B}$:

Adding ab to both sides of $aB < Ab$ we get

$ab + aB < Ab + ab \Rightarrow a(b+B) < (A+a)b \Rightarrow$

$b+B>0$
$b>0$

$$\frac{a\cancel{(b+B)}}{b\cancel{(b+B)}} < \frac{(A+a)\cancel{b}}{\cancel{b}(b+B)} \Rightarrow \frac{a}{b} < \frac{A+a}{b+B}$$

Example: $\underbrace{\text{If } a>0}_{\text{Always true}}, \ \underbrace{\text{then } a^{-1}>0}_{\text{Prove this is true.}}.$

Assume that $a>0$ and $a^{-1} \leq 0$.

$\underbrace{a^{-1}<0}_{\text{case II}}$ or $\underbrace{a^{-1}=0}_{\text{Case I}}$.

case I:

Suppose that $a^{-1}=0$. We know that $a \cdot a^{-1}=1$.

But $a^{-1}=0$ so $a \cdot a^{-1}=0 \Rightarrow 0=1$ contradiction!

So $a^{-1} \neq 0$.

$\left.\begin{array}{r} \rightarrow \text{state } 1 \\ \text{false} \rightarrow \text{state } 2 \end{array}\right\}$ contradiction

case II:

Now suppose that $a^{-1}<0$. Multiplying both sides by $a>0$, we get $a \cdot a^{-1} < a \cdot 0$ or

$$1 = a \cdot a^{-1} < a \cdot 0 = 0 \Rightarrow 1 < 0.$$

So $a^{-1}>0$.

Example: If x and y are positive, then $x<y$ if and only if $x^2<y^2$.

- If $x<y$ then $x^2<y^2$ $\Longrightarrow$
- If $x^2<y^2$ then $x<y$ $\Longleftarrow$

- If $x<y$ then $x^2<y^2$

Multiply both sides of $x<y$ times x we get $x^2<xy$.

Multiply both sides of $x<y$ times y we get $xy<y^2$

By transitivity $x^2<y^2$.

- If $x^2<y^2$ then $x<y$

$x<y$ $\quad$ $x+y>0$

$x-y<0$

example, $(x+y)^{-1} > 0$. Multiplying both sides of (1) by $(x+y)^{-1}$ we get

$x-y<0 \Rightarrow x<y.$

add y to both sides

$\cancel{(x+y)^{-1}}\,\cancel{(x+y)}(x-y) < 0\,(x+y)^{-1} = 0$

$x-y<0$

$a = x+y > 0$

$a^{-1} = (x+y)^{-1} > 0$