

► The induction principle is a property of the natural numbers that is used to prove statements that are true for all $n \in \mathbb{N}$.

It is quite useful when the statement can be expressed recursively.

Example $P(n) : a_n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$

$$a_{n+1} = \sum_{k=1}^{n+1} k = \underbrace{1+2+3+\dots+n+n+1}_{a_n = \sum_{k=1}^n k}$$

$$a_n = \sum_{k=1}^n k = 1+2+3+\dots+n$$

$$a_{n+1} = a_n + (n+1)$$

The logo for Cartagena99, featuring the text 'Cartagena99' in a stylized, bold font with a blue and orange gradient background.

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

Principle of Induction

Suppose that for each $n \in \mathbb{N}$, $P(n)$ is a statement about the natural number n .

Suppose also that

- 1. $P(1)$ is true.
- 2. If $P(n)$ is true, then $P(n+1)$ is also true.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Example. For each $n \in \mathbb{N}$, $P(n): \sum_{k=1}^n k = \frac{n(n+1)}{2}$.

$P(1)$ is true

$n=1$: On one hand, $1+2$

$$\sum_{k=1}^1 k = 1$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

$P(1)$ is true

I.H. — $P(n+1)$

If $P(n)$ is true, then $P(n+1)$ is true

Now, assume that $P(n)$ is true (I.H.).

We need to prove that $P(n+1)$ is true.

$$P(n+1) : \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^n k + (n+1) \stackrel{\text{I.H.}}{=} \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Therefore, by induction,

$P(n)$ is true for all $n \in \mathbb{N}$.

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

Example. If $x \neq 1$, then

$$P(n): \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x} \quad \forall n \in \mathbb{N}.$$

$\mathbb{N} = \{1, 2, 3, \dots\}$

1. On one hand, $\sum_{k=0}^1 x^k = x^0 + x = 1 + x$

On the other hand, $\frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{1-x} = 1+x$

So $P(1)$ is true.

2. Assume that $P(n)$ is true.

I need to show that $P(n+1)$ is true.

$$P(n+1): \sum_{k=0}^{n+1} x^k = 1 - x^{n+2}$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

...

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

$$\sum_{k=0}^{n+1} x^k = \sum_{k=0}^n x^k + x^{n+1} \stackrel{\text{I.H.}}{=} \frac{1-x^{n+1}}{1-x} + x^{n+1}$$

$$= \frac{1-x^{n+1} + (1-x)x^{n+1}}{1-x} = \frac{1-\cancel{x^{n+1}} + \cancel{x^{n+1}} - x^{n+2}}{1-x}$$

$$= \frac{1-x^{n+2}}{1-x}$$

So $P(n+1)$ is true.

Therefore, by induction, $P(n)$ is true.

$$\sum_{k=0}^{n+1} x^k = \underbrace{x^0 + x^1 + x^2 + \dots + x^n}_{\text{I.H.}} + x^{n+1}$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

...

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

Example For all $n \in \mathbb{N}$

$\mathbb{N} = \{1, 2, 3, \dots\}$

$$a^n - b^n = (a-b) \sum_{k=1}^n a^{n-k} b^{k-1}$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 - b^3 = (a-b) \sum_{k=1}^3 a^{3-k} b^{k-1} = (a-b)(a^2 + ab + b^2)$$

P(1)? On one hand, $a^1 - b^1$
On the hand, $(a-b) \sum_{k=1}^1 a^{1-k} b^{k-1} = (a-b)$

So $P(1)$ is true.

Induction step

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

N.T.S: $a^{n+1} - b^{n+1} = (a-b) \sum_{k=1}^{n+1} a^{n+1-k} b^{k-1}$ is true.

$\underbrace{\hspace{15em}}_{P(n+1)}$

$$(a-b) \sum_{k=1}^{n+1} a^{n+1-k} b^{k-1} =$$

$$(a-b) \left[\sum_{k=1}^n a^{n+1-k} b^{k-1} + a^{n+1-(n+1)} b^{(n+1)-1} \right]$$

$$= (a-b) \sum_{k=1}^n a^{n+1-k} b^{k-1} + (a-b) b^n$$

$\underbrace{\hspace{10em}}_{\sum a^{n-k} b^{k-1}}$

$$= a(a-b) \sum_{k=1}^n a^{n-k} b^{k-1} + (a-b) b^n$$

$\underbrace{\hspace{15em}}_{P.H: a^n - b^n}$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

...

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

Example. Prove by induction

$$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

$$n > 1$$

$$n = 2$$

$$2 > 1 \rightarrow \sqrt{2} > \sqrt{1} \rightarrow \sqrt{2+1} > \sqrt{1+1}$$

$$P(2): 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt{2}} > \frac{\sqrt{1} + 1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

So $P(2)$ is true.

Induction Step:

$$I.H.: 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} \text{ is true}$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70

$$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \stackrel{I.H.}{>} \sqrt{n} + \frac{1}{\sqrt{n+1}}$$

$$= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}}$$

$$n^2 + n > n^2 \quad \leftarrow = \frac{\sqrt{n^2 + n} + 1}{\sqrt{n+1}}$$

$$\downarrow$$

$$\sqrt{n^2 + n} > \sqrt{n^2}$$

$$\sqrt{n^2 + n} + 1 > \sqrt{n^2} + 1 > \frac{\sqrt{n^2} + 1}{\sqrt{n+1}}$$

$$= \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

So $P(n+1)$ is true

and we have proved the desired result by induction.

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS
CALL OR WHATSAPP: 689 45 44 70