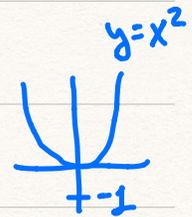


$$\mathbb{N} \supset \mathbb{Z} \supset \mathbb{Q} \supset \mathbb{R} \supset \mathbb{C}$$

We begin by introducing a new object i with the property

$$i^2 = -1$$

$$x^2 = -1$$



► Since $x^2 = -1$ has no solution in \mathbb{R} ,
 $i \notin \mathbb{R}$.

Definition

A complex number z is defined as

$$z = a + ib \quad a, b \in \mathbb{R}.$$

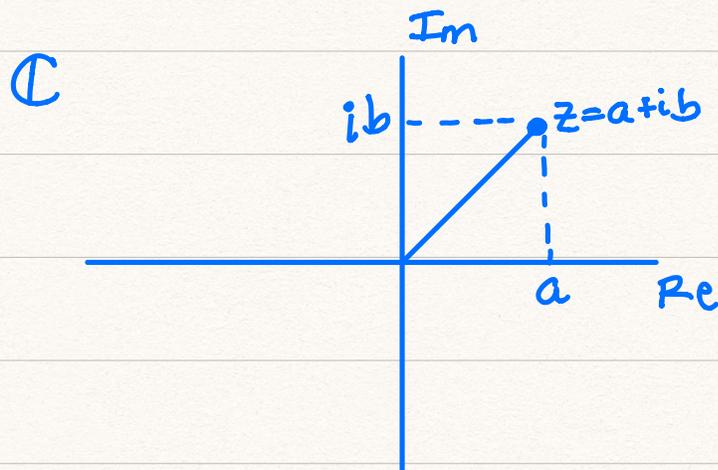
The set of all complex numbers \mathbb{C} is known as the complex plane.

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- ▶ Each complex number $z=a+ib$ is identified with the point $(a,b)\in\mathbb{R}^2$.



- ▶ a is called the real part of z and b is called the imaginary part of z .

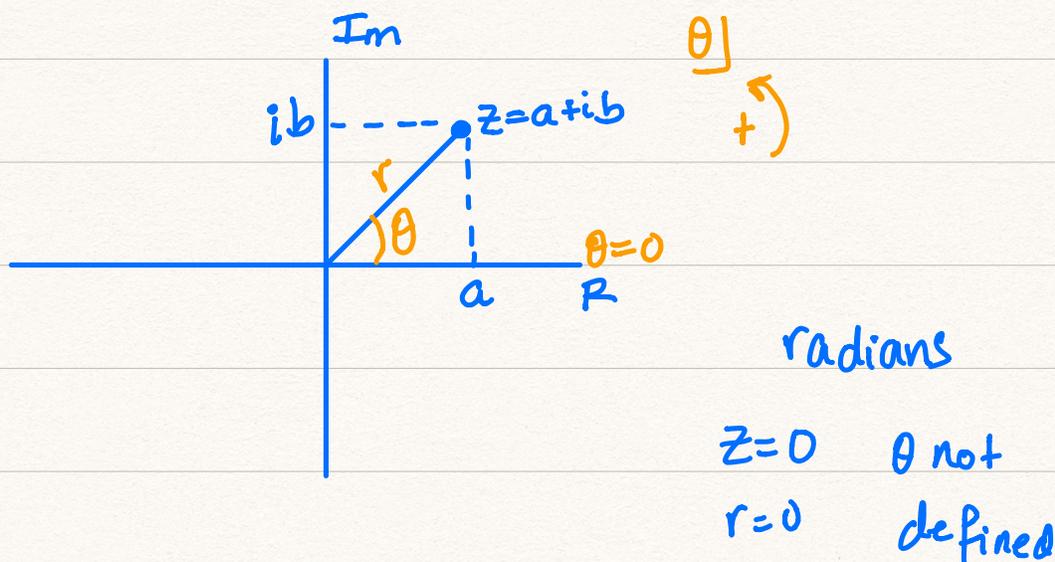
$$\operatorname{Re} z = a \quad \operatorname{Im} z = b$$

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Rectangular and polar representation



$$r = \sqrt{a^2 + b^2} \quad \theta = \arctan b/a$$

$$a = r \cos \theta \quad b = r \sin \theta$$

$$0 \leq \theta < 2\pi$$

► Rectangular : $z = a + ib$

► Polar : $z = r \cos \theta + i r \sin \theta$

$z = a + ib$
with
 $b=0$

Definition

$$z = a + ib = r \cos \theta + i r \sin \theta$$

$z \in \mathbb{R}$

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Euler Formula

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \theta \in \mathbb{R}$$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\begin{aligned} Z = a + ib &= r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

Example:

Express the following numbers in polar form:

(a) $-2 - i3$ (b) $2 + i3$ (c) $-2 + i$ (d) $1 - i3$

(a) $r = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

$\theta = \arctan\left(\frac{-3}{-2}\right) \approx 4^\circ$

π or 180°
 75°

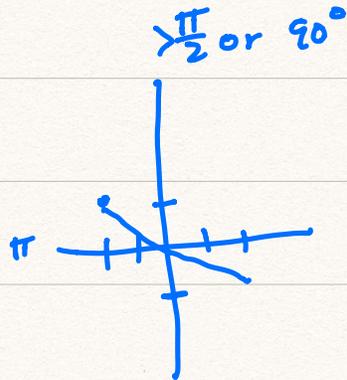
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$$b) 2 + i3 = \sqrt{13} e^{i \arctan 3/2}$$

$$(c) r = \sqrt{(-2)^2 + 1} = \sqrt{5}$$
$$\theta = \arctan(-1/2) + \pi$$

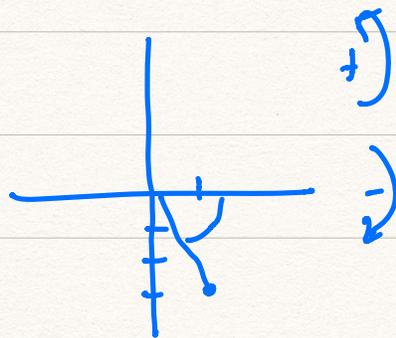


$$-2 + i = \sqrt{5} e^{i(\arctan(-1/2) + \pi)}$$

$$(d) 1 - i3$$

$$r = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$

$$\theta = \arctan(-3/1)$$



$$1 - i3 = \sqrt{10} e^{i \arctan(-3)}$$

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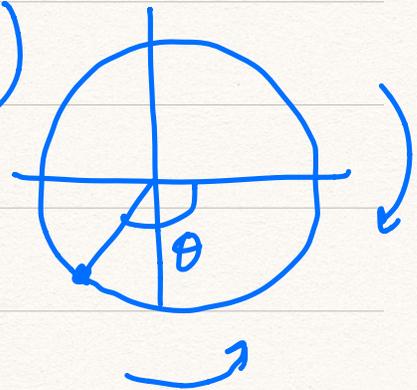
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Example. Represent the following numbers in the complex plane and express them in rectangular form:

(a) $2e^{i\pi/3}$ (b) $4e^{-i3\pi/4}$ (c) $2e^{i\pi/2}$ (d) $3e^{-3\pi i}$
(e) $2e^{i4\pi}$ (f) $2e^{-i4\pi}$

$$\begin{aligned} \text{(a)} \quad 2e^{i\pi/3} &= 2(\cos \pi/3 + i \sin \pi/3) \\ &= 2(1/2 + i\sqrt{3}/2) \\ &= 1 + i\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 4e^{-i3\pi/4} &= 4(\cos(-3\pi/4) + i \sin(-3\pi/4)) \\ &= 4(-\sqrt{2}/2 - i\sqrt{2}/2) \\ &= -2\sqrt{2} - i2\sqrt{2} \end{aligned}$$

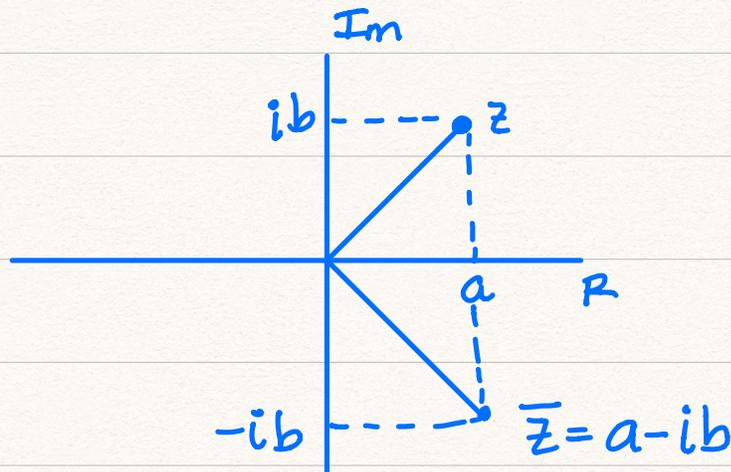


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Conjugate of a complex number



Definition

The complex conjugate of a number $z = a + ib$ is the number $\bar{z} = a - ib$

► Polar : $\bar{z} = r e^{-i\theta}$

► $z \bar{z} = (a + ib)(a - ib) = a^2 + b^2 = r^2 = |z|^2$

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$$\blacktriangleright \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$\blacktriangleright z + \bar{z} = 2\operatorname{Re}z, \quad z - \bar{z} = 2i\operatorname{Im}z$$

$$a + ib + a - ib = 2a \quad z = e^{i\theta}$$

Example. For $z_1 = 2e^{i\pi/4}$ and $z_2 = 8e^{i\pi/3}$,
find (a) $2z_1 - z_2$ (b) $\frac{1}{z_1}$ (c) $\frac{z_1}{z_2^2}$ (d) $\frac{\sqrt[3]{z_2}}{z_2^{1/3}}$

$$\begin{aligned} \text{(a) } 2z_1 - z_2 &= 4e^{i\pi/4} - 8e^{i\pi/3} \\ &= 4\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) - 8\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= 2\sqrt{2} + i2\sqrt{2} - 4 - i4\sqrt{3} \\ &= (2\sqrt{2} - 4) + i(2\sqrt{2} - 4\sqrt{3}) \end{aligned}$$

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$$(b) \frac{1}{z_1} = \frac{1}{2e^{i\pi/4}} = \frac{1}{2} e^{-i\pi/4}$$

\uparrow
 r

$\theta = -\pi/4$

$re^{i\theta}$
 $\frac{3\pi - 8\pi}{12}$

$$(c) \frac{z_1}{z_2^2} = \frac{2e^{i\pi/4}}{(8e^{i\pi/3})^2} = \frac{2e^{i\pi/4}}{64e^{i2\pi/3}} = \frac{1}{32} e^{i(\pi/4 - 2\pi/3)}$$

$$= \frac{1}{32} e^{-i5\pi/12}$$

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Example. For $z_1 = 3 + i4$ and $z_2 = 2 + i3$
determine $z_1 z_2$ and z_1 / z_2

$$\frac{z_1}{z_2} = \frac{3+i4}{2+i3} \cdot \frac{2-i3}{2-i3} = \frac{(3+i4)(2-i3)}{4+9}$$

$a+ib$
 $re^{i\theta}$

The logo for Cartagena99 features the text 'Cartagena99' in a stylized, dark green font. The '99' is significantly larger and more prominent than the 'Cartagena' part. The text is set against a background of light blue and orange geometric shapes, including a large blue triangle and an orange shape below it.

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$n^2 - n$ is even

$n(n-1)$

if n is odd, then $n-1$ is even so $n^2 - n$ is even.

if n is even, then the result is always even.

if n is a perfect square, then $\sqrt{n} \in \mathbb{Q}$

The logo for Cartagena99, featuring the text 'Cartagena99' in a stylized, bold font. The 'C' is large and blue, while the rest of the text is in a dark green color. The logo is set against a light blue and orange background.

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(b) $\sqrt{n} \notin \mathbb{Q}$ if n is not a perfect square (HINT: write $n = k^2 r$, where r does not contain any square factor),

If n is not a perfect square, then at least one of its factors is not a square. So we can write $n = k^2 r$ where r does not contain any square factors.

Now, we argue by contradiction. Suppose that n is not a perfect square and $\sqrt{n} \in \mathbb{Q}$.

Then we can write $\sqrt{n} = p/q$, $p, q \in \mathbb{N}$, where p and q have no common factors (p/q is in its simplest form).

Then $n = p^2/q^2 = k^2 r$ or, equivalently, $r = p^2/k^2$. But this is impossible because

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Roots of complex numbers

$$z = a + ib = r e^{i\theta + 2\pi im}$$

$$m \in \mathbb{Z}$$

$$r = \sqrt{a^2 + b^2}$$

arctan

$$\tan \theta = \frac{b}{a}$$

$$\theta = \tan^{-1} \frac{b}{a}$$

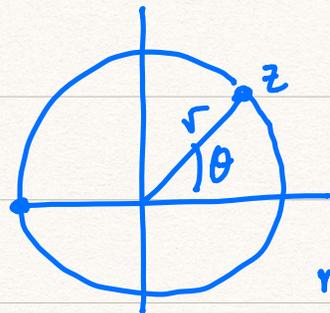
$$z^\alpha = (r e^{i\theta})^\alpha = r^\alpha e^{i\theta\alpha + 2\pi i\alpha m}$$

$$\alpha \in \mathbb{R}$$

$$\sqrt{z} = z^{1/2}$$

$$\frac{5\pi}{3} \Rightarrow 300^\circ$$

$$x^3 + 1 = 0$$



$$r = 1$$

$$(-1)^{1/3} = \left(e^{\pi i + 2\pi im} \right)^{1/3} = e^{\pi/3 i + \frac{2\pi}{3} im}$$

$$m=1: \frac{\pi}{3} i + \frac{2\pi}{3} i$$

$$m=2: \frac{\pi}{3} i + \frac{4\pi}{3} i$$



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