Suppose that f is defined on an interval (a,b), except possibly for some point $\xi \in (a,b)$. We say that $f(x)$ tends (or converges) to a limit l as x tends to ξ and write

$$f(x) \to l \quad \text{as} \quad x \to \xi$$

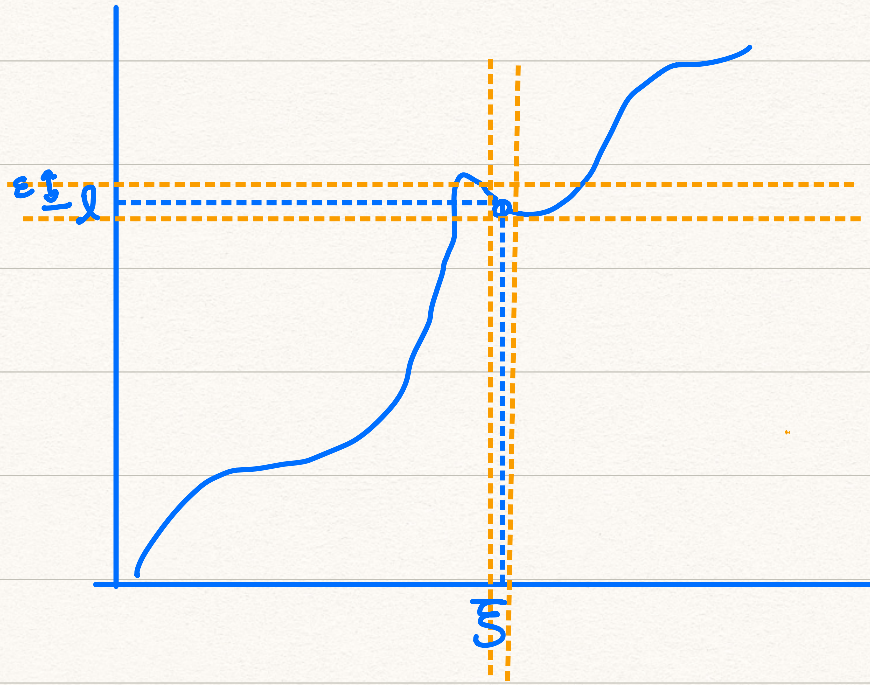
or, alternatively,

$$\lim_{x \to \xi} f(x) = l$$

if the following criterion is satisfied.

Given any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ $\quad l - \varepsilon < f(x) < l + \varepsilon$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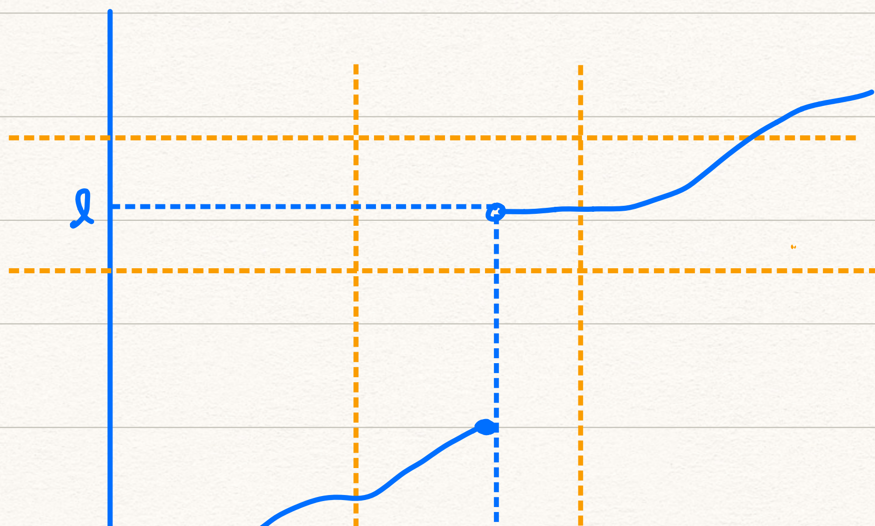
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Properties of Limits

Let f and g be defined on an interval (a,b) except possibly at $\xi \in (a,b)$. Suppose that $f(x) \to l$ and $g(x) \to m$ as $x \to \xi$ and Suppose that λ and μ are any real numbers. Then

1. $\lambda f(x) + \mu g(x) \longrightarrow \lambda l + \mu m$ as $x \to \xi$.
2. $f(x) g(x) \longrightarrow lm$ as $x \to \xi$.
3. $f(x)/g(x) \longrightarrow l/m$ as $x \to \xi$ (provided $m \neq 0$).

- For any polynomial $P(x)$

$$\lim_{x \to \xi} P(x) = P(\xi), \quad \forall \xi \in \mathbb{R}$$

Example. Calculate the following limits.

a) $\lim_{x\to 1} \frac{x^2+4}{x^2-4}$

b) $\lim_{x\to 0} \frac{x^{73}+5x^{42}+9}{3x^{23}+7}$

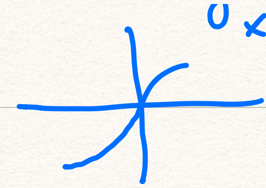
c) $\lim_{x\to 0} \frac{\frac{1}{(1-x)^3} - 1}{x}$

d) $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$

For any function $g(x)\to 0$ as $x\to \xi$ we have

$$\lim_{x\to \xi} \frac{\sin g(x)}{g(x)} = 1, \qquad \lim_{x\to \xi} \frac{1-\cos g(x)}{g(x)^2} = \frac{1}{2}$$

$0 \quad x\to 0$



a) $\lim_{x\to 1} \frac{x^2+4}{x^2-4} = \frac{5}{-3} = \frac{-5}{3}$

b) $\lim_{x\to 0} \frac{x^{73}+5x^{42}+9}{3x^{23}+7} = \frac{9}{7}$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

c) $\lim_{x\to 0} \frac{\frac{1}{(1-x)^3} - 1}{x} = \lim_{x\to 0} \frac{1-(1-x)^3}{x(1-x)^3}$

$= \lim_{x\to 0} \frac{1-(1-3x+3x^2-x^3)}{x(1-x)^3} = \lim_{x\to 0} \frac{3x-3x^2+x^3}{x(1-x)^3}$

$= \lim_{x\to 0} \frac{x(3-3x+x^2)}{x(1-x)^3} = 3$

Geogebra

d) $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim_{x\to a} \frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}$

$= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$

$(x-a)(x+a) = x^2 + \cancel{ax} - \cancel{ax} - a^2$

$= x^2 - a^2$

$(x-a)^2 = (x-a)(x-a) = x^2 - ax - ax + a^2$

$= x^2 - 2ax + a^2$

$\neq x^2 - a^2$

Examples

a) $\lim_{x\to 0} \frac{(\sin 2x^3)^2}{x^6}$ b) $\lim_{x\to 0} \frac{\tan x^2 + 2x}{x + x^2}$

c) $\lim_{x\to 0} \frac{\ln(1-2x)}{\sin x}$ d) $\lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{\sin x}{\sin x - x}}$

e) $\lim_{x\to 0} (1+\sin x)^{2/x}$ f) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$

One-sided limits

- We say that the left-handed limit of a function $f: A \to \mathbb{R}$ when x approaches ξ is l, and denote it

a) $$\lim_{x \to 0} \frac{(\sin 2x^3)^2}{x^6} = \frac{0}{0}$$

$$= \lim_{x \to 0} \left(\frac{2}{2} \frac{\sin 2x^3}{x^3} \right)^2$$

$$= \lim_{x \to 0} \left(2 \frac{\sin 2x^3}{2x^3} \right)^2 = 2^2 = 4$$

$$\lim_{x \to \xi} \frac{\sin g(x)}{g(x)} = 1$$

$\xi = 0 \quad g(x) = 2x^3$

$g \to 0$ when $x \to 0$

b) $$\lim_{x \to 0} \frac{\tan x^2 + 2x}{x + x^2} = \lim_{x \to 0} \frac{\tan x^2 + 2x}{x(1+x)}$$

$$= \lim_{x \to 0} \frac{\frac{\tan x^2}{x} + 2}{1+x} = \lim_{x \to 0} \frac{\frac{\sin x^2}{x \cos x^2} + 2}{1+x}$$

$$= \lim_{x \to 0} \frac{\frac{x}{x} \frac{\sin x^2}{x} \frac{1}{\cos x^2} + 2}{1+x} = \lim_{x \to 0} \frac{x \frac{\sin x^2}{x^2} \frac{1}{\cos x^2} + 2}{1+x}$$

$$= 2$$

if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(x)-\ell|<\varepsilon$$

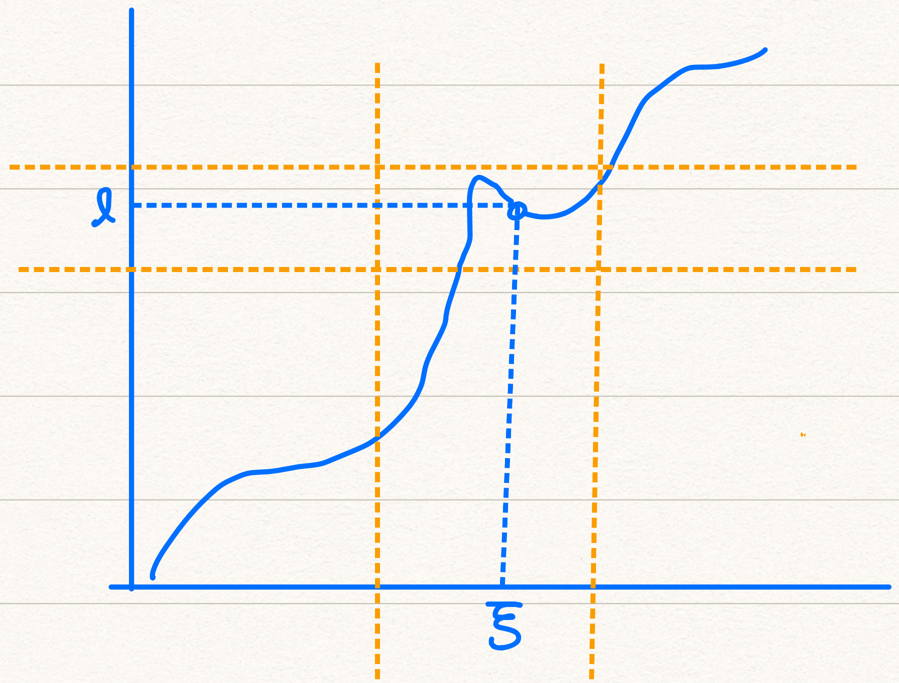
provided that $\xi-\delta<x<\xi$

- Similarly, we say that the right-handed limit of a function $f:A\to\mathbb{R}$ when x approaches ξ is ℓ, and denote it $\lim_{x\to\xi^+} f(x)=\ell$ or $f(x)\to\ell$ as $x\to\xi^+$

if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(x)-\ell|<\varepsilon$$

provided that $\xi<x<\xi+\delta$.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Proposition

$$\lim_{x \to \xi} f(x) = l \iff \lim_{x \to \xi^-} f(x) = \lim_{x \to \xi^+} f(x) = l$$

Infinite Limits

We write sample definitions. The student will have little difficulty in supplying the definitions in other cases.

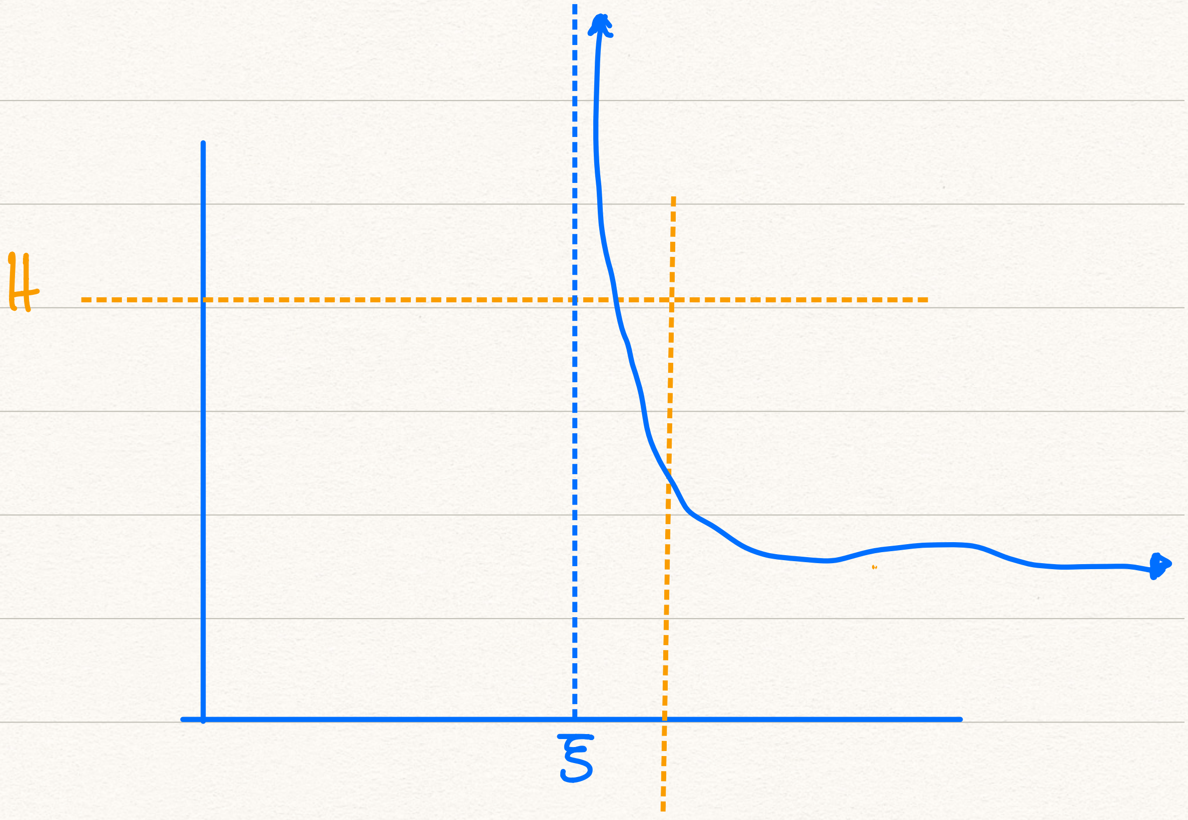
- We say that $f(x) \to +\infty$ as $x \to \xi^+$ if, given any $H>0$, we can find a $\delta>0$ such that

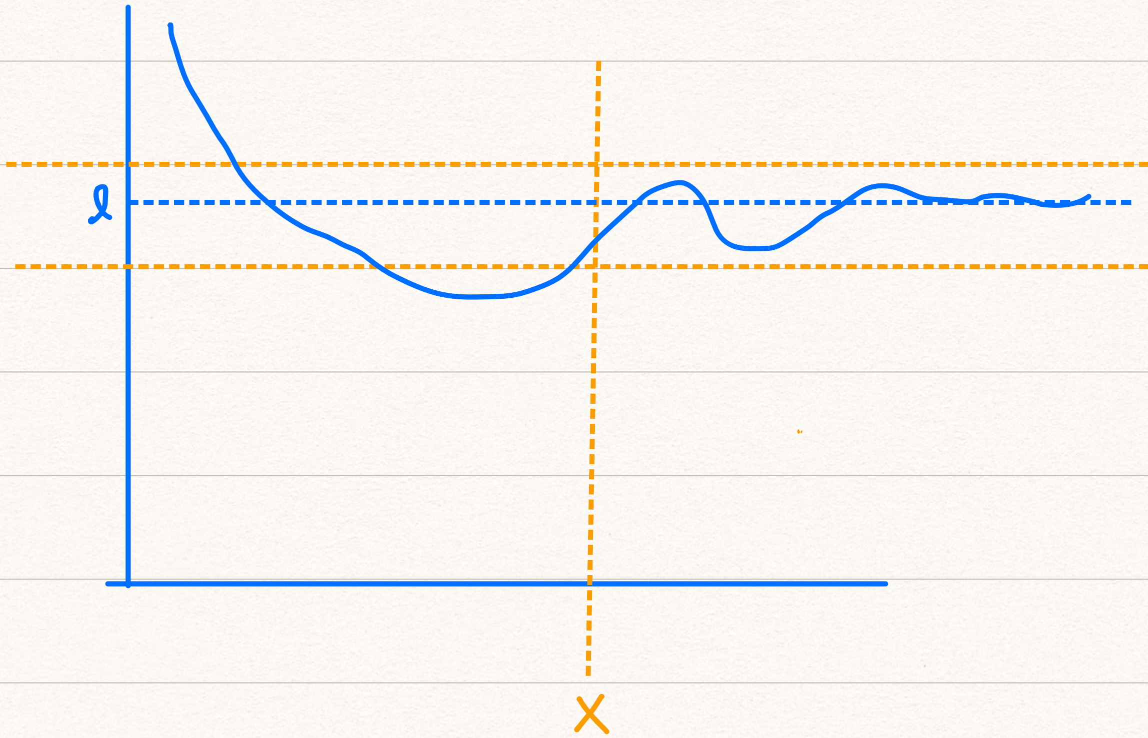
$$f(x) > H$$

- We say that $f(x) \to \ell$ as $x \to \infty$ if, given any $\varepsilon > 0$, we can find an X such that

$$|f(x) - \ell| < \varepsilon$$

provided that $x > X$.





Examples.

a.) $\lim_{x\to\infty} \sin x$

b) $\lim_{x\to\infty} \dfrac{x^3+4x-7}{7x^2-\sqrt{2x^6+x^5}}$

c). $\lim \quad x-2$

Sandwich Rule

If $\lim_{x\to\xi} g(x) = \lim_{x\to\xi} h(x) = \ell$, and if

$g(x) \le f(x) \le h(x)$ holds for all x in an interval

containing ξ, then $\lim_{x\to\xi} f(x) = \ell$

Example

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\lim_{x\to a} \sqrt{x} = \sqrt{a}$$