

Suppose that f is defined on an interval (a,b) , except possibly for some point $\xi \in (a,b)$.

We say that $f(x)$ tends (or converges) to a limit l as x tends to ξ and write

$$f(x) \rightarrow l \text{ as } x \rightarrow \xi$$

or, alternatively,

$$\lim_{x \rightarrow \xi} f(x) = l$$

if the following criterion is satisfied.

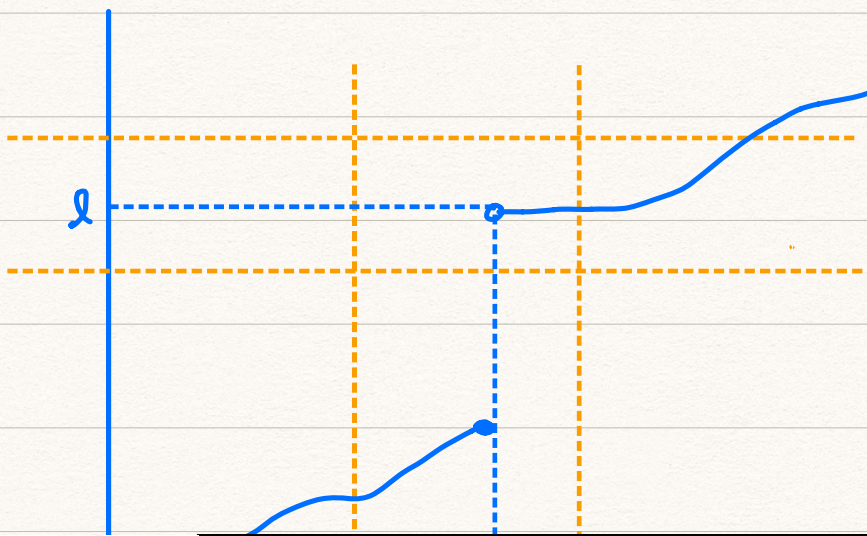
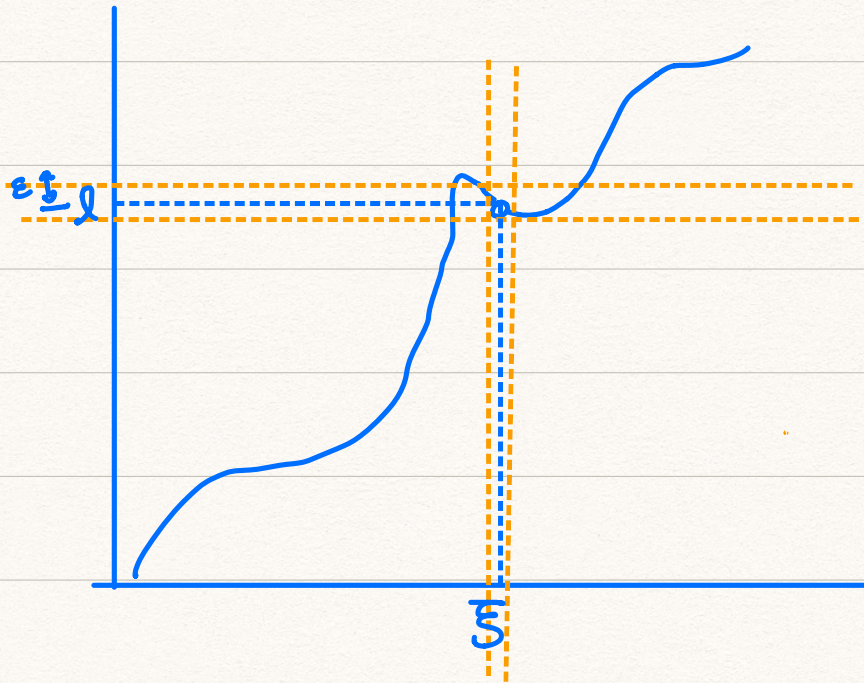
Given any $\epsilon > 0$, we can find a $\delta > 0$ such that

$$|f(x) - l| < \epsilon \quad l - \epsilon < f(x) < l + \epsilon$$

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Properties of Limits

Let f and g be defined on an interval (a,b) except possibly at $\xi \in (a,b)$. Suppose that

$f(x) \rightarrow l$ and $g(x) \rightarrow m$ as $x \rightarrow \xi$ and

Suppose that λ and μ are any real numbers.

Then

1. $\lambda f(x) + \mu g(x) \rightarrow \lambda l + \mu m$ as $x \rightarrow \xi$.

2. $f(x)g(x) \rightarrow lm$ as $x \rightarrow \xi$.

3. $f(x)/g(x) \rightarrow l/m$ as $x \rightarrow \xi$ (provided $m \neq 0$).

► For any polynomial $P(x)$

$$\lim_{x \rightarrow \xi} P(x) = P(\xi), \quad \forall \xi \in \mathbb{R}$$

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Example. Calculate the following limits.

$$a) \lim_{x \rightarrow 1} \frac{x^2 + 4}{x^2 - 4}$$

$$b) \lim_{x \rightarrow 0} \frac{x^{73} + 5x^{42} + 9}{3x^{23} + 7}$$

$$c) \lim_{x \rightarrow 0} \frac{1 - 1}{(1-x)^3} \cdot x$$

$$d) \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

For any function $g(x) \rightarrow 0$ as $x \rightarrow \xi$ we have

$$\lim_{x \rightarrow \xi} \frac{\sin g(x)}{g(x)} = 1,$$

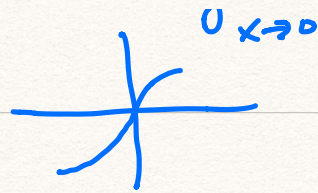
$$\lim_{x \rightarrow \xi} \frac{1 - \cos g(x)}{g(x)^2} = \frac{1}{2}$$

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$$a) \lim_{x \rightarrow 1} \frac{x^2 + 4}{x^2 - 4} = \frac{5}{-3} = -\frac{5}{3}$$



$$b) \lim_{x \rightarrow 0} \frac{x^{73} + 5x^{42} + 9}{3x^{23} + 7} = \frac{9}{7}$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$c) \lim_{x \rightarrow 0} \frac{1}{(1-x)^3} - 1 = \lim_{x \rightarrow 0} \frac{1 - (1-x)^3}{x(1-x)^3}$$

$$= \lim_{x \rightarrow 0} \frac{1 - (1 - 3x + 3x^2 - x^3)}{x(1-x)^3} = \lim_{x \rightarrow 0} \frac{3x - 3x^2 + x^3}{x(1-x)^3}$$

$$= \lim_{x \rightarrow 0} \frac{x(3 - 3x + x^2)}{x(1-x)^3} = 3$$

Geogebra

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$$d) \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \lim_{x \rightarrow a} \frac{\cancel{x} - a}{(\cancel{x} - a)(\sqrt{x} + \sqrt{a})}$$

$$= \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

$$\begin{aligned} (x-a)(x+a) &= x^2 + \cancel{ax} - \cancel{ax} - a^2 \\ &= x^2 - a^2 \end{aligned}$$

$$\begin{aligned} (x-a)^2 &= (x-a)(x-a) = x^2 - ax - ax + a^2 \\ &= x^2 - 2ax + a^2 \\ &\neq x^2 - a^2 \end{aligned}$$

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Examples

$$a) \lim_{x \rightarrow 0} \frac{(\sin 2x^3)^2}{x^6}$$

$$b) \lim_{x \rightarrow 0} \frac{\tan x^2 + 2x}{x + x^2}$$

$$c) \lim_{x \rightarrow 0} \frac{\ln(1-2x)}{\sin x}$$

$$d) \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right)^{\frac{\sin x}{\sin x - x}}$$

$$e) \lim_{x \rightarrow 0} (1 + \sin x)^{2/x}$$

$$f) \lim_{x \rightarrow 0} \frac{1 - \sqrt{1-x^2}}{x^2}$$

One-sided limits

► We say that the left-handed limit of a function $f: A \rightarrow \mathbb{R}$ when x approaches ξ is l , and denote it

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$$a) \lim_{x \rightarrow 0} \frac{(\sin 2x^3)^2}{x^6} = \frac{0}{0} \neq$$

$$= \lim_{x \rightarrow 0} \left(\frac{2 \sin 2x^3}{2 x^3} \right)^2$$

$$= \lim_{x \rightarrow 0} \left(2 \frac{\sin 2x^3}{2x^3} \right)^2 = 2^2 = 4$$

$$\lim_{x \rightarrow \xi} \frac{\sin g(x)}{g(x)} = 1$$

$$\xi = 0 \quad g(x) = 2x^3$$

$g \rightarrow 0$ when $x \rightarrow 0$



$$b) \lim_{x \rightarrow 0} \frac{\tan x^2 + 2x}{x + x^2} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{\tan x^2 + 2x}{x(1+x)}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan x^2}{x} + 2}{1+x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x^2}{x \cos x^2} + 2}{1+x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{x}{x} \frac{\sin x^2}{x} \frac{1}{\cos x^2} + 2}{1+x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x^2}{x^2} \frac{1}{\cos x^2} + 2}{1+x}$$

$$= 2$$

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if for every $\varepsilon > 0$ there exists a $\delta > 0$
such that

$$|f(x) - l| < \varepsilon$$

provided that $\xi - \delta < x < \xi$

► Similarly, we say that the right-handed
limit of a function $f: A \rightarrow \mathbb{R}$ when x
approaches ξ is l , and denote it

$$\lim_{x \rightarrow \xi^+} f(x) = l \quad \text{or} \quad f(x) \rightarrow l \quad \text{as} \quad x \rightarrow \xi^+$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$
such that

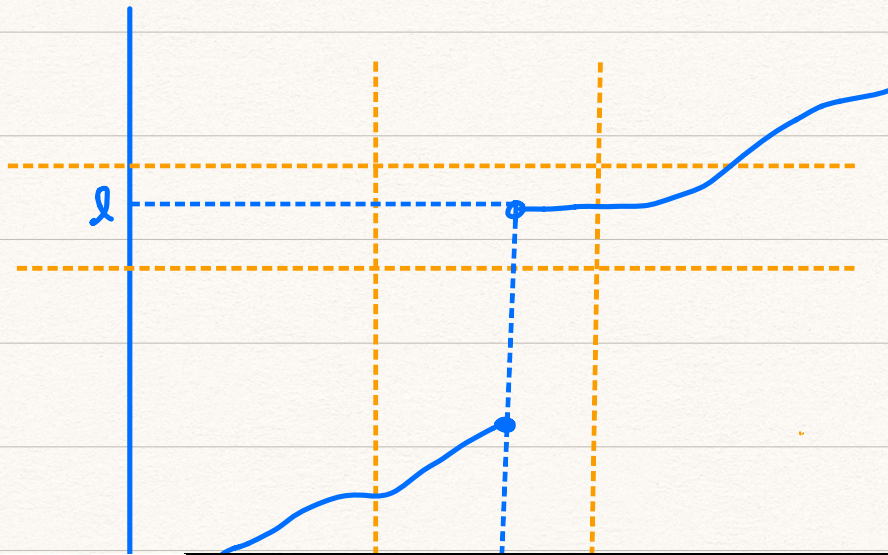
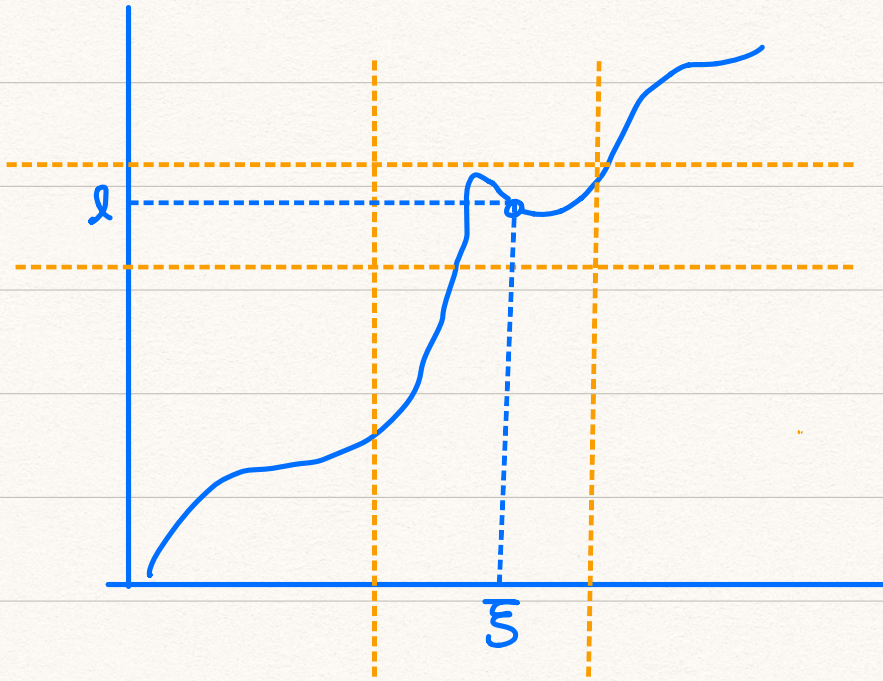
$$|f(x) - l| < \varepsilon$$

provided that $\xi < x < \xi + \delta$.

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Proposition

$$\lim_{x \rightarrow \xi} f(x) = l \iff \lim_{x \rightarrow \xi^-} f(x) = \lim_{x \rightarrow \xi^+} f(x) = l$$

Infinite Limits

We write sample definitions. The student will have little difficulty in supplying the definitions in other cases.

- We say that $f(x) \rightarrow +\infty$ as $x \rightarrow \xi^+$ if, given any $H > 0$, we can find a $\delta > 0$ such that
$$f(x) > H$$

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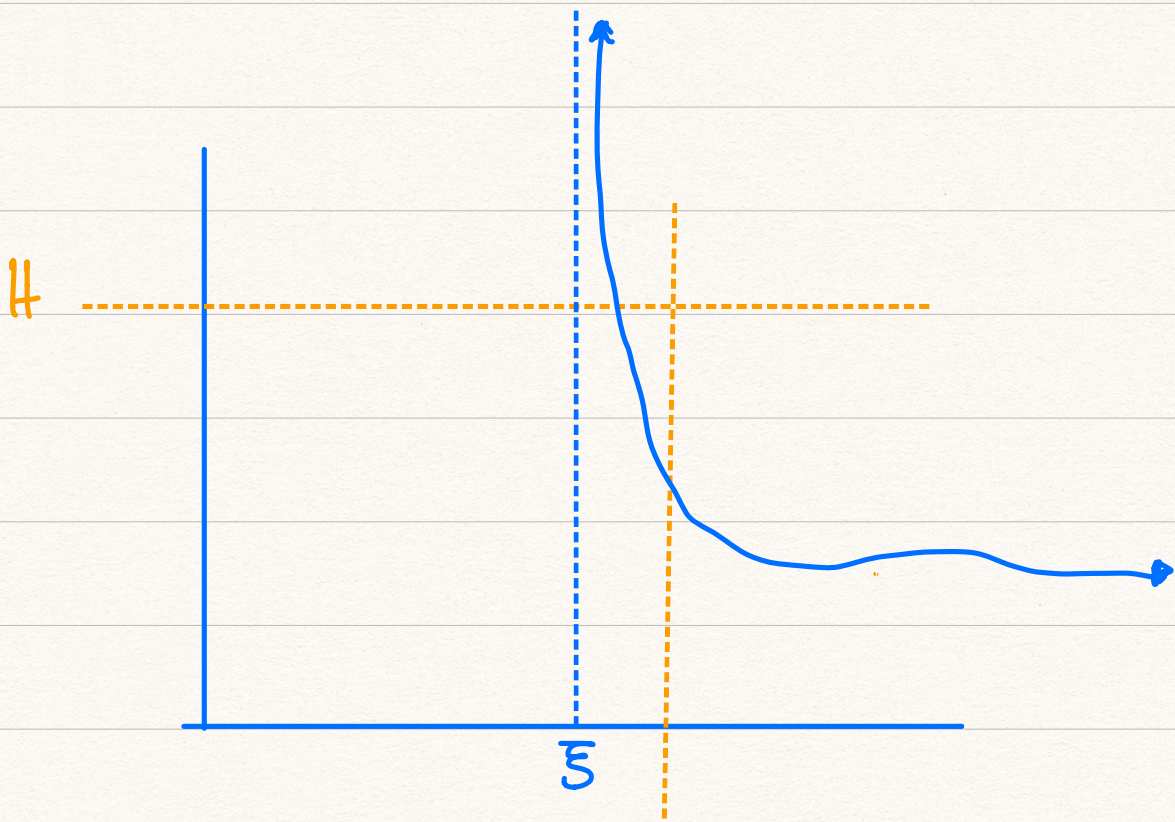
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► We say that $f(x) \rightarrow l$ as $x \rightarrow \infty$ if, given any $\epsilon > 0$, we can find an X such that

$$|f(x) - l| < \epsilon$$

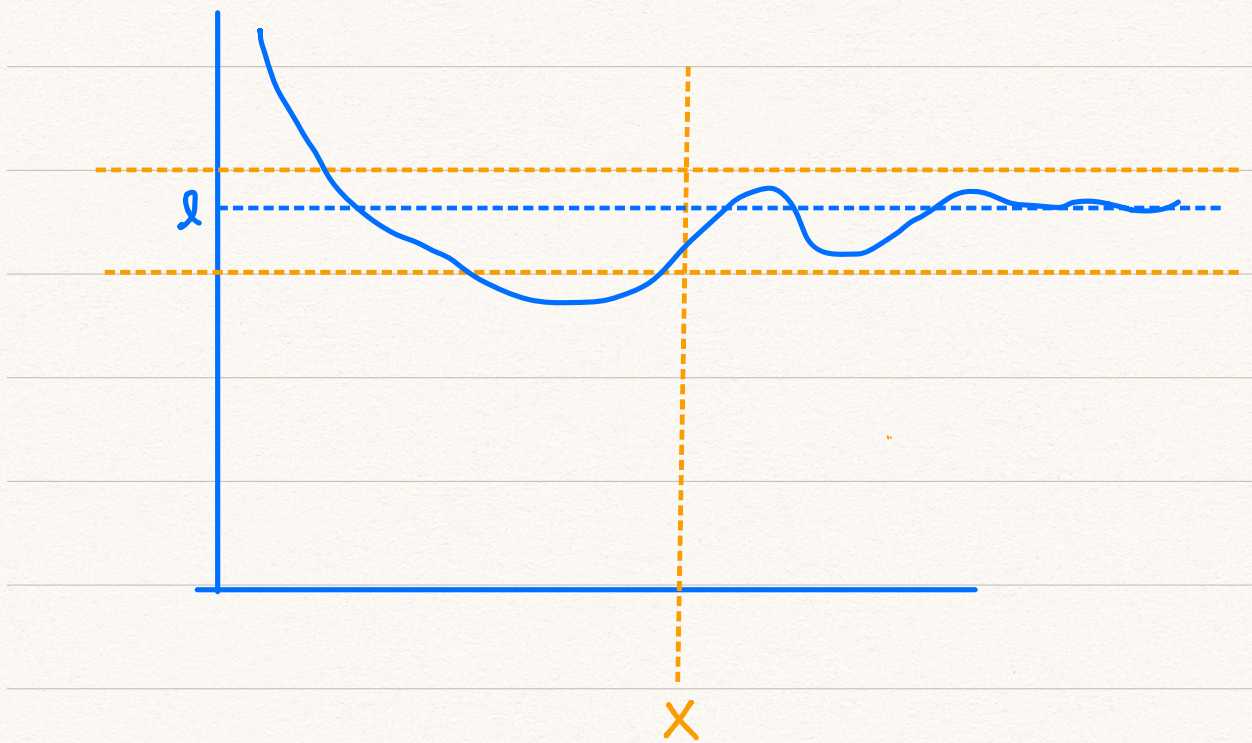
provided that $x > X$.



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Examples.

a.) $\lim_{x \rightarrow \infty} \sin x$

b.) $\lim_{x \rightarrow \infty} \frac{x^3 + 4x - 7}{7x^2 - \sqrt{2x^6 + x^5}}$

c.) $\lim_{x \rightarrow 2} x - 2$

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Sandwich Rule

If $\lim_{x \rightarrow \xi} g(x) = \lim_{x \rightarrow \xi} h(x) = l$, and if $g(x) \leq f(x) \leq h(x)$ holds for all x in an interval containing ξ , then $\lim_{x \rightarrow \xi} f(x) = l$

Example

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$$

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