Calculus: Chapter 4

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In this chapter we study a fundamental concept for infinitesimal calculus and mathematics in general: the continuity of functions.

Definition 1 (continuity):
Suppose that $f(x)$ is defined for all x of an open interval that contains c. Then, the function is *continuous* if and only if

$$\lim_{x\to c} f(x) = f(c).$$

Comments:
(1) In contrast to the definition of limits, it is necessary that the function exists at a point c for being continuous there (otherwise $f(c)$ would not exist there). Furthermore, while the existence of the limit in c is necessary for continuity, it is not sufficient: the limit must coincide with $f(c)$.

(2) We can express the definition also in terms of ϵ and δ:
A function is continuous if and only if for all $\epsilon > 0$ exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \quad \text{if} \quad 0 < |x - c| < \delta,$$

where the limit value L has been replaced by $f(c)$.

(3) The methods to calculate limits apply also for checking for continuity.

(4) Continuity is a property of a function at one value $x = c$. Nevertheless, it is possible to generalise the concept to intervals and often we refer to *continuous functions*, functions which are continuous in their domain.

(5) In the examples for limits where we could insert $x = c$, actually we were checking for continuity.

(6) If a function is continuous in c, we therefore know that the limit there exists.

(7) We encounter the same cases as with the limits (function tends to $\pm\infty$, function oscillates, lateral limits do not coincide, etc.), with the simple question added: If the limit exists, does it coincide with $f(c)$? Therefore, we can also define lateral continuity.

Definition 2 (lateral continuity):
A function f is:

$$\begin{aligned} &\text{left continuous in } x = c \qquad \text{if and only if } \lim_{x\to c-} f(x) = f(c), \\ &\text{right continuous in } x = c \qquad \text{if and only if } \lim_{x\to c+} f(x) = f(c). \end{aligned}$$

Theorem 1 (continuity and lateral continuity):
A function is continuous at a point if and only if it is left and right continuous there.

Definition 3 (continuity in intervals):
A function f is continuous in

$$\begin{aligned} (a,b) &\quad \text{if } f \text{ is continuous for all } x \in (a,b); \\ [a,b) &\quad \text{if } f \text{ is continuous in } (a,b) \text{ and right continuous in } a; \\ (a,b] &\quad \text{if } f \text{ is continuous in } (a,b) \text{ and left continuous in } b; \\ [a,b] &\quad \text{if } f \text{ is continuous in } (a,b),\ \text{right continuous in } a \text{ and left continuous in } b; \end{aligned}$$

Comments:
(1) Continuity in a closed interval actually only means left and right continuity in its boundaries. This is not a contradiction to Theorem 1 (at most an abuse of notation) since there continuity at *points* is discussed, not in intervals.

(2) Note also that, for example, for an open interval $I = (a,b)$, obviously $a \notin I$ and the function cannot be continuous at a point outside its domain.

Let us classify the discontinuities:
(1) $f(c)$ and $\lim_{x\to c} f(x)$ both exist, but do not coincide. This discontinuity is called *evitable*, because we have the possibility to redefine the function by setting $f(c) = \lim_{x\to c} f(x)$ as condition. By doing so, we change the function in one point only.

(2) The lateral limits exist but not not coincide. This *jump discontinuity* is not evitable. The value $f(c)$ may exist or not, and may coincide with one of the limits or not.

(3) At least one lateral limit tends to $\pm\infty$ (or oscillates, like for $\lim_{x\to 0} f(x) = \sin(1/x)$) and hence the limit does not exist. This represents an *essential discontinuity*.

In Fig. 1 we show the most important cases. A simple rule to find out whether a function is continuous is to draw its graph: only if you can do it without lifting the pen, it may be continuous.

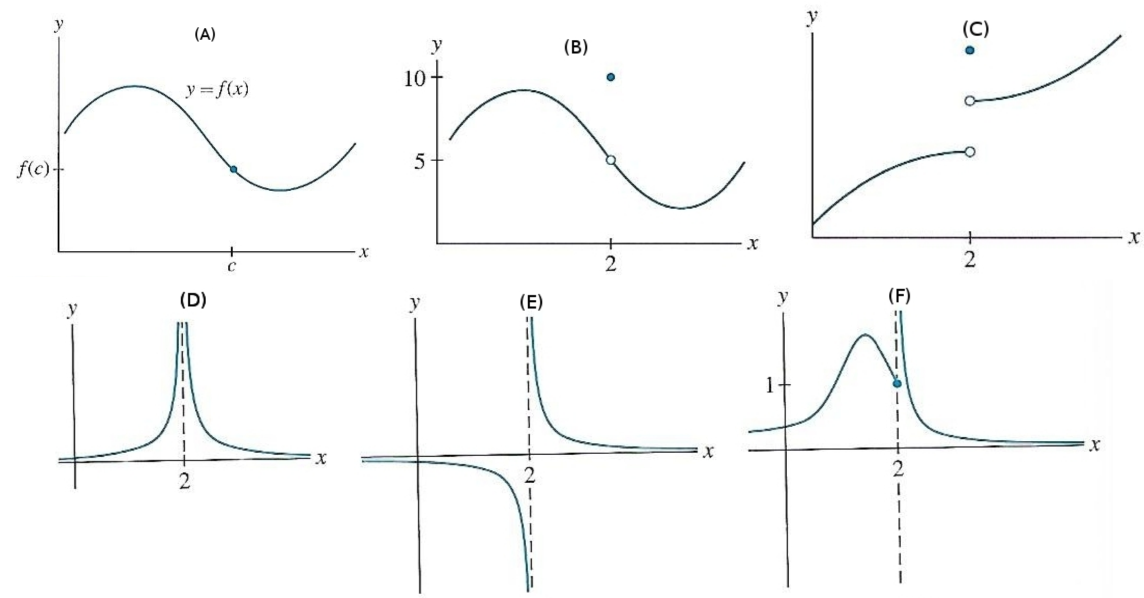

Figure 1: Graphs of a continuous function (A), an evitable discontinuity (B), a jump discontinuity (C) and several cases of essential discontinuity (D-F), from p. 62 (A), 63 (B,C) and 64 (D-F) of [1].

Example 1: Study the continuity of $f(x)$, given by

$$f(x) = \begin{cases} f_1(x) = \dfrac{x^2 - 4x + 4}{x - 2} & \text{if } x \neq 2, \\ f_2(x) = -4 & \text{if } x = 2. \end{cases}$$

For $x \neq 2$ we can simplify:

$$\frac{x^2 - 4x + 4}{x - 2} = \frac{(x-2)^2}{x - 2} = x - 2.$$

As $x = 2$ is the point excluded from the domain of $f_1(x)$ according to the definition of $f(x)$, we can reformulate:

$$f(x) = \begin{cases} f_1(x) = x - 2 & \text{if } x \neq 2, \\ f_2(x) = -4 & \text{if } x = 2. \end{cases}$$

Now we proceed to study the continuity de $f(x)$. Both f_1 and f_2 represent polynomials which are continuos in $\mathbb{R}$ and hence in their domains. The point to consider is $x = 2$. We have to calculate $\lim_{x \to 2} f(x)$ and determine whether it coincides with $f(2)$, (i.e., to calculate $\lim_{x \to 2} f_1(x)$ and compare with $f_2(2)$). In this example, $\lim_{x \to 2} f(x) = 0 \neq f(2) = -4$

and hence the function is not continuous, but with an evitable discontinuity since we could redefine $f_2(2) = 0$, rendering $f(x)$ continuous.

We have studied limits as $x \to \pm\infty$ and we discussed whether the function approaches a finite value (horizontal asymptote) or not. There is no equivalent concept for continuity as there is no value $f(\pm\infty)$ to compare with.

3.1 Properties of continuous functions

Given the intimate relation between limits and continuity, it is not surprising to find that many properties of limits translate directly to continuous functions:

Theorem 2 (basic properties of continuous functions):
Suppose that $f(x)$ and $g(x)$ are continuous in $x = c$, then the following functions are also continuous in $x = c$:
(i) $f(x) + g(x)$ and $f(x) - g(x)$;
(ii) $kf(x)$, $k \in \mathbb{R}$;
(iii) $f(x)g(x)$;
(iv) $f(x)/g(x)$ if $g(c) \neq 0$.

Theorem 3 (some fundamental continuous functions):
(i) The polynomial $P(x)$ is continuous on the real line (defined and continuous for all real values of x);
(ii) The rational function $P(x)/Q(x)$ (with $P(x)$ and $Q(x)$ being polynomials) is continuous in its domain (all values $x = c$ such that $Q(c) \neq 0$);
(iii) $f(x) = x^{1/n}$ is continuous in its domain, for n being a positive integer number;
(iv) $f(x) = \sin x$ and $f(x) = \cos x$ are continuous on the real line;
(v) $f(x) = b^x$ is continuous on the real line (for $b > 0$, $b \neq 1$);
(vi) $f(x) = \log_b x$ is continuous on the positive real line (for $b > 0$, $b \neq 1$).

We have to clarify what happens in the case of composite functions.

Theorem 4 (composite functions):
If $g(x)$ is continuous in $x = c$ and $f(y)$ is continuous in $y = g(c)$, then the composite function $f(g(x))$ is continuous in $x = c$.

These theorems assure the continuity of functions built by the operations of addition, subtraction, multiplication, division and composition of elementary functions (e.g. polynomials, roots and rational functions, exponential and logarithmic functions). We always have to take into account that the domain is not necessarily the real line and that it is possible that isolated values (denominator zero) or intervals (e.g. logarithms with non-positive argument; roots with negative argument) are excluded.

Example 2: Justify that $h(x) = \sin(x^2)$ is a continuous function.

Function h is a composite function: $h(x) = f(g(x))$, with $f(y) = \sin(y)$ and $g(x) = x^2$. Theorem 3 establishes that both f and g are continuous functions in their domain, the real line $\mathbb{R}$. Consequently, according to Theorem 4, the composite function $\sin(x^2)$ is continuous for all $x \in \mathbb{R}$. Additional comment: the composite function in inverse order, $g(f(x)) = (\sin(x))^2 = \sin^2(x)$ is also a continuous function due to the same argument.

We finish the chapter with two important theorems about continuous functions.

Theorem 5 (boundedness):
Let f be a continuous function on $[a, b]$ compact. Then,
(i) f is bounded on $[a, b]$, i.e, there exist $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$.
(ii) f takes both minimum and maximum values on $[a, b]$, i.e., there exist $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

Comments:
(1) A compact interval $[a, b]$ is closed and bounded (neither $a \to -\infty$ nor $b \to +\infty$).

(2) Theorem 5 (ii) is also known as the Weierstrass theorem. It is important for proving properties of continuous and differentiable functions.

Theorem 6 (intermediate value theorem, IVT):
Let f be a continuous function on $[a, b]$ with $f(a) \neq f(b)$. Then, for all u between $f(a)$ and $f(b)$ there exist at least one $c \in (a, b)$ such that $f(c) = u$.

Comments:
(1) In Fig. 2 we illustrate the IVT.
(2) The IVT seems to state an obvious property: a continuous function cannot make jumps. For example, if a child grows in a year from 1,23m to 1,37m, there is a date in that year on which the child was exactly 1,28m tall (and any other height between 1,23m and 1,37m).

(3) A direct consequence of the IVT is Bolzano's theorem which confirms that if $f(a)$ and $f(b)$ are non-zero and of opposite sign, then $f(x)$ has to have at least one root between a and b, i.e., there exist an $r \in (a, b)$ with $f(r) = 0$.

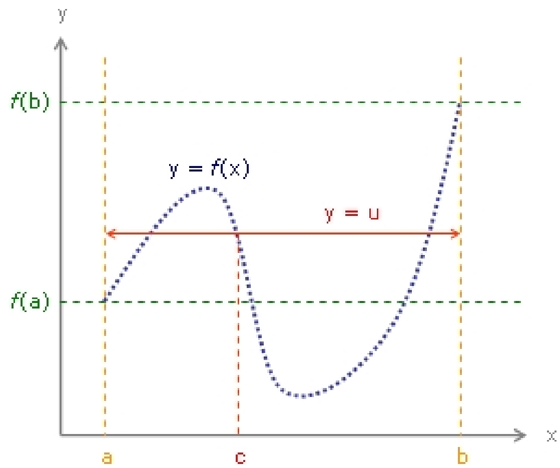

Figure 2: Illustration of the intermediate value theorem. Source: [2].

Bibliografía

[1] Jon Rogawski, *Cálculo*, Ed. Reverté, segunda edición, Barcelona, 2012.
[2] CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=414399