Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica
Lecture 6

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Outline

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Limits

The **concept of limit** is very important in calculus. It enable to speak about derivatives and integrals, the main and most powerful concept in this area of study. One dimensional limits can be extended to several dimension in a similar way.

Limit of a scalar-valued function of a single variable

- For a scalar-valued function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ of a single variable we know the statement

$$\lim_{x \to a} f(x) = L$$

The limit of $f(x)$
as x approaches a is L

- Intuitively, we can make the numerical value of $f(x)$ arbitrarily close to L by keeping x sufficiently close (but not equal) to a.

Limit of a vector-valued function

- This idea generalizes to functions $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

- We can make the vector $\mathbf{f}(\mathbf{x})$ arbitrarily close to the limit vector $\mathbf{L}$ by keeping the vector $\mathbf{x}$ sufficiently close (but not equal) to $\mathbf{a}$.
- The word "close" means that the distance between $\mathbf{f}(\mathbf{x})$ and $\mathbf{L}$ is small.

This offers a first definition of limit
using the notation for distance

Definition 2.1: Intuitive Definition of Limit

- Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$
- Consider the equation

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

- It means that we can make $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\|$ arbitrarily small (i.e., near zero) by keeping $\|\mathbf{x} - \mathbf{a}\|$ sufficiently small (but nonzero).

Note

- If f is a scalar-valued function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ the vector length $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\|$ can be replaced by the absolute value

$$|f(\mathbf{x}) - L|$$

- Similarly, if f is a function of just one variable, then $\|\mathbf{x} - \mathbf{a}\|$ can be replaced by

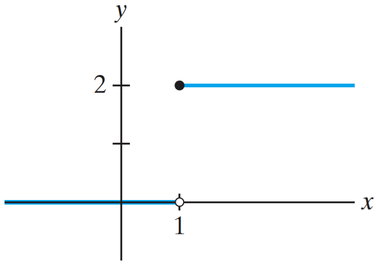
$$|x - a|$$

Example 1

- Suppose that $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2 & \text{if } x \geq 1 \end{cases}$$

- What should $\lim_{x \to 1} f(x)$ be?



Example 2

- Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathbf{f}(\mathbf{x}) = \mathbf{5x}$$

That is, **f** is five times
the **identity function**

- It should be obvious intuitively that

$$\lim_{\mathbf{x}\to\mathbf{i}+\mathbf{j}} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{i}+\mathbf{j}} \mathbf{5x} = 5\mathbf{i} + 5\mathbf{j}$$

- Indeed, if we write $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$, then

$$\begin{aligned} &\|\mathbf{f}(\mathbf{x}) - (5\mathbf{i} + 5\mathbf{j})\| = \|(5x\mathbf{i} + 5y\mathbf{j}) - (5\mathbf{i} + 5\mathbf{j})\| \\ = \quad &\|5(x-1)\mathbf{i} + 5(y-1)\mathbf{j}\| = \sqrt{25(x-1)^2 + 25(y-1)^2} \\ = \quad &5\sqrt{(x-1)^2 + (y-1)^2} = 5\|x - (\mathbf{i} + \mathbf{j})\| \end{aligned}$$

Example 2

$$\|\mathbf{f}(\mathbf{x}) - (5\mathbf{i} + 5\mathbf{j})\| = 5\sqrt{(x-1)^2 + (y-1)^2}$$

- This last quantity can be made as small as we wish by keeping

$$\|\mathbf{x} - (\mathbf{i} + \mathbf{j})\| = \sqrt{(x-1)^2 + (y-1)^2}$$

sufficiently small.

Definition 2.2: Rigorous Definition of Limit

- Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function
- Expression

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

  means that given any $\varepsilon > 0$, you can find a $\delta > 0$ (which will, in general, depend on ε) such that if $\mathbf{x} \in X$ and $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon$
- The condition $0 < \|\mathbf{x} - \mathbf{a}\|$ simply means that we care only about values $\mathbf{f}(\mathbf{x})$ when $\mathbf{x}$ is near $\mathbf{a}$, but not equal to $\mathbf{a}$.
- This definition is not easy to use in practice.

The evaluation of the limit of a function
of more than one variable is, in general, a difficult task

Outline

Closed Ball and Open Ball

- Recall the vector equation

$$\|\mathbf{x} - \mathbf{a}\| = r$$

where $\mathbf{x}$ and $\mathbf{a}$ are in $\mathbb{R}^3$ and $r > 0$.

- It defines a sphere of radius r centered at $\mathbf{a}$.
- We modify this equation so that it becomes the inequality,

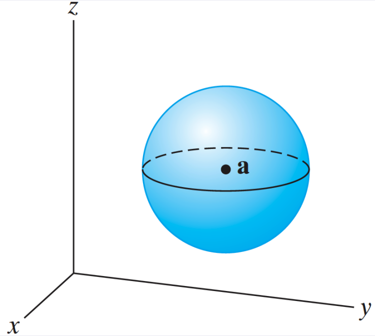
$$\|\mathbf{x} - \mathbf{a}\| \leq r$$

- The points $\mathbf{x} \in \mathbb{R}^3$ that satisfy it fill out what is called a closed ball.

Closed Ball and Open Ball

$$\|\mathbf{x} - \mathbf{a}\| \leq r$$

A closed ball



Closed Ball and Open Ball

- Similarly, we can consider the strict inequality

$$\|\mathbf{x} - \mathbf{a}\| < r$$

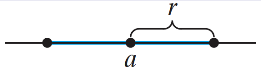
- It describes points $\mathbf{x} \in \mathbb{R}^3$ that are a distance of less than r from $\mathbf{a}$.
- Such points determine an open ball of radius r centred at $\mathbf{a}$.

A solid ball
without the boundary sphere

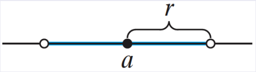
Closed Ball and Open Ball

- The same definition of closed and open balls can be directly apply to $\mathbb{R}^n$.
- But we cannot draw sketches when $n > 3$.
- For $n = 1$, they are intervals,

A closed ball in $\mathbb{R}$



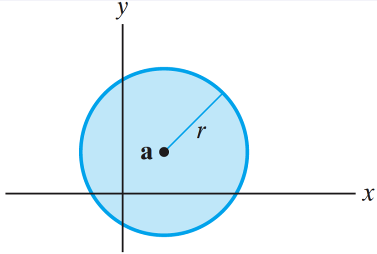
An open ball in $\mathbb{R}$



Closed Ball and Open Ball

- The same definition of closed and open balls can be directly apply to $\mathbb{R}^n$
- But we cannot draw sketches when $n > 3$
- For $n = 2$ they are disks

A closed ball in $\mathbb{R}^2$



Closed Ball and Open Ball

- The same definition of closed and open balls can be directly apply to $\mathbb{R}^n$
- But we cannot draw sketches when $n > 3$
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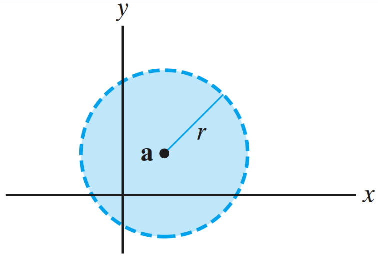
An open ball in $\mathbb{R}^2$

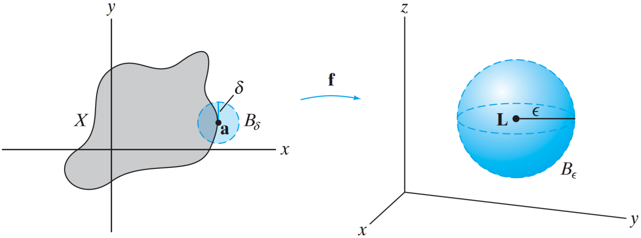


Geometric interpretation

- Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function
- We look for the geometric meaning of the statement

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

- Given any $\varepsilon > 0$, you can find a corresponding $\delta > 0$ such that
  - If points $\mathbf{x} \in X$ are inside an open ball of radius δ centered at $\mathbf{a}$
  - Then the corresponding points $\mathbf{f}(\mathbf{x})$ will remain inside an open ball of radius $\varepsilon > 0$ centered at $\mathbf{L}$



Outline

Theorem 2.4: Uniqueness of Limits

If a limit exists,
it is unique

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. If

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

and also,

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{M}$$

Then,

$$\mathbf{L} = \mathbf{M}$$

Theorem 2.5: Algebraic Properties of the Limits

- Let $\mathbf{F}, \mathbf{G} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions
- Let $f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be scalar-valued functions
- Let $k \in \mathbb{R}$ be a scalar

1. If
$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{L} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{G}(\mathbf{x}) = \mathbf{M}$$
Then
$$\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{F} + \mathbf{G})(\mathbf{x}) = \mathbf{L} + \mathbf{M}$$
2. If
$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{L}$$
Then
$$\lim_{\mathbf{x}\to\mathbf{a}} k\mathbf{F}(\mathbf{x}) = k\mathbf{L}$$

Theorem 2.5: Algebraic Properties of the Limits

- Let $\mathbf{F}, \mathbf{G} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions
- Let $f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be scalar-valued functions
- Let $k \in \mathbb{R}$ be a scalar

3. If

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M$$

Then

$$\lim_{\mathbf{x}\to\mathbf{a}} (fg)(\mathbf{x}) = LM$$

4. If

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L, \quad g(\mathbf{x}) \neq 0 \quad \text{for } \mathbf{x} \in X, \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M \neq 0$$

Then

$$\lim_{\mathbf{x}\to\mathbf{a}} (f/g)(\mathbf{x}) = L/M$$

Example 11

Evaluate

$$\lim_{(x,y)\to(a,b)} x^2 + 2xy - y^3$$

- By intuition (if not faith),

$$\lim_{(x,y)\to(a,b)} x = a \quad \text{and} \quad \lim_{(x,y)\to(a,b)} y = b$$

- From these facts, it follows from Theorem 2.5 that

$$\lim_{(x,y)\to(a,b)} x^2 + 2xy - y^3 = \lim x^2 + \lim 2xy + \lim(-y^3)$$
$$= (\lim x)^2 + 2(\lim x)(\lim y) - (\lim y)^3 = a^2 + 2ab - b^3$$

Example 13

Evaluate

$$\lim_{(x,y)\to(-1,0)} \frac{x^2 + xy + 3}{x^2y - 5xy + y^2 + 1}$$

- Proceeding in a similar way to Example 11

$$\lim_{(x,y)\to(-1,0)} x^2 + xy + 3 = 4$$

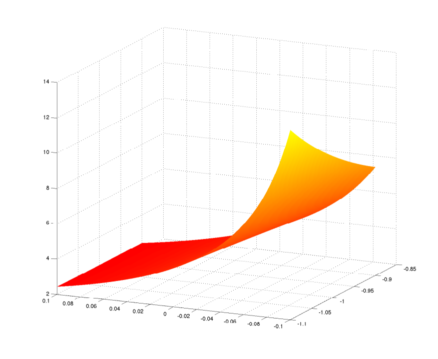
$$\lim_{(x,y)\to(-1,0)} x^2y - 5xy + y^2 + 1 = 1 \neq 0$$

- Thus

$$\lim_{(x,y)\to(-1,0)} \frac{x^2 + xy + 3}{x^2y - 5xy + y^2 + 1} = \frac{4}{1} = 4$$

Example 13

$$\lim_{(x,y)\to(-1,0)} \frac{x^2 + xy + 3}{x^2 y - 5xy + y^2 + 1} = 4$$



Example 14

Not all limits of quotient expressions are as simple to evaluate as that of Example 13

Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^4}{x^2 + y^4}$$

- For one hand, it holds

$$\lim_{(x,y)\to(0,0)} x^2 + y^4 = 0$$

- On the other hand, it also holds

$$\lim_{(x,y)\to(0,0)} x^2 - y^4 = 0$$

Example 14

Not all limits of quotient expressions are as simple to evaluate as that of Example 13

Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^4}{x^2 + y^4}$$

- Therefore, as $(x, y) \to 0$, the expression $\frac{x^2-y^4}{x^2+y^4}$ becomes indeterminate
- We cannot use Theorem 2.5 to evaluate this limit.
- We use a different approach.

Example 14

Not all limits of quotient expressions are as simple to evaluate as that of Example 13

Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^4}{x^2 + y^4}$$

- Note that

$$\lim_{x\to 0,\ \text{along } y=0} \frac{x^2 - y^4}{x^2 + y^4} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

- In a similar way

$$\lim_{y\to 0,\ \text{along } x=0} \frac{x^2 - y^4}{x^2 + y^4} = \lim_{y\to 0} \frac{-y^4}{y^4} = -1$$

Example 14

Not all limits of quotient expressions
are as simple to evaluate as that of **Example 13**

Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^4}{x^2 + y^4}$$

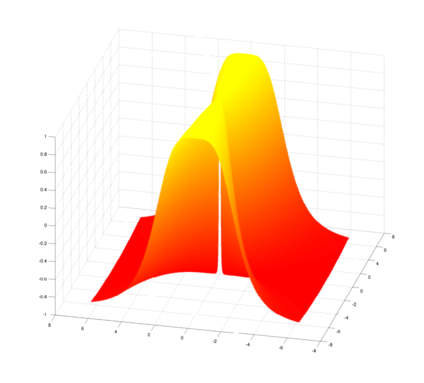
$$\lim_{x\to 0,\ \text{along } y=0} \frac{x^2 - y^4}{x^2 + y^4} = 1 \neq -1 = \lim_{y\to 0,\ \text{along } x=0} \frac{x^2 - y^4}{x^2 + y^4}$$

Thus, this limit
does not exist

Example 14

Evaluate,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^4}{x^2 + y^4}$$



Theorem 2.6

- Suppose $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function.
- Suppose also that $\mathbf{L} = (L_1, \ldots, L_m)$.
- Then

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

if and only if

$$\lim_{\mathbf{x} \to \mathbf{a}} f_i(\mathbf{x}) = L_i \quad \text{for} \quad i = 1, \ldots, m$$

Evaluating the limit of a vector-valued function is equivalent to evaluating the limits of its scalar-valued component functions

Example 8

Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

1. Compute the limit through lines of the form $y = mx$. What is the limit candidate? May it have limit?
2. Compute the limit through the line $x = 0$. Is the result in accordance with step 1?
3. Try to compute the limit changing to polar coordinates.
4. Does the function have limit?
5. Repeat the same steps for the function in example 14,

$$f(x, y) = {}^{x^2 - y^4}\!/_{x^2 + y^4}$$

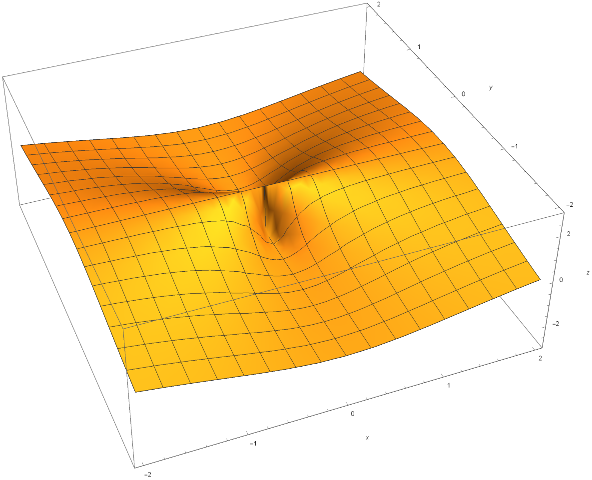


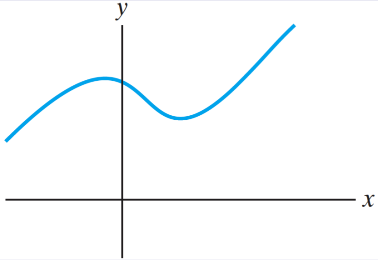
Figure: $\frac{x^2-y^2}{x^2+y^2}$

Outline

Continuity for scalar-valued functions of a single variable

A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is **continuous** if its graph can be drawn without taking the pen off the paper

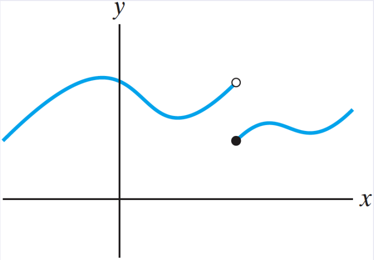
Continuous function $y = f(x)$



Continuity for scalar-valued functions of a single variable

A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is **continuous** if
its graph can be drawn without taking the pen off the paper

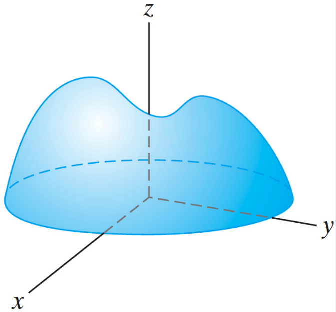
Not continuous function $y = f(x)$



Continuity for scalar-valued functions of two variables

A function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is **continuous** if its graph (in $\mathbb{R}^3$) has no breaks in it

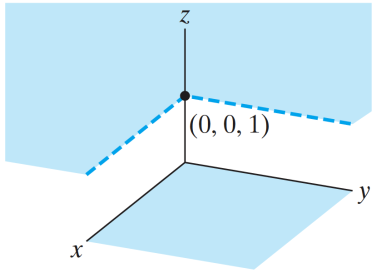
Continuous function $z = f(x, y)$



Continuity for scalar-valued functions of two variables

A function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is **continuous** if its graph (in $\mathbb{R}^3$) has no breaks in it

Not continuous function $z = f(x, y)$



Continuity for vector-valued functions

- This graphical approach to continuity is geometric and intuitive.
- But it does have real and fatal flaws:
  - We can not visualize graphs of functions of more than two variables

    How will we be able to tell
    if a function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous?
  - A graph of a function of two variables is not always sufficient to make a visual determination of continuity (as Example 14).

Definition 2.7: Continuity of functions of several variables

- Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{a} \in X$
- Then $\mathbf{f}$ is said to be continuous at $\mathbf{a}$ if either $\mathbf{a}$ is an isolated point of X or if

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$$

- If $\mathbf{f}$ is continuous at all points of its domain X, then we simply say that

f is **continuous**

Example 16

- Consider the function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2+xy-2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- Therefore

$$f(0,0) = 0$$

- But

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2+xy-2y^2}{x^2+y^2} \quad \text{does not exist} \; .$$

- Hence, f is not continuous at $(0,0)$.

Continuity for vector-valued functions

- One way of thinking about continuous functions is that

  They are the ones whose limits
  are easy to evaluate

- When $\mathbf{f}$ is continuous, the limit of $\mathbf{f}$ as $\mathbf{x}$ approaches $\mathbf{a}$ is just $\mathbf{f}(\mathbf{a})$.
- The functions that will be of primary interest to us will be continuous
  - Polynomial functions in n variables are continuous (see Example 17)
  - Linear mappings are continuous (see Example 18)
  - $\cdots$
- But recall,

  Not all functions
  are continuous

Useful Theorems

- In practice, we usually establish continuity of a function through the use of Theorems 2.5 and 2.6
    1. The sum $\mathbf{F} + \mathbf{G}$ of two functions $\mathbf{F}, \mathbf{G} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ that are continuous at $\mathbf{a} \in X$ is also continuous at $\mathbf{a}$
    2. For all $k \in \mathbb{R}$, the scalar multiple $k\mathbf{F}$ of a function $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ that is continuous at $\mathbf{a} \in X$ is also continuous at $\mathbf{a}$
    3. The product fg and the quotient f/g $(g \neq 0)$ of two scalar-valued functions $f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ that are continuous at $\mathbf{a} \in X$ are also continuous at $\mathbf{a}$
    4. $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{a} \in X$ if and only if its component functions $F_i : X \subseteq \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ are all continuous at $\mathbf{a}$

Theorem 2.8: Continuity of the Composite Function

- Assume $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : Y \subseteq \mathbb{R}^m \to \mathbb{R}^p$ are continuous functions such that,

$$\text{range } \mathbf{f} \subseteq Y$$

- Then the composite function $\mathbf{g} \circ \mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is defined and is also continuous.

Example 19

- The function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\mathbf{f}(x, y) = (x + y, x^2 y, y \sin(xy))$$

is continuous.

- In view of the properties we can see this by checking that the three component functions

$$f_1(x, y) = x + y, \quad f_2(x, y) = x^2 y \quad \text{and} \quad f_3(x, y) = y \sin(xy)$$

are each continuous (as scalar-valued functions)

Example 19

$$f_1(x,y) = x + y. \quad f_2(x,y) = x^2 y \quad \text{and} \quad f_3(x,y) = y\sin(xy)$$

- f_1 and f_2 are continuous, since they are polynomials in the two variables x and y
- The function f_3 is the product of two further functions

$$f_3(x,y) = g(x,y)h(x,y)$$

where

$$\begin{aligned} g(x,y) &= y \\ h(x,y) &= \sin(xy) \end{aligned}$$

- g is clearly continuous (It is a polynomial).
- h is also continuous since is the composition of two continuous functions.

Example 19

$$f_1(x,y) = x + y, \quad f_2(x,y) = x^2 y \quad \text{and} \quad f_3(x,y) = y\sin(xy)$$

- The function f_3 is the product of two further functions

$$f_3(x,y) = g(x,y)h(x,y)$$

  where
$$\begin{aligned} g(x,y) &= y \\ h(x,y) &= \sin(xy) \end{aligned}$$

- g is clearly continuous (It is a polynomial).
- h is also continuous since is the composition of two continuous functions.
- Thus, h, hence f_3, and, consequently, $\mathbf{f}$ are all continuous on the domain $\mathbb{R}^2$.